Hilbert's problems and contemporary mathematical logic

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Are there "BIG" open problems in mathematical logic besides the P vs. NP problem?

Ideals for the qualification "big":

- The Riemann hypothesis.
- The Continuum hypothesis.
- Fermat's last theorem.
- The Poincare conjecture.

A quote from the introduction to S.Shelah's "Logical dreams" (Bull.AMS, 2003):

... so this selection will be riddled with prejudices but at least they are mine; hopefully some others will be infuriated enough to offer differing opinions.

Attributes of ”big” problems:

- a problem not a program
  - there is a clear criterion when the problem is solved
  - not:
    - develop a theory of ...
    - explain the phenomenon that ...

- a problem not a program

- simple to state

  - can be explained to a mathematician over a cup of coffee

  - does not need a specialists knowledge, just a "general mathematical education"

- a problem not a program
- simple to state
- concerns fundamental concepts
  - not: take a super-duper-compact-alpha-this or that …

- a problem not a program
- simple to state
- concerns fundamental concepts
- challenging
  - there is an evidence that it defies the current methods
  - may represent a number of similar questions
  - at least one, better two, generations old
  - its solution will likely yield a new insight into the fundamental concepts of logic

- a problem not a program
- simple to state
- concerns fundamental concepts
- challenging
- of interest to many

Set theory

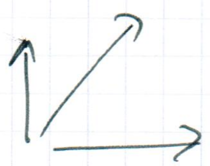
Several big programs, including:

- inner models program (core models for large cardinals),
- descriptive set theory and various forms of determinacy,
- Woodin's Ω-logic,
- Shelah's "Logical dreams" state several tens of "dreams", which are mostly programs, not problems,

but no problem meeting the requirements above.

$(\mathbb{Q}, <) \simeq$ any DLO

[Cantor]



vector spaces $/\mathbb{Q}$

$\to$ dim 1

$\to$ dim 2

$\vdots$

$\to$ dim $\aleph_0$

$\mathbb{M} \models Th(\mathbb{Z})$

$2^{\aleph_0}$ iso types $\leftarrow$

may code any $X \subseteq \mathbb{N}$

$\boxed{\text{Vaught's conjecture (1961):}}$

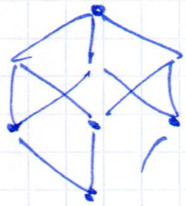
A countable complete theory has - up to an isomorphism - either a finite number of countably infinite models, countably many or continuum many.

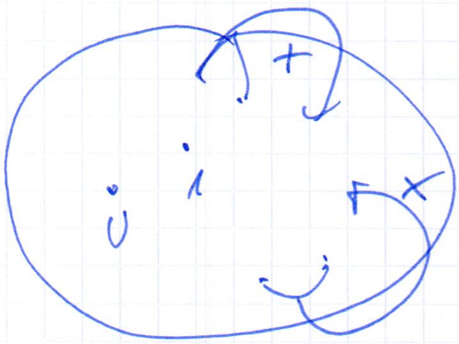
Facts:

- Trivially true if CH holds.
- Finite number: 0, 1, not 2, 3, 4, 5, ...
- Morley (1970): true if you allow also $\aleph_1$.

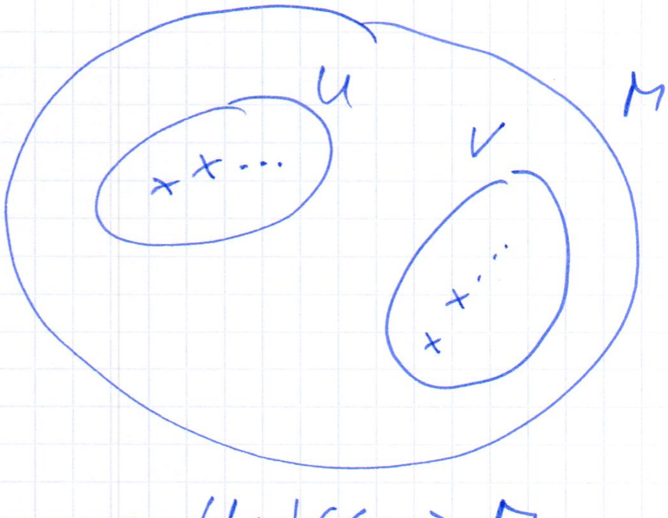
R.Knight announced a counter-example in 2002, 2007, ... , 2015 but it was not as of now accepted by model theorists.

