Contents

1 Preface

This is a typeset version of the lectures Alex Wilkie gave in Semester One of the MSc course in Mathematical Logic and the Theory of Computation at the University of Manchester in the academic year 2009 / 2010. I would like to say thanks to Daniel Kirsch, whose web application `Detexify`[1] has made it a lot easier for me to look up the LaTeX codes of all those neat math symbols.

Every phrase or word is highlighted in **bold** when it is being defined.

2 Structures

First see Predicate Calculus for a formal treatment of the objects created from a formal language. Note that the following will be held as the fundamental symbols of this course which is different to Predicate Calculus.

(i) The **logical connectives** will be "$\wedge$" and "$\neg$".

(ii) The **existential quantifier** "$\exists$" rather than "$\forall$".

2.1 Definition - Similarity Type

A **similarity type** consists of three sets I, J, K (possibly some/all empty) and two functions $\rho : I \to \mathbb{N}_{>0}, \mu : J \to \mathbb{N}_{>0}$, and is sometimes written as a 5-tuple $\sigma = \langle I, J, K, \rho : I \to \mathbb{N}_{>0}, \mu : J \to \mathbb{N}_{>0}\rangle$.

2.2 Definition - Structure

$\mathfrak{A} = \langle A; \{R_i\}_{i\in I}; \{f_j\}_{j\in K}; \{c_k\}_{k\in K}\rangle$ is a **structure** of similarity type $\sigma = \langle I, J, K, \rho : I \to \mathbb{N}_{>0}, \mu : J \to \mathbb{N}_{>0}\rangle$ if and only if

1. A is a non-empy set called the domain of $\mathfrak{A}$, written $dom(\mathfrak{A})$
2. $\forall i \in I, R_i \subseteq A^{\rho(i)}$
3. $\forall j \in J, f_j : A^{\mu(j)} \to A$
4. $\forall k \in K, c_k \in A$

Examples

1.1 (a),(b): $\langle \mathbb{R}; +; 0\rangle$, $\langle \mathbb{R}; \cdot; 1\rangle$
These are structures with $I = \emptyset\, (\rho = \emptyset), J = \{1\}, \mu(1) = 2, K = \{1\}$.
$\left.\begin{smallmatrix} f_1 \text{ is } +,\cdot \\ e_1 \text{ is } 0,1 \end{smallmatrix}\right\}$ respectively.
For 1.2: $\langle \mathbb{R}; <; +; 0\rangle$
$I = \{1\}, \rho(1) = 2, J = \{1\}, \mu(1) = 2, K = \{1\}$
Here: R_1 is $< (\subseteq R^2)$, i.e. $<= \{(a, b)|a, b \in \mathbb{R}, a < b\}$
f_1 is $+$, e_1 is 0.
A is $\mathbb{R}$ in these examples.
For 1.4 $\langle G; \cdot, ^{-1}; e\rangle$,
the similarity type is given by $I = \emptyset, J = \{1, 2\}, \mu(1) = 2, \mu(2) = 1, K = \{1\}$. Here, f_1 is $\cdot$, f_2 is $^{-1}$, e_1 is e.

2.3 Definition - Class of all Structures

Let σ be a similarity type, K_σ denotes the class of all structures of similarity type σ.

[1] which can be accessed via `http://detexify.kirelabs.org`

2.4 Definition - Isomorphic

Let $\mathfrak{A}, \mathfrak{B} \in K_\sigma$, where $\mathfrak{A} = \langle A; \{R_i\}_{i\in I}; \{f_j\}_{j\in K}; \{c_k\}_{k\in K}\rangle$ and $\mathfrak{B} = \langle B; \{S_i\}_{i\in I}; \{g_j\}_{j\in K}; \{d_k\}_{k\in K}\rangle$, let $\pi : \mathfrak{A} \to \mathfrak{B}$, then we say that π is an **isomorphism** from $\mathfrak{A}$ to $\mathfrak{B}$ if and only if

1. it is bijective
2. $\forall i \in I, \forall a_1, ..., a_{\rho(i)} \in A$
   $(a_1, ..., a_{\rho(i)}) \in R_i \Leftrightarrow (\pi(a_1), ..., \pi(a_{\rho(i)})) \in S_i$
3. $\forall j \in J, \forall a_1, ..., a_{\mu(j)} \in A$
   $\pi(f_j(a_1, ..., a_{\mu(j)})) = g_j(\pi(a_1), ..., \pi(a_{\mu(j)}))$
4. $\forall k \in K, \pi(c_k) = d_k$

We write $\pi : \mathfrak{A} \cong \mathfrak{B}$ if and only if π is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, and we write $\mathfrak{A} \cong \mathfrak{B}$ if and only if there exists $\pi : \mathfrak{A} \cong \mathfrak{B}$.

Example

Let $\mathfrak{A} = \langle \mathbb{R}; <; +; 0\rangle$, $\mathfrak{B} = \langle \mathbb{R}_{>0}; <; \cdot; 1\rangle$.
Then $\mathfrak{A} \cong \mathfrak{B}$.
Consider the function $\pi : \mathbb{R} \to \mathbb{R}_{>0}$ given by $\pi(x) = e^x$.

- It is a bijection from $\mathbb{R}$ to $\mathbb{R}_{>0}$.
- $\forall x, y \in \mathbb{R}\, x < y \leftrightarrow e^x < e^y$, i.e. $\langle x, y\rangle \in R_1$ iff $\langle e^x, e^y\rangle \in S_1$.
- $\forall x, y \in \mathbb{R}\, \pi(f_1(x, y)) = \pi(x + y) = e^{x+y} = e^x \cdot e^y = \pi(x) \cdot \pi(y) = g_1(\pi(x), \pi(y))$, so (c) holds.

Also, $\pi(e_1) = \pi(0) = e^0 = 1 = d_1$. So (d) holds.

2.5 Non-Example

Let $\mathfrak{A} = \langle \mathbb{Q}; <\rangle, \mathfrak{B} = \langle \mathbb{R}; <\rangle$.
Then $\mathfrak{A} \ncong \mathfrak{B}$ since there isn't even a bijection from $\mathbb{Q}$ to $\mathbb{R}$.

2.6 Definition - Embedding

In definition 2.4 π is an **embedding** if and only if π satisfies (2) - (4) and is injective, written $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$.

2.7 Definition - Substructure

In definition 2.4 $\mathfrak{A}$ is a **substructure** of $\mathfrak{B}$ written $\mathfrak{A} \subseteq \mathfrak{B}$ if and only if $A \subseteq B$ and the identity $Id_A : A \to B$ is an embedding from $\mathfrak{A}$ to $\mathfrak{B}$. Note that the identity is clearly injective so we only need check (2) - (4) for substructures.