$\Rightarrow$ even cardinalities

2^n $\Leftarrow$  Bool. algebra



fields $\Rightarrow$ prime^n

composite numbers $\Leftarrow$ 

$g : U \times V \longleftrightarrow M$

$Spec(\varphi) := \{n \in N^+ \mid \varphi$ has a model of size $n\}$

Scholz 1952: characterize spectra.

- Various characterizations are known, e.g. the class of spectra = NE = NTime($2^{O(n)}$) (Jones-Selman 1970).

Observation: Spectra are closed under $S \cap S',\ S \cup S', S + S', S \cdot S'$.

The spectrum problem (Asser 1955):

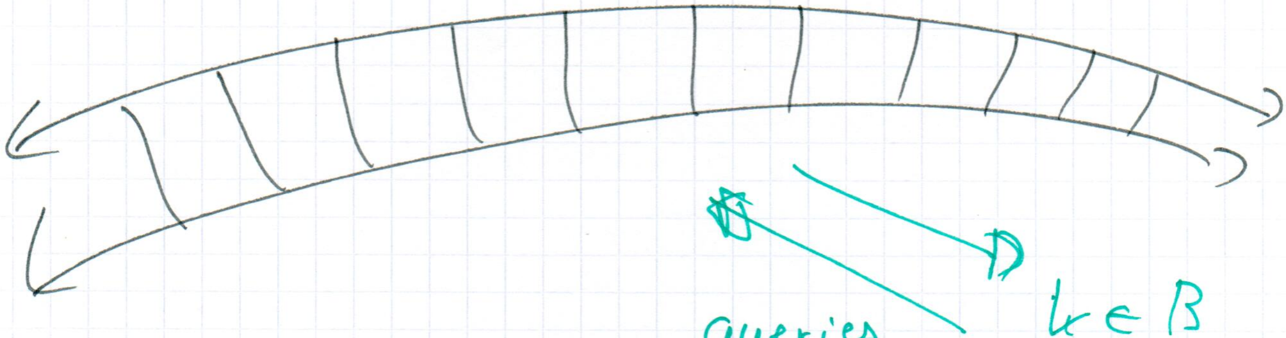
$$S \text{ spectrum} \longrightarrow N^+ \setminus S \text{ spectrum?}$$

Fact: NO $\rightarrow$ $NE \neq coNE$ $\rightarrow$ $P \neq NP$

$n \in_? A$

$\Downarrow$

Turing machine



queries

$k \in_? B$

oracle

$\Downarrow$

yes/no

$A \leq_T B$

[Post 1944]

Turing degrees and their poset $(\mathcal{D}, <)$:

- quasi-ordering $\leq_T$ factored by

$$A \equiv_T B \text{ iff } A \leq_T \wedge B \leq_T A \ ,$$

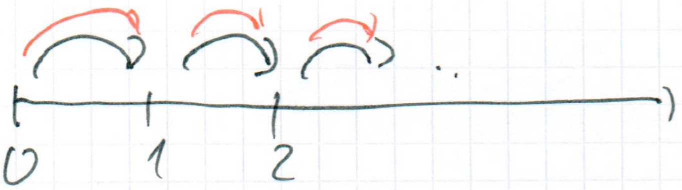
$\boxed{\text{The automorphism problem:}}$

Does $(\mathcal{D}, <)$ allow a non-trivial automorphism?

Remarks:

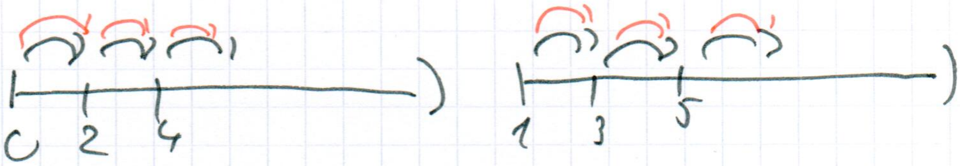
- B.Cooper announced an affirmative solution in 1999 but it is not accepted by the community.

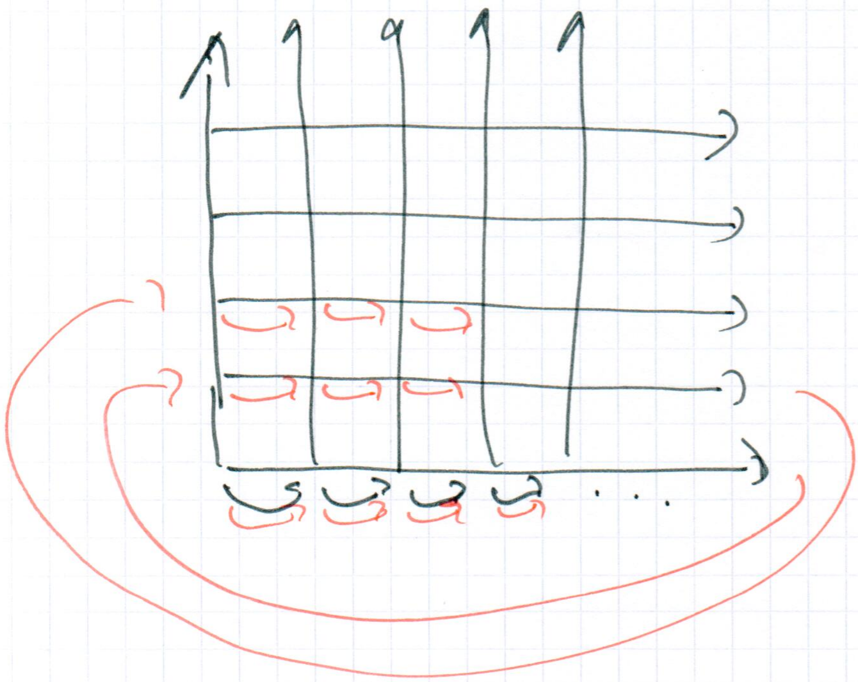
- The bi-interpretability conjecture of T.Slaman and H.Woodin (2005) implies the negative answer.



ω

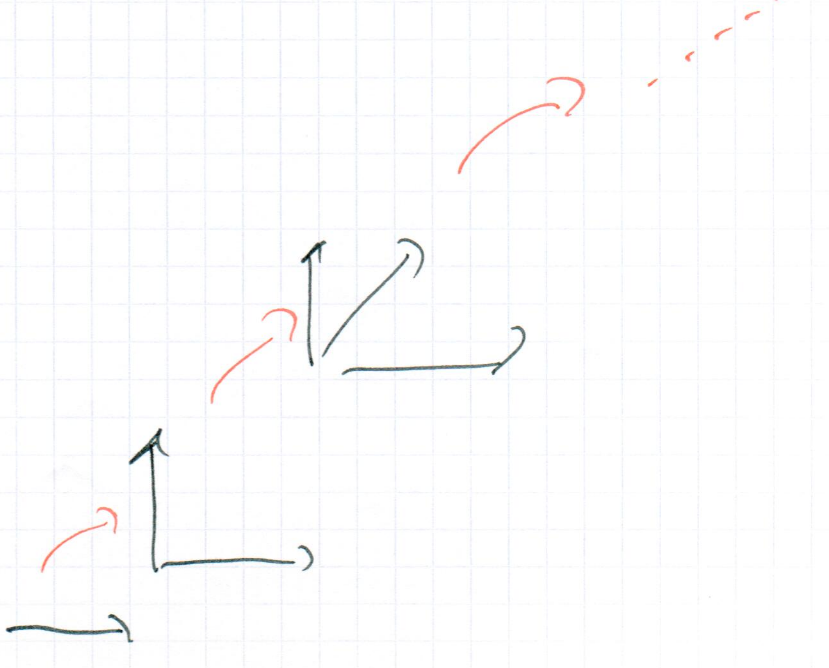
$\omega + \omega$





ω^2

ω^ω



An ordinal analysis was done for:

- PA (G.Gentzen 1936)
- Π^1_1-CA (G.Takeuti 1967)
- Π^1_2-CA (M.Rathjen and T.Arai 1990s)

$\boxed{\text{Ordinal analysis problem:}}$

Extend the ordinal analysis to Π^1_3-CA.