Examples

1. Let $\mathfrak{A} = \langle \mathbb{R}; +, \cdot; 0\rangle$
   $\mathfrak{B} = \langle \mathbb{C}; +, \cdot; 0\rangle$.
   Then $\mathfrak{A} \subseteq \mathfrak{B}$.
   But if $\mathfrak{A}' = \langle \mathbb{R}; +, \cdot; 1\rangle$ we have $\mathbb{R} \subseteq \mathbb{C}$, but $\mathfrak{A}' \nsubseteq \mathfrak{B}$ since it's not the case that $\mathrm{id}_A(1) = 0$.
2. If $\mathcal{G}_1 = \langle G_1; \circ, ^{-1}; e_1\rangle$,
   $\mathcal{G}_2 = \langle G_2; \circ', ^{-1'}; e_2\rangle$
   are groups and $G_1 \subseteq G_2$, then $\mathcal{G}_1 \subseteq \mathcal{G}_2$ precisely if $\mathcal{G}_1$ is a subgroup of $\mathcal{G}_2$.

3 A Little Universal Algebra

3.1 Remark

Let $\sigma = \langle I, J, K, \rho : I \to \mathbb{N}_{>0}, \mu : J \to \mathbb{N}_{>0}\rangle$ be a similarity type s.t. $J = \emptyset = K, \mathfrak{B} \in K_\sigma, \mathfrak{B} = \langle B; \{R_i\}_{i\in I}\rangle$, and $A \subseteq B$, then $\mathfrak{A} := \langle A; \{R_i \cap A^{\rho(i)}\}_{i\in I}\rangle \subseteq \mathfrak{B}$.

Proof:

$\mathfrak{A}$ satisfies 2.4 (3) - (4) vacuously. $\forall i \in I, \forall a_1, ..., a_{\rho(i)} \in A$,

$$\begin{aligned}(a_1, ..., a_{\rho(i)}) \in R_i \cap A^{\rho(i)} &\Leftrightarrow (a_1, ..., a_{\rho(i)}) \in R_i \\ &\Leftrightarrow (Id_A(a_1), ..., Id_A(a_{\rho(i)})) \in R_i\end{aligned}$$

thus (2) also holds. ■

3.2 Lemma

Let $\sigma = \langle I, J, K, \rho : I \to \mathbb{N}_{>0}, \mu : J \to \mathbb{N}_{>0}\rangle$ be a similarity type s.t. $J = \emptyset, \mathfrak{B} \in K_\sigma, \mathfrak{B} = \langle B; \{R_i\}_{i\in I}; \emptyset; \{c_k\}_{k\in K}\rangle$, and $A \subseteq B$, then $\mathfrak{A} := \langle A \cup \{c_k | k \in K\}; \{R_i \cap (A^{\rho(i)} \cup \{c_k | k \in K\}\}_{i\in I}; \emptyset; \{c_k\}_{k\in K}\rangle \subseteq \mathfrak{B}$.

Proof:

$\mathfrak{A}$ satisfies 2.4 (2) - (3) by the same argument in the proof of 3.1 and (4) holds trivially. ■

3.3 Remarks

Let $\sigma = \langle I, J, K, \rho : I \to \mathbb{N}_{>0}, \mu : J \to \mathbb{N}_{>0}\rangle$ be a similarity type with $J \neq \emptyset$, let $\mathfrak{A}, \mathfrak{B} \in K_\sigma$ s.t. $\mathfrak{A} \subseteq \mathfrak{B}$ (with conventional notion), then

1. $\forall a_1, ..., a_{\mu(j)} \in A, \forall j \in J, g_j(a_1, ..., a_{\mu(j)}) \in A$, i.e. A must be **closed** under the function g_j
2. $\forall j \in J, g_j|_A = f_j$
3. $\forall k \in K, d_k \in A$

Proof:

$\mathfrak{A} \subseteq \mathfrak{B}$ so by 2.4 (3) we have $\forall j \in J, \forall a_1, ..., a_{\mu(j)} \in A, Id_A(f_j(a_1, ..., a_{\mu(j)})) = g_j(Id_A(a_1), ..., Id_A(a_{\mu(j)}))$
$\Rightarrow (f_j(a_1, ..., a_{\mu(j)})) = g_j(a_1, ..., a_{\mu(j)})$, this proves part (2)
by definition of a structure $(f_j(a_1, ..., a_{\mu(j)})) \in A$
$\therefore g_j(a_1, ..., a_{\mu(j)}) \in A$
Finally (3) follows trivially from 2.4 (4) ■

3.4 Theorem

Let $\sigma = \langle I, J, K, \rho : I \to \mathbb{N}_{>0}, \mu : J \to \mathbb{N}_{>0}\rangle$ be a similarity type, Let $\mathfrak{B}$ be a structure and $\emptyset \neq S \subseteq B :=$ dom($\mathfrak{B}$), then there is a unique **smallest** substructure $\mathfrak{A}$ of $\mathfrak{B}$ such that $S \subseteq A :=$ dom($\mathfrak{A}$), where **smallest** means that if $\mathfrak{A}' \subseteq \mathfrak{B}$ with $S \subseteq A' :=$ dom($\mathfrak{A}$'), then $A \subseteq A'$.

Proof:

Let $\sigma = \langle I, J, K, \rho : I \to \mathbb{N}_{>0}, \mu : J \to \mathbb{N}_{>0}\rangle$ be the similarity type s.t. $\mathfrak{B} \in K_\sigma$ and let $\mathfrak{B} = \langle B; \{S_i\}_{i\in I}; \{g_j\}_{j\in K}; \{d_k\}_{k\in K}\rangle$.
Define $\mathscr{S} := \{X | S \cup \{d_k : k \in K\} \subseteq X \subseteq B$ and $\forall j \in J, g_j(X) \subseteq X\}$, $A := \cap\mathscr{S}$.
$s \in S \cup \{d_k | k \in K\}$
$\Rightarrow \forall X \in \mathscr{S}, s \in X$
$\Rightarrow s \in A$,
$\therefore S \cup \{d_k | k \in K\} \subseteq A$.

$\forall j \in J, \forall a_1, ..., a_{\mu(j)} \in A, \forall X \in S, a_1, ..., a_{\mu(j)} \in X$ by definition of A, so $g_j(a_1, ..., a_{\mu(j)}) \in X$ by definition of S.

$\Rightarrow g_j(a_1, ..., a_{\mu(j)}) \in A$ thus A is closed under the functions of $\mathfrak{B}$, so if we define $f_j := g_j|_A$ as the functions of a stucture $\mathfrak{A}$ with domain A, then $\mathfrak{A}$ satisfies (3) of 2.4. Then by 3.1 we can define relations and constants for $\mathfrak{A}$ s.t. $\mathfrak{A}$ satisfies 2.4 (2) and (4).
$\therefore \mathfrak{A} \subseteq \mathfrak{B}$
Let $\mathfrak{A}' \subseteq \mathfrak{B}$ s.t. $S \subseteq A' :=$ dom($\mathfrak{A}$'), then by 3.3 (3) $\{d_k | k \in K\} \subseteq A'$, then by 3.3 (1) and the definition of $\mathscr{S}$ we have $A' \in \mathscr{S}$.
$\therefore A \subseteq A'$ ■

NOTE:

The following two theorems will in future be used without reference.