Experts believe that this is very likely the generic case for doing an ordinal analysis of the entire second order arithmetic Z_2.

Summary:

- Vaught's conjecture.
- The spectrum problem.
- Automorphism of $(\mathcal{D}, <)$.
- Ordinal analysis of Π^1_3-CA.

This is not a very satisfactory list: most of the problems miss several of the required attributes.

Formal lists:

- "Mathematics: Frontiers and perspectives", IMU, Eds. V.Arnold, M.Atiyah, P.Lax and B.Mazur, 1999.

  [no logic here, only Smale mentions the P vs. NP problem]

- "The millennium prize problem", AMS/Clay Math. Inst., Eds. J.Carlson, A.Jaffe and A.Wiles.

  [contains the P vs. NP problem]

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 4, Pages 407–436
S 0273-0979(00)00881-8
Article electronically published on June 26, 2000

MATHEMATICAL PROBLEMS

DAVID HILBERT

Lecture delivered before the International Congress of Mathematicians at Paris in 1900.

Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries? What particular goals will there be toward which the leading mathematical spirits of coming generations will strive? What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?

History teaches the continuity of the development of science. We know that every age has its own problems, which the following age either solves or casts aside as profitless and replaces by new ones. If we would obtain an idea of the probable development of mathematical knowledge in the immediate future, we must let the unsettled questions pass before our minds and look over the problems which the science of to-day sets and whose solution we expect from the future. To such a review of problems the present day, lying at the meeting of the centuries, seems to me well adapted. For the close of a great epoch not only invites us to look back into the past but also directs our thoughts to the unknown future.

The deep significance of certain problems for the advance of mathematical science in general and the important rôle which they play in the work of the individual investigator are not to be denied. As long as a branch of science offers an abundance of problems, so long is it alive; a lack of problems foreshadows extinction or the cessation of independent development. Just as every human undertaking pursues certain objects, so also mathematical research requires its problems. It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon.

It is difficult and often impossible to judge the value of a problem correctly in advance; for the final award depends upon the grain which science obtains from the problem. Nevertheless we can ask whether there are general criteria which mark a good mathematical problem. An old French mathematician said: “A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.” This clearness and ease of comprehension, here insisted on for a mathematical theory, I should still more demand for a mathematical problem if it is to be perfect; for what is clear and easily comprehended attracts, the complicated repels us.

Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts. It should be to us a guide

Reprinted from Bull. Amer. Math. Soc. **8** (July 1902), 437–479.

Originally published as *Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematike-Congress zu Paris 1900*, Gött. Nachr. 1900, 253-297, Vandenhoeck & Ruprecht, Göttingen. Translated for the *Bulletin*, with the author’s permission, by Dr. Mary Winston Newson, 1902.

23 problems including:

1. The continuum hypothesis.

2. The consistency of PA.

(5. On Lie groups - a non-standard approach.)