3.5 Theorem

Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ be structures of the same similarity type (with conventional notation), let $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ and $\varphi : \mathfrak{B} \hookrightarrow \mathfrak{C}$, then $\varphi \circ \pi : \mathfrak{A} \hookrightarrow \mathfrak{C}$.

Proof:

(2.4 (1) Part)
From set theory $\varphi \circ \pi$ is injective as both φ and π are.
(2.4 (3) Part)
Since $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ we have $\forall j \in J, \forall a_1, ..., a_{\mu(j)} \in A$
$\pi(f_j(a_1, ..., a_{\mu(j)})) = g_j(\pi(a_1), ..., \pi(a_{\mu(j)}))$
$\Rightarrow \varphi(\pi(f_j(a_1, ..., a_{\mu(j)}))) = \varphi(g_j(\pi(a_1), ..., \pi(a_{\mu(j)})))$
Since $\varphi : \mathfrak{B} \hookrightarrow \mathfrak{C}$ and $\pi(a_1), ..., \pi(a_{\mu(j)}) \in B$ we have
$\varphi(g_j(\pi(a_1), ..., \pi(a_{\mu(j)}))) = h_j(\varphi(\pi(a_1)), ..., \varphi(\pi(a_{\mu(j)})))$
$\therefore \varphi(\pi(f_j(a_1, ..., a_{\mu(j)}))) = h_j(\varphi(\pi(a_1)), ..., \varphi(\pi(a_{\mu(j)})))$
(2.4 (2) and (4) Part) **- Exercise 1(A)(i) for week 3**

3.6 Theorem

Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ be structures of the same similarity type (with conventional notation).

1. if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathfrak{C}$ then $\mathfrak{A} \subseteq \mathfrak{C}$.
2. if $\mathfrak{A} \cong \mathfrak{B}$ and $\mathfrak{B} \cong \mathfrak{C}$ then $\mathfrak{A} \cong \mathfrak{C}$.
3. if $\pi : \mathfrak{A} \cong \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathfrak{C}$ then $\pi : \mathfrak{A} \hookrightarrow \mathfrak{C}$.
4. if $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ then $\exists \mathfrak{D} \subseteq \mathfrak{B}$ s.t. $\pi : \mathfrak{A} \cong \mathfrak{D}$.
5. if $\mathfrak{B} \subseteq \mathfrak{C}, \mathfrak{A} \subseteq \mathfrak{C}$ and $A \subseteq B$, then $\mathfrak{A} \subseteq \mathfrak{B}$.

Proof: Exercise

Note (4) is exercise 1(A)(ii) for week 3 and (5) is exercise 1(B) for week 3.

4 Interpretations

The equality symbol of the object language will be written $\doteq$, and so if a meta-variable φ is a $\mathscr{L}_\sigma$-string we will write φ = *string*. The definitions for interpretations here can be shown to be equivalent to those in Predicate Logic (see Predicate Logic).

4.1 Definition - The Interpretation of Terms

Let $\mathfrak{A}$ be a structure (with conventional notation), and τ be an $\mathscr{L}_\sigma$-term. Let n be **suitable** for τ, where n is **suitable** means that $\forall p$ s.t. $v_p \in \mathrm{Var}(\tau)$, $n \geq p$. The **Interpretation** of the term τ denoted $\tau^{\mathfrak{A}}$ is a function $\tau^{\mathfrak{A}} : A^n \to A$ s.t.

1. $\tau = v_p \Rightarrow \forall a_1, ..., a_n \in A, v_p^{\mathfrak{A}}(a_1, ..., a_n) := a_p$
2. $\exists k \in K$, s.t. $\tau = c_k \Rightarrow \forall a_1, ..., a_n \in A, c_k^{\mathfrak{A}}(a_1, ..., a_n) := d_k$
3. if $\tau = F_j(\tau_1, ..., \tau_{\mu(j)})$ then if $\tau_1^{\mathfrak{A}} : A^n \to A, ..., \tau_{\mu(j)}^{\mathfrak{A}} : A^n \to A$ are defined then $\forall a_1, ..., a_n \in A, \tau^{\mathfrak{A}}(a_1, ..., a_n) := f_j(\tau_1(a_1, ..., a_n), ..., \tau_{\mu(j)}(a_1, ..., a_n))$.

This definition makes sense by Unique Readability[2] for terms.

Example

Consider $\tau = F_2(v_3, F_1(F_2(v_6, c_1)))$ and consider the structure $\mathfrak{A} = \langle \mathbb{R}; +, -; 0 \rangle$ [3]
bf Take n=6 (NB $\mathrm{Var}(\tau) = \{v_3, v_6\}$)
Then for $r_1, \ldots, r_6 \in R$

$$\begin{aligned} \tau^{\mathfrak{A}}(r_1, \ldots, r_6) &= +(r_3, -(+(r_6, 0))) \\ &= r_3 + (-(r_6 + 0)) \\ &= r_3 - r_6. \end{aligned}$$

4.2 Lemma

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures (with conventional notation), let τ be an $\mathscr{L}_\sigma$-term, let n be suitable for τ and let $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$. Then for all $a_1, ..., a_n \in A, \pi(\tau^{\mathfrak{A}}(a_1, ..., a_n)) = \tau^{\mathfrak{B}}(\pi(a_1), ..., \pi(a_n)))$.

Proof:

We prove by induction on the length of τ (i.e. the number of symbols in τ).

(Base Cases)

$$\begin{aligned} \tau = v_p \quad &\Rightarrow \quad \tau^{\mathfrak{A}}(a_1, ..., a_n) = a_p \quad \text{by4.1(1)} \\ &\Rightarrow \quad \pi(\tau^{\mathfrak{A}}(a_1, ..., a_n)) = \pi(a_p) \\ &\qquad = \tau^{\mathfrak{B}}(\pi(a_1), ..., \pi(a_n)) \quad \text{by4.1(1)} \\ \exists k \in K \quad \text{s.t.} \quad \tau = c_k \quad &\Rightarrow \quad \tau^{\mathfrak{A}}(a_1, ..., a_n) = c_k \quad \text{by4.1(2)} \\ &\Rightarrow \quad \pi(\tau^{\mathfrak{A}}(a_1, ..., a_n)) = \pi(c_k) \\ &\qquad = d_k \\ &\qquad = \tau^{\mathfrak{B}}(\pi(a_1), ..., \pi(a_n)) \quad \text{by4.1(2)} \end{aligned}$$

As required.

[2]see predicate logic for all the unique readability theorems
[3]F_2 is $+$, F_1 is $-$, $\mu(2) = 2, \mu(1) = 1$

(Inductive Step)

Assume true for all $\mathscr{L}_\sigma$-terms with length less than n. Let τ have exactly length n and let $\tau = F_j(\tau_1, ..., \tau_{\mu(j)})$, where $\tau_1, ..., \tau_{\mu(j)}$ are $\mathscr{L}_\sigma$-terms. Now by Unique Readability τ is uniquely determined by $\tau_1, ..., \tau_{\mu(j)}$, thus $\tau_1, ..., \tau_{\mu(j)}$ must all have length less than n. Let $(a_1, ..., a_{\mu(j)}) =: \bar{a} \in A^{\mu(j)}$, then

$$\begin{aligned}
&\tau^{\mathfrak{A}}(\bar{a}) = f_j(\tau_1^{\mathfrak{A}}(\bar{a}), ..., \tau_{\mu(j)}^{\mathfrak{A}}(\bar{a})) \quad \text{by4.1(3)} \\
\Rightarrow \quad &\pi(\tau^{\mathfrak{A}}(\bar{a})) = \pi(f_j(\tau_1^{\mathfrak{A}}(\bar{a}), ..., \tau_{\mu(j)}^{\mathfrak{A}}(\bar{a}))) \\
&= g_j(\pi(\tau_1^{\mathfrak{A}}(\bar{a}), ..., \tau_{\mu(j)}^{\mathfrak{A}}(\bar{a}))) \quad \text{by2.4(3)} \\
&= g_j(\tau_1^{\mathfrak{B}}(\pi(a_1), ..., \pi(a_{\mu(j)})), ..., \tau_{\mu(j)}^{\mathfrak{B}}(\pi(a_1), ..., \pi(a_{\mu(j)}))) \quad \text{by the inductive hypothesis} \\
&= \tau^{\mathfrak{B}}(\pi(a_1), ..., \pi(a_{\mu(j)})) \quad \text{by4.1(3), i.e. by definition of } \tau^{\mathfrak{B}}
\end{aligned}$$

■

4.3 Definition - Interpretation of $\mathscr{L}_\sigma$-atomic formulas

If φ is an $\mathscr{L}_\sigma$-atomic formula and $\mathrm{Var}(\varphi) \subseteq \{v_1, ..., v_n\}$ then we write φ as $\varphi(v_1, ..., v_n)$. Let $\mathfrak{A}$ be an $\mathscr{L}_\sigma$-structure (with conventional notation) and $a_1, ..., a_n \in A$ we say "φ is true in $\mathfrak{A}$ at $(a_1, ..., a_n)$" and write $\mathfrak{A} \vDash \varphi[a_1, ..., a_n]$ if and only if

1. if $\varphi = \tau_1 \doteq \tau_2$ where τ_1 and τ_2 are $\mathscr{L}_\sigma$-terms then $\tau_1^{\mathfrak{A}}(a_1, ..., a_n) = \tau_2^{\mathfrak{A}}(a_1, ..., a_n)$ or
2. if $\varphi = P_i(\tau_1, ..., \tau_{\rho(i)})$ where for each $i \in I, \tau_i$ is a $\mathscr{L}_\sigma$-term then $(\tau_1^{\mathfrak{A}}(a_1, ..., a_n), ..., \tau_{\rho(i)}^{\mathfrak{A}}(a_1, ..., a_n)) \in R_i$.

This definition makes sense by Unique Readability for atomic formulas.

Example

Take $\mathfrak{A} = \langle \mathbb{R}; <; -, +; 0\rangle$[4].
Take $n = 6$: Then, for φ_1 : $F_1(v_4, F_2(v_3, F_1(F_2(v_6, c_1))))$, φ_2 : $F_2(v_1, v_2) \doteq F_2(v_2, v_1)$, we have for all $r_1, \ldots, r_6 \in \mathbb{R}, \mathfrak{A} \models \varphi_1[r_1, \ldots, r_n]$ iff $r_4 < r_3 + (-(r_6 + 0))$ iff $r_4 < r_3 - r_6$.

(One can easily show that the atomic formulas of the type $P_1(\tau_1, \tau_2)$ express strict inequality between homogeneous linear functions of the variables, with integer coefficients.)

Take $n = 2, \mathfrak{A} \models \varphi_2[r_1, r_2]$ iff $r_1 + r_2 = r_2 + r_1$ (which is always true)

4.4 Lemma

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures (with conventional notation), let φ be an $\mathscr{L}_\sigma$-atomic formula, let n be suitable for φ and let $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$. Then for all $a_1, ..., a_n \in A, \mathfrak{A} \vDash \varphi[a_1, ..., a_n] \Leftrightarrow \mathfrak{B} \vDash \varphi[\pi(a_1), ..., \pi(a_n)]$.

Proof:

Write $\overline{a} = (a_1, ..., a_n), \overline{\pi(a)} = (\pi(a_1), ..., \pi(a_n))$

$$\begin{aligned}
\mathfrak{A} \vDash \tau_1 \doteq \tau_2[\overline{a}] \quad &\Leftrightarrow \quad \tau_1^{\mathfrak{A}}(\overline{a}) = \tau_2^{\mathfrak{A}}(\overline{a}) \quad \text{by 4.3 (i)} \\
&\Leftrightarrow \quad \pi(\tau_1^{\mathfrak{A}}(\overline{a})) = \pi(\tau_2^{\mathfrak{A}}(\overline{a})) \quad \text{since } \pi \text{ is an 1-1 function} \\
&\Leftrightarrow \quad \tau_1^{\mathfrak{B}}(\overline{\pi(a)}) = \tau_2^{\mathfrak{B}}(\overline{\pi(a)}) \quad \text{by 4.2} \\
&\Leftrightarrow \quad \mathfrak{B} \vDash \tau_1 \doteq \tau_2[\overline{\pi(a)}] \quad \text{by 4.3 (i)} \\
\mathfrak{A} \vDash P_i(\tau_1, ..., \tau_{\rho(i)})[\overline{a}] \quad &\Leftrightarrow \quad (\tau_1^{\mathfrak{A}}(\overline{a}), ..., \tau_{\rho(i)}^{\mathfrak{A}}(\overline{a})) \in R_i \quad \text{by 4.3 (ii)} \\
&\Leftrightarrow \quad (\pi(\tau_1^{\mathfrak{A}}(\overline{a}), ..., \pi(\tau_{\rho(i)}^{\mathfrak{A}}(\overline{a})))) \in S_i \quad \text{by 2.4 (ii)} \\
&\Leftrightarrow \quad (\tau_1^{\mathfrak{B}}(\overline{\pi(a)}), ..., \tau_{\rho(i)}^{\mathfrak{B}}(\overline{\pi(a)})))) \in S_i \quad \text{by 4.2} \\
&\Leftrightarrow \quad \mathfrak{B} \vDash P_i(\tau_1, ..., \tau_{\rho(i)})[\overline{\pi(a)}] \quad \text{by 4.3 (ii)}
\end{aligned}$$

■

[4] where R_1 is $<$, f_1 is $-$, f_2 is $+$, e_1 is 0

4.5 Lemma

Let n be suitable for the $\mathscr{L}_\sigma$-atomic formula φ and $a_1, ..., a_n, a'_1, ..., a'_n \in A$ s.t. $a_l = a'_l$ for all l s.t. $v_l \in \text{var}(\varphi)$, then $\mathfrak{A} \vDash \varphi[a_1, ..., a_n] \Leftrightarrow \mathfrak{A} \vDash \varphi[a'_1, ..., a'_n]$.