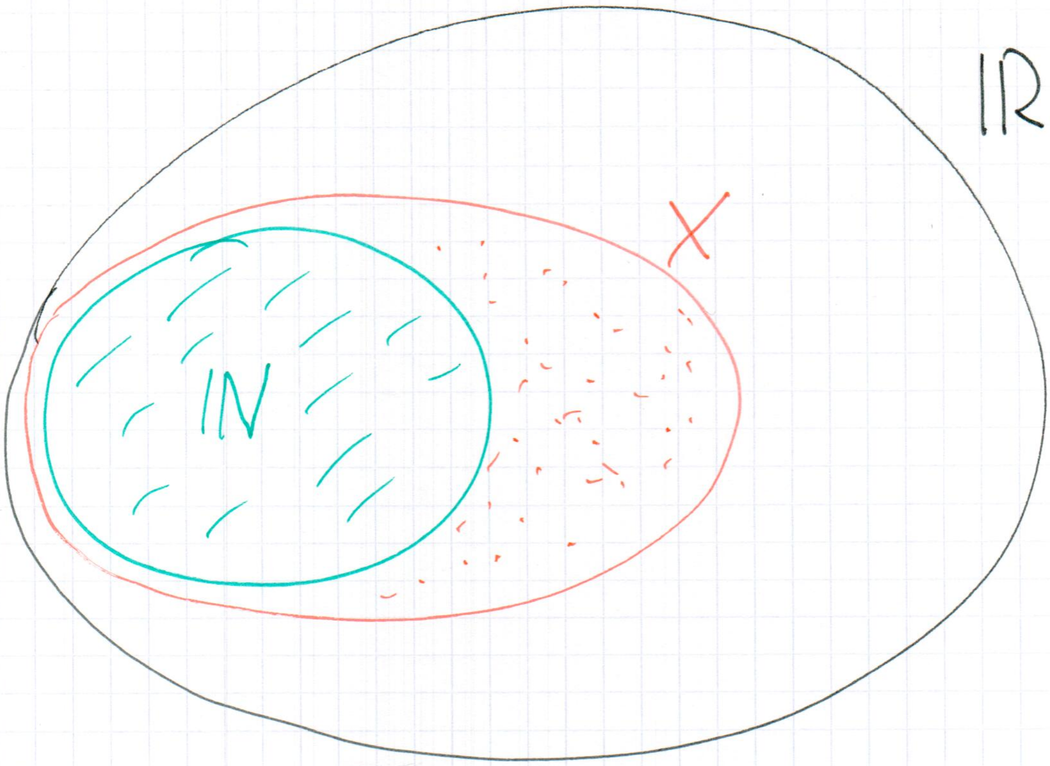
10. An algorithm for solvability of Diophantine equations.

(17. On real polynomials with non-negative values - model completeness.)

A separate problem from 1928:

- Entscheidungsproblem.

1st problem: CH



Cantor 1878: $X \approx \mathbb{N}$ or $X \approx \mathbb{R}$?

Gödel 1938/40: not refutable

Cohen 1963: not provable

The 2nd problem. Peano arithmetic PA:

- Finitely many axioms concerning $0, 1, +, \cdot$.

- The scheme of induction for all formulas φ:

$$\neg\varphi(0) \vee \exists x(\varphi(x) \wedge \neg\varphi(x+1)) \vee \forall x \varphi(x)$$

Facts:

- PA is "equivalent" to finite set theory and formalizes the syntax of first-order logic.

- K.Gödel (1931): PA does not prove its own consistency

- G.Gentzen (1934): A little bit extra induction suffices for a finitary proof.

The 10th problem.

Asks for an algorithm that would decide if a Diophantine equation:

$$(*) \qquad p(x, y, \ldots) = 0$$

is solvable in integers.

Facts:

- Y.Matiyasevich (1970, building on work by J.Robinson, M.Davis and H.Putnam):
  - Every r.e. subset of N is Diophantine, i.e. definable as:

    $$\{k \in N \mid \exists x_1, \ldots, x_n p(x_1, \ldots, x_n, k) = 0\}.$$

- Using the halting problem: There is no algorithm solving (*).

The Entscheidungsproblem.

Devise an algorithm for deciding the logical validity of first-order formulas.

Facts:

- Possible for several classes of formulas.
- Ramsey's theorem (1930) appeared in this context first.
- A.Church (1936) and A.Turing (1936-7): Impossible in general (there is a reduction of the halting problem to logical validity).

Were Hilbert's logic problems:

- The Continuum hypothesis.
- Consistency of PA.
- The 10th problem.
- Entscheidungsproblem.

solved completely or something remains?

Continuum hypothesis:

- Various programs "to settle" CH by new axioms.

Consistency of PA:

- Gödel's Con_T is a Π^0_1-statement.

- There are natural combinatorial Π^0_2-statements that are true but unprovable in ... you name it.

Problem: Find a natural combinatorial, number theoretic, ... Π^0_1-statement that is true but unprovable in PA.

E.g. the Riemann hypothesis can be formulated as Π^0_1 (M.Davis 1974).

The 10th problem:

- The problem is open for polynomial equations over Q (i.e., for homogeneous polynomials over Z).

Facts:

- J.Robinson (1948): Z is definable in Q.
- J.Koenigsmann (2016): It can be defined by a universal formula and by a $\forall\exists$-formula with 1 universal quantifier.
- An existential definition would imply the unsolvability of the problem over Q.