Proof: Exercise

4.6 Lemma - Converse of 4.4

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures (with conventional notation), let φ be an $\mathscr{L}$-atomic formula, let n be suitable for φ and let $\pi : \mathfrak{A} \to \mathfrak{B}$. If for all $a_1, ..., a_n \in A, \mathfrak{A} \vDash \varphi[a_1, ..., a_n] \Leftrightarrow \mathfrak{B} \vDash \varphi[\pi(a_1), ..., \pi(a_n)]$ then $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$.

Proof: Exercise

4.7 Definition - Substitution

Terms: Let τ, λ be $\mathscr{L}_\sigma$-terms and $p \geq 1$ then

1. if $\tau = c_k$ then define $(c_k)^\lambda_p := c_k$
2. if $\tau = v_q$ then define
$$(v_q)^\lambda_p := \begin{cases} \lambda & \text{if } p = q \\ v_q & \text{if } p \neq q \end{cases}$$
3. if $\tau = F_j(\tau_1, ..., \tau_{\mu(j)})$ where $\tau_1, ..., \tau_{\mu(j)}$ are $\mathscr{L}_\sigma$-terms and $(\tau_1)^\lambda_p, ..., (\tau_{\mu(j)})^\lambda_p$ have already been defined then define $(F_j(\tau_1, ..., \tau_{\mu(j)}))^\lambda_p := F_j((\tau_1)^\lambda_p, ..., (\tau_{\mu(j)})^\lambda_p)$.

Atomic Formulas: Let φ be an $\mathscr{L}_\sigma$-atomic formula, λ an $\mathscr{L}_\sigma$-term and $p \geq 1$ then

1. if $\varphi = \tau_1 \doteq \tau_2$ where τ_1, τ_2 are $\mathscr{L}_\sigma$-terms then $(\tau_1 \doteq \tau_2)^\lambda_p := (\tau_1)^\lambda_p \doteq (\tau_2)^\lambda_p$
2. if $\varphi = P_i(\tau_1, ..., \tau_{\mu(j)})$ where $\tau_1, ..., \tau_{\mu(j)}$ are $\mathscr{L}_\sigma$-terms then($P_i(\tau_1, ..., \tau_{\rho(i)}))^\lambda_p := P_i((\tau_1)^\lambda_p, ..., (\tau_{\rho(i)})^\lambda_p)$.

Formulas: Let φ, ψ be $\mathscr{L}_\sigma$-quantifier free formulas already defined, λ an $\mathscr{L}_\sigma$-term and $p \geq 1$ then

1. $(\neg\varphi)^\lambda_p := \neg(\varphi)^\lambda_p$
2. $(\varphi \wedge \psi)^\lambda_p := (\varphi)^\lambda_p \wedge (\psi)^\lambda_p$
3. $$(\exists v_j \psi)^\lambda_p := \begin{cases} \exists v_j (\psi)^\lambda_p & \text{if } j \neq p \\ \exists v_j \psi & \text{if } j = p \end{cases}$$

Substitution is well defined by Unique Readability.

4.8 Lemma

Notation as above and $\mathfrak{A}$ any σ-structure.

1. $(\tau)^\lambda_p$ is an $\mathscr{L}_\sigma$-term, moreover for all $\overline{a} := (a_1, ..., a_n) \in A^n, (\tau)^{\lambda\mathfrak{A}}_p(\overline{a}) = \tau^\mathfrak{A}(a_1, ..., a_{p-1}, \lambda^\mathfrak{A}(\overline{a}), a_{p+1}, a_n)$.
2. $(\varphi)^\lambda_p$ is an $\mathscr{L}_\sigma$-quantifier free formula and if n is suitable for φ and λ then for all $\overline{a} := (a_1, ..., a_n) \in A^n$,
$$\mathfrak{A} \vDash (\varphi)^\lambda_p[\overline{a}] \Leftrightarrow \mathfrak{A} \vDash \varphi[a_1, ..., a_{p-1}, \lambda^\mathfrak{A}(\overline{a}), a_{p+1}, ..., a_n]$$

See later for similar result for formulas.

Proof: Exercise - on week 4 problems

4.9 Definition - Interpretation of $\mathscr{L}_\sigma$-formulas

Let φ, ψ be $\mathscr{L}_\sigma$-formulas and n be suitable for φ and ψ. Let $\mathfrak{A}$ be an $\mathscr{L}_\sigma$-structure and $a_1, ..., a_n \in A$ we define "φ is true in $\mathfrak{A}$ at $(a_1, ..., a_n)$" and $\mathfrak{A} \vDash \varphi[a_1, ..., a_n]$ inductively by

1. $\mathfrak{A} \vDash (\neg\varphi)[a_1, ..., a_n] \Leftrightarrow \mathfrak{A} \vDash \varphi[a_1, ..., a_n]$ does not hold
2. $\mathfrak{A} \vDash (\varphi \wedge \psi)[a_1, ..., a_n] \Leftrightarrow \mathfrak{A} \vDash \varphi[a_1, ..., a_n]$ and $\mathfrak{A} \vDash \psi[a_1, ..., a_n]$
3. $\mathfrak{A} \vDash \exists v_p \varphi[a_1, ..., a_n] \Leftrightarrow \mathfrak{A} \vDash \varphi[a_1, ..., a_{p-1}, b_p, a_{p+1}, ..., a_n]$ for some $b_p \in A$

This definition makes sense by Unique Readability for formulas.

4.10 Example

Let $I = K = \emptyset, J = \{1\}, \mu(1) = 2$.
$v_1 \doteq F_1(v_2, v_2), v_1 \doteq F_1(v_3, F_1(v_3, v_3))$ are atomic formulas.
$(v_1 \doteq F_1(v_2, v_2) \wedge v_1 \doteq F_1(v_3, F_1(v_3, v_3)))$ is a formula.
then we get
$\varphi : \exists v_2 \underbrace{\exists v_3 (v_1 \doteq F_1(v_2, v_2) \wedge \underbrace{v_1 \doteq F_1(v_3, F_1(v_3, v_3))}_{\theta})}_{\psi}$ is a formula.

Then $\mathrm{Var}(\varphi) = \{v_1, v_2, v_3\}, \mathrm{FrVar}(\varphi) = \{v_1\}$.
Let $\mathfrak{A} = \langle \mathbb{Z}; + \rangle$. Let $n = 3$. Let $a_1, a_2, a_3 \in \mathbb{Z}$.
Then $\mathfrak{A} \models \varphi[a_1, a_2, a_3] \Leftrightarrow$ for some $b_2 \in \mathbb{Z}, \mathfrak{A} \models \psi[a_1, b_2, a_3]$ (by 4.9)
$\Leftrightarrow$ for some $b_2 \in \mathbb{Z}$(for some $b_3 \in \mathbb{Z}$, $\mathfrak{A} \models \theta[a_1, b_2, b_3]$)
$\Leftrightarrow$ for some $b_2, b_3 \in \mathbb{Z}, (\mathfrak{A} \models v_1 \doteq F_1(v_2, v_2)[a_1, b_2, b_3]$ and $\mathfrak{A} \models v_1 \doteq F_1(v_3, F_1(v_3, v_3))[a_1, b_2, b_3])$
$\Leftrightarrow$ for some $b_2, b_3 \in \mathbb{Z}, (a_1 = b_2 + b_2$ and $a_1 = b_3 + b_3 + b_3)$
$\Leftrightarrow a_1$ is even and a_1 is divisible by 3
$\Leftrightarrow a_1$ is divisible by 6.