Entscheidungsproblem:

A natural modification turns it into a major open problem:

- Decide the logical validity of propositional formulas by a non-trivial algorithm.

Remarks:

- Non-trivial:
  - in good taste (C.F.Gauss),
  - feasible (complexity th.).
- This is equivalent to the question whether problems in NP are decidable by feasible algorithms.

Hilbert's Twenty-Fourth Problem

Rüdiger Thiele

1. INTRODUCTION. For geometers, Hilbert's influential work on the foundations of geometry is important. For analysts, Hilbert's theory of integral equations is just as important. But the address "Mathematische Probleme" [**37**] that David Hilbert (1862–1943) delivered at the second International Congress of Mathematicians (ICM) in Paris has tremendous importance for all mathematicians. Moreover, a substantial part of Hilbert's fame rests on this address from 1900 (the year after the American Mathematical Society began to publish its *Transactions*). It was by the rapid publication of Hilbert's paper [**37**] that the importance of the problems became quite clear, and it was the American Mathematical Society that very quickly supplied English-language readers with both a report on and a translation of Hilbert's address. (In Paris, the United States and England were represented by seventeen and seven participants, respectively.)

Indeed, this collection of twenty-three unsolved problems, in which Hilbert tried "to lift the veil behind which the future lies hidden" [**37**, p. 437] has occupied much attention since that time, with many mathematicians watching each contribution attentively and directing their research accordingly. Hermann Weyl (1885–1955) once remarked that "We mathematicians have often measured our progress by checking which of Hilbert's questions had been settled in the meantime" [**110**, p. 525]. (See also [**31**] and [**115**].)

Hilbert and his twenty-three problems have become proverbial. As a matter of fact, however, because of time constraints Hilbert presented *only* ten of the problems at the Congress. Charlotte Angas Scott (1858–1931) reported on the Congress and Hilbert's presentation of ten problems in the *Bulletin of the American Mathematical Society* [**91**]. The complete list of twenty-three problems only appeared in the journal *Göttinger Nachrichten* in the fall of 1900 [**37**], and Mary Winston Newson (1869–1959) translated the paper into English for the *Bulletin* in 1901 [**37**]. Already by September 1900, George Bruce Halsted (1853–1922) had written in this MONTHLY that Hilbert's beautiful paper on the problems of mathematics "is epoch-making for the history of mathematics" [**34**, p. 188]. In his report on the International Congress, Halsted devoted about forty of the article's eighty lines to the problems. As to the actual speech, no manuscript was preserved, nor was the text itself ever published.

Recently, Ivor Grattan-Guinness presented an interesting overview of Hilbert's problems in the *Notices of the American Mathematical Society*, discussing the form in which each of the twenty-three problems was published [**30**]. Yet, in dealing with the celebrated problems from this viewpoint, he failed to mention the most interesting problem of Hilbert's collection: the canceled twenty-fourth. Hilbert included it neither in his address nor in any printed version, nor did he communicate it to his friends Adolf Hurwitz (1859–1919) and Hermann Minkowski (1864–1909), who were proofreaders of the paper submitted to the *Göttinger Nachrichten* and, more significantly, were direct participants in the developments surrounding Hilbert's ICM lecture.

So, for a century now, the twenty-fourth problem has been a Sleeping Beauty. This article will try to awaken it, thus giving the reader the chance to be the latter-day Prince (or Princess) Charming who can take it home and solve it. This paper also aims to convince the reader of the utility of the history of mathematics in the sense to which

The cancelled 24th problem of Hilbert:

”The 24th problem in my Paris lecture was to be:

Criteria of simplicity, or proof of the greatest simplicity of certain proofs. Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof.

... (continues) ... ”

Fach-idiocy claims

of solutions immediately appeared:

- Cut-elimination.
- Reverse mathematics.
- Automated theorem provers.
- "Combinatorial" proof theory.
- ... .

What does it mean when we say:

* Proofs P and Q are similar.

* P is more general than Q.

* P can be derived from Q.

* P is sound but I do not understand it.

* P is long but has a simple idea.

* ...