Example

So we write the formula of 4.10 as $\varphi(v_1)$. Let $\chi(v_1)$ be the formula $\neg\varphi(v_1)$. Let $\mathfrak{A} = \langle \mathbb{Z}; + \rangle, \mathfrak{B} = \langle \mathbb{R}; + \rangle$. Notice that $\mathfrak{A} \models \chi[5]$. However, $\mathfrak{B} \models \varphi[5]$, so not $\mathfrak{B} \models \chi[5]$. Thus embeddings (even identity embeddings) do not in general *preserve* all formulas – i.e. 4.11 does not hold for all formulas.

4.11 Lemma

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures (with conventional notation), let φ be an $\mathscr{L}$-quantifier free formula let n be suitable for φ and let $\pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$. Then for all $a_1, ..., a_n \in A, \mathfrak{A} \vDash \varphi[a_1, ..., a_n] \Leftrightarrow \mathfrak{B} \vDash \varphi[\pi(a_1), ..., \pi(a_n)]$.

Proof

We are given the base case from 4.4, so it remains to check the inductive step corresponding to the constructions 4.9 (1) and (2):-

$$\begin{aligned}
\mathfrak{A} \vDash \neg\varphi_1[\overline{a}] &\Leftrightarrow \text{ not } \mathfrak{A} \vDash \varphi_1[\overline{a}] \quad \text{by definition} \\
&\Leftrightarrow \text{ not } \mathfrak{B} \vDash \varphi_1[\overline{\pi(a)}] \quad \text{by IH and contraposition} \\
&\Leftrightarrow \mathfrak{B} \vDash \neg\varphi_1[\overline{\pi(a)}] \quad \text{by definition.} \\
\mathfrak{A} \vDash (\varphi_1 \wedge \varphi_2)[\overline{a}] &\Leftrightarrow \mathfrak{A} \vDash \varphi_1[\overline{a}] \text{ and } \mathfrak{A} \vDash \varphi_2[\overline{a}] \quad \text{by definition} \\
&\Leftrightarrow \mathfrak{B} \vDash \varphi_1[\overline{\pi(a)}] \text{ and } \mathfrak{B} \vDash \varphi_2[\overline{\pi(a)}] \quad \text{by IH} \\
&\Leftrightarrow \mathfrak{B} \vDash (\varphi_1 \wedge \varphi_2)[\overline{\pi(a)}] \quad \text{by definition.}
\end{aligned}$$

■

4.12 Lemma

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures (with conventional notation), let φ be an $\mathscr{L}_\sigma$-formula, let n be suitable for φ and let $\pi : \mathfrak{A} \cong \mathfrak{B}$. Then for all $a_1, ..., a_n \in A, \mathfrak{A} \vDash \varphi[a_1, ..., a_n] \Leftrightarrow \mathfrak{B} \vDash \varphi[\pi(a_1), ..., \pi(a_n)]$.

Proof: Exercise - On Mid Term Exam 09/10

5 Preamble to The Compactness Theorem

5.1 Definitions - Chain, Upper Bound and Maximal Element

Let $X \neq \emptyset$, $\mathscr{S} \subseteq P(X) - \{\emptyset\}$, and let $\mathscr{C} \subseteq \mathscr{S}$

- $\mathscr{C}$ is a **chain** in $\mathscr{S}$ if and only if for all $A, B \in \mathscr{C}$, $A \subseteq B$ or $B \subseteq A$
- if $\mathscr{C}$ is a chain, then $Y \subseteq X$ is an **upper bound** for $\mathscr{C}$ if and only if for all $A \in \mathscr{C}, A \subseteq Y$.
- $Z \in \mathscr{S}$ is a **maximal element** if and only if whenever $Z' \in \mathscr{S}$ and $Z \subseteq Z'$ then $Z' = Z$.

5.2 Lemma - Zorn's Lemma

Let $X \neq \emptyset$ and $\mathscr{S} \subseteq P(X) - \{\emptyset\}$, if every non-empty chain in $\mathscr{S}$ has an upper bound $Y \in \mathscr{S}$ then $\mathscr{S}$ has a maximal element. ∎

5.3 Definition - Model

Let σ be a similarity type, $\mathfrak{A} \in \mathcal{K}_\sigma$ and Σ a set of $\mathscr{L}_\sigma$-sentences, we say $\mathfrak{A}$ is a **model** for Σ and write $\mathfrak{A} \vDash \Sigma$ if and only if for every $\varphi \in \Sigma, \mathfrak{A} \vDash \varphi$.

5.4 Definition - Satisfiable (and Finitely Satisfiable)

Let σ be a similarity type and Σ be a set of $\mathscr{L}_\sigma$-sentences, we say Σ is **satisfiable** if and only if there exists $\mathfrak{A} \in \mathcal{K}_\sigma$ s.t. $\mathfrak{A}$ is a model for Σ. We say Σ' is **finitely satisfiable** if and only if every finite subset of Σ is Satisfiable.

6 Back to $\mathscr{L}_\sigma$

6.1 Lemma - 4.5 for formulas

Let n be suitable for the $\mathscr{L}_\sigma$-formula φ and $a_1, ..., a_n, a'_1, ..., a'_n \in A$ s.t. $a_l = a'_l$ for all l s.t. $v_l \in \mathrm{FrVar}(\varphi)$, then $\mathfrak{A} \vDash \varphi[a_1, ..., a_n] \Leftrightarrow \mathfrak{A} \vDash \varphi[a'_1, ..., a'_n]$.

Proof: Exercise

6.2 Lemma

Let φ be an $\mathscr{L}_\sigma$-<u>formula</u> and $\mathfrak{A}$ a $\mathscr{L}_\sigma$-structure (with conventional notation) then $(\varphi)^\lambda_p$ is an $\mathscr{L}_\sigma$-formula and if n is suitable for φ and λ then for all $\overline{a} := (a_1, ..., a_n) \in A^n$,

$$\mathfrak{A} \vDash (\varphi)^\lambda_p[\overline{a}] \quad \Leftrightarrow \quad \mathfrak{A} \vDash \varphi[a_1, ..., a_{p-1}, \lambda^{\mathfrak{A}}(\overline{a}), a_{p+1}, ..., a_n]$$

Proof: Exercise

7 The Compactness Theorem

Here is the full version without proof because it is non-examinable, the proof can be found in a separate document.

7.1 Theorem - The Compactness Theorem

Let σ be a similarity type and let Σ be a set of finitely satisfiable $\mathscr{L}_\sigma$-sentences, then Σ is satisfiable. ■

7.2 Corollary

Let σ be a similarity type and T any set of $\mathscr{L}_\sigma$-sentences, if for every $N \in \mathbb{N}, N \geq 1, T$ has a model $\mathfrak{A}_N$ such that $\text{dom}(\mathfrak{A}_N)$ is a finite set with at least N elements, then there exists a model $\mathfrak{A}$ of T such that $\text{dom}(\mathfrak{A})$ is infinite.

Proof:

For $N \geq 1$, let χ_N denote the following sentence of $\mathscr{L}_\sigma$:

$$\exists v_1 \exists v_2 \ldots \exists v_N \Big(\bigwedge_{1 \leq p < q \leq N} \neg v_p \doteq v_q \Big).$$

Note that for *any* $\mathscr{L}_\sigma$-structure $\mathfrak{A}$, we have $\mathfrak{A} \models \chi_N \Leftrightarrow \text{dom}(\mathfrak{A})$ has $\geq N$ elements. -(*)
Let $T^* := T \cup \{\chi_N : N \geq 1\}$.
Claim: T^* is finitely satisfiable.
Proof: Let $\Delta \subseteq_{\text{fin}} T^*$.
Then there is $N_0 \geq 1$, such that $\Delta \subseteq T \cup \{\chi_1, \ldots, \chi_{N_0}\}$. But $\mathfrak{A}_{N_0} \models T$ and $\text{dom}(\mathfrak{A}_{N_0})$ has at least N_0 elements.
By (*), $\mathfrak{A}_{N_0} \models \chi_N$ for all $N \leq N_0$.
$\therefore \mathfrak{A}_{N_0} \models \Delta$. ■ claim.
$\therefore$ by claim and Compactness Theorem, T^* has a model, $\mathfrak{A}$ say.
Certainly $\mathfrak{A} \models T$ since $T \subseteq T^*$. Also for *all* $N \geq 1, \mathfrak{A} \models \chi_N$.
$\therefore$ by (*), $\text{dom}(\mathfrak{A})$ must be infinite. ■

8 Elementary embeddings

8.1 Conventions

- We will no longer refer to similarity types, but just to languages: "Let $\mathscr{L}$ be a language ...", "Let $\mathfrak{A}$ be an $\mathscr{L}$-structure".
- $\forall v_i \varphi$ is an abbreviation for $\neg\exists v_i \neg\varphi$. One can easily check that

$$\mathfrak{A} \models \forall v_i \varphi[a_1, \dots, a_n] \Leftrightarrow \text{ for all } b_i \in \mathrm{dom}(\mathfrak{A}), \mathfrak{A} \models \varphi[a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n]$$

- $\mathrm{dom}(\mathfrak{A}) = A, \mathrm{dom}(\mathfrak{B}) = B, \dots$
  $\mathrm{dom}(\mathfrak{A}_1) = A_1, \dots$

Fix a language $\mathscr{L}$.

8.2 Definition - Elementary Embedding

Let $\mathfrak{A}, \mathfrak{B}$ be $\mathscr{L}$-structures.

(i) A function $\pi : A \to B$ is called an **elementary embedding**, written $\pi : A \preceq B$, if for all formulas $\varphi(v_1, \dots, v_n)$ of $\mathscr{L}$ and all $a_1, \dots, a_n \in A$ we have

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n] \Leftrightarrow \mathfrak{B} \models \varphi[\pi(a_1), \dots, \pi(a_n)].$$

(ii) If $A \subseteq B$ and $\mathrm{id}_A : \mathfrak{A} \preceq \mathfrak{B}$, we say that $\mathfrak{A}$ is an **elementary substructure** of $\mathfrak{B}$ and write $\mathfrak{A} \preceq \mathfrak{B}$.

8.2.1 Remark:

An elementary embedding *is* an embedding by 4.6 In particular, if $\mathfrak{A} \preceq \mathfrak{B}$, then $\mathfrak{A} \subseteq \mathfrak{B}$.

8.3 Tarski's lemma

Let $\mathfrak{A}, \mathfrak{B}$ be $\mathscr{L}$-structures.
Then $\mathfrak{A} \preceq \mathfrak{B} \Leftrightarrow$

(i) $\mathfrak{A} \subseteq \mathfrak{B}$; and

(ii) whenever $\varphi(v_1, \dots, v_n)$ is an $\mathscr{L}$-formula, $j \leq n, a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n \in A$, and $a'_j \in B$ and $\mathfrak{B} \models \varphi[a_1, \dots, a_{j-1}, a'_j, a_{j-1}, \dots, a_n]$ then there exists $a_j \in A$, $\mathfrak{B} \models \varphi[a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n]$.

Remark:

Note that (ii) refers to truth in $\mathfrak{B}$ only (and not truth in $\mathfrak{A}$).

Proof of Tarski's Lemma

"$\Rightarrow$": Suppose $\mathfrak{A} \preceq \mathfrak{B}$. Then by 8.2.1, we have (i).

For (ii), let $\varphi, a_1, \dots, a_{j-1}, a'_j, a_{j+1}, \dots, a_n$ be as in the hypothesis of (ii). Let ψ be the formula $\exists v_j \varphi$. Then we have $\mathfrak{B} \models \psi[a_1, \dots, a_{j-1}, a'_j, a_{j+1}, \dots, a_n]$.
Let a''_j be any element of A. Then since $v_j \notin \mathrm{FrVar}(\psi)$, we have

$$\mathfrak{B} \models \psi[a_1, \dots, a_{j-1}, a''_j, a_{j+1}, \dots, a_n] \quad \text{(by 6.1)}$$

Since $\mathfrak{A} \preceq \mathfrak{B}$ we have $\mathfrak{A} \models \psi[a_1, \dots, a_{j-1}, a''_j, a_{j+1}, \dots, a_n]$.
$\therefore$ there is some $a_j \in A$ such that $\mathfrak{A} \models \varphi[a_1, \dots, a_n]$ (by definition of "$\models$"). Again, since $\mathfrak{A} \preceq \mathfrak{B}$, we have $\mathfrak{B} \models \varphi[a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n]$ as required.

"$\Leftarrow$": Assume (i),(ii) hold. We show by induction on $\varphi(\bar{v})$, that for all $\bar{a} \in A$,

$$\mathfrak{A} \models \varphi[\bar{a}] \Leftrightarrow \mathfrak{B} \models \varphi[\bar{a}] \tag{1}$$

(Convention: $\bar{a} = a_1, \ldots, a_n, \bar{v} = v_1, \ldots, v_n$ and "$\bar{a} \in A$" means $a_1 \in A, \ldots, a_n \in A$.)

(1) holds for atomic (even QF) φ by 4.4, by (i).

Also if (1) is true for φ and for ψ then it is an easy exercise to show that it is true for the formulas $\neq \varphi$ and $(\varphi \wedge \psi)$. Finally, suppose (1) is true for φ. We must show that it is true for the formula $\exists v_j \varphi$. Let $\bar{a} = a_1, \ldots, a_n \in A$.

"$\Rightarrow$": $\mathfrak{A} \models \exists v_j \varphi[\bar{a}] \Rightarrow$ for some $a'_j \in A, \mathfrak{A} \models \varphi[a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n]$
$\Rightarrow$ for some $a'_j \in A, \mathfrak{B} \models \varphi[a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n]$ (by ind. hyp.)
$\Rightarrow$ for some $a'_j \in B, \mathfrak{B} \models \varphi[a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n]$ (as $A \subseteq B$).
$\Rightarrow \mathfrak{B} \models \exists v_j \varphi[a_1, \ldots, a_j, \ldots, a_n]$ as required.

"$\Leftarrow$": $\mathfrak{B} \models \exists v_j \varphi[\bar{a}] \Rightarrow$ for some $b_j \in B, \mathfrak{B} \models \varphi[a_1, \ldots, a_{j-1}, b_j, a_{j+1}, \ldots, a_n]$
$\Rightarrow$ for some $a'_j \in A, \mathfrak{B} \models \varphi[a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n]$ (by (ii))
$\Rightarrow$ for some $a'_j \in A, \mathfrak{A} \models \varphi[a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n]$ (by ind. hyp.)
$\Rightarrow \mathfrak{A} \models \exists v_j \varphi[a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n]$ as required. (Definition of "$\models$").

8.4 Remark

Note that if $\mathfrak{A}, \mathfrak{B}$ are $\mathscr{L}$-structures and $\pi : \mathfrak{A} \cong \mathfrak{B}$ then $\pi : \mathfrak{A} \preceq \mathfrak{B}$ by 4.12.

8.5 Definition - Automorphism

If $\pi : \mathfrak{A} \cong \mathfrak{A}$, then we say that π is an **automorphism** of $\mathfrak{A}$.

8.6 Corollary of Tarski's Lemma

Let $\mathfrak{A}, \mathfrak{B}$ be $\mathscr{L}$-structures with $\mathfrak{A} \subseteq \mathfrak{B}$. Suppose that for all finite subsets X of A and all $b \in B$, there exists an automorphism $\pi : \mathfrak{B} \cong \mathfrak{B}$ such that $\pi(a) = a$ for all $a \in X$ and $\pi(b) \in A$. Then $\mathfrak{A} \preceq \mathfrak{B}$.

Proof:

We must establish (ii) of Tarski's lemma. Assume the hypothesis of (ii). Let $X = \{a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n\}$ and $b = a'_j$. Let π be an automorphism of $\mathfrak{B}$ with $\pi(a_i) = a_i$ for $i = 1, \ldots, n, i \neq j$ and $\pi(a'_j) \in A$. By 8.4 applied to $\pi : \mathfrak{B} \cong \mathfrak{B}$, we have, since $\mathfrak{B} \models \varphi[a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n]$, that $\mathfrak{B} \models \varphi[\pi(a_1), \ldots, \pi(a_{j-1}), \pi(a'_j), \pi(a_{j+1}), \ldots, \pi(a_n)]$, i.e. $\mathfrak{B} \models \varphi[a_1, \ldots, a_{j-1}, \pi(a'_j), a_{j+1}, \ldots, a_n]$ (since π is identity on X).
Since $\pi(a'_j) \in A$, we are done, with $a_j = \pi(a'_j)$.

8.6.1 Example

Let $\mathscr{L}$ be the empty language (i.e. $I = J = K = \emptyset$). Let $\mathfrak{A}, \mathfrak{B}$ be $\mathscr{L}$-structures with $\mathfrak{A} \subseteq \mathfrak{B}$. (So $\mathfrak{A} = \langle A \rangle, \mathfrak{B} = \langle B \rangle, A \subseteq B$). Then $\mathfrak{A} \preceq \mathfrak{B} \Leftrightarrow$ either A is infinite or A is finite and $A = B$.

Proof:

Case 1: A is infinite. We establish the hypothesis of 8.6, so suppose $X \subseteq_{\text{fin}} A$ and $b \in B$. We must find a $\pi : \mathfrak{B} \cong \mathfrak{B}$, i.e. just a bijection $\pi : B \to B$, such that $\pi(a) = a$ (for all $a \in X$) and $\pi(b) \in A$. If $b \in A$, take $\pi = \text{id}_B$.

Assume $b \notin A$. Let $c \in A \setminus X$ (possible since A is infinite). Define $\pi : B \to B$ (for $x \in B$) by

$$\pi(x) = \begin{cases} x & \text{if } x \in B \setminus \{c, b\} \text{ (in particular if } x \in X) \\ c & \text{if } x = b \\ b & \text{if } x = c \end{cases}$$

Clearly π is bijective, therefore an automorphism of $\mathfrak{B}$, and has required properties.
Therefore by 8.6, $\mathfrak{A} \preceq \mathfrak{B}$.

Case 2: A is finite.
Clearly if $A = B$, then $\mathfrak{A} \cong \mathfrak{B}$, so $\text{id}_A : \mathfrak{A} \preceq \mathfrak{B}$.

Suppose $A \neq B$. Say $|A| = N$. Therefore $|B| \geq N+1$. $\mathfrak{B} \models \underbrace{\exists v_1 \dots \exists v_{N+1} \bigwedge_{1 \leq i < j \leq N+1} \neg v_i \doteq v_j}_{\chi_{N+1}}$.

But not $\mathfrak{A} \models \chi_{N+1}$. Therefore $\mathfrak{A} \not\preceq \mathfrak{B}$ since $\mathfrak{A} \models \neg\chi_{N+1}$ and $\mathfrak{B} \models \chi_{N+1}$.

8.6.2 Exercises

Suppose that $\pi : \mathfrak{A} \preceq \mathfrak{B}$. Then we know (since π is an embedding by 4.6) that $\pi[A]$ is the domain of a unique substructure of $\mathfrak{B}$, $\mathfrak{A}'$ say. So $\mathfrak{A}' \subseteq \mathfrak{B}$. Then $\pi : \mathfrak{A} \cong \mathfrak{A}'$ and $\mathfrak{A}' \preceq \mathfrak{B}$.

9 Theories

Let $\mathscr{L}$ be a language.

9.1 Definition - $\mathscr{L}$-theory

An $\mathscr{L}$**-theory** is a satisfiable set of $\mathscr{L}$-sentences.

9.2 Definition - Logical Consequence

Let Σ be a set of $\mathscr{L}$-sentences and φ an $\mathscr{L}$-sentence. We say that "φ is a **logical consequence** of" Σ and write $\Sigma \vDash \varphi$ if and only if for every model $\mathfrak{A}$ of Σ we also have $\mathfrak{A} \vDash \varphi$.

9.2.1 Remark

Let Σ be a set of $\mathscr{L}$-sentences, then Σ is an $\mathscr{L}$-theory if and only if for no $\mathscr{L}$-sentence φ we have both $\Sigma \vDash \varphi$ and $\Sigma \vDash \neg\varphi$.

Proof:

($\Rightarrow$): Let $\mathfrak{A} \vDash \Sigma$, if $\Sigma \vDash \varphi$ *and* $\Sigma \vDash \neg\varphi$ then $\mathfrak{A} \vDash \varphi$ and $\mathfrak{A} \vDash \neg\varphi$ ↯ .
($\Leftarrow$): if not $\Sigma \vDash \varphi$ then for some $\mathfrak{A} \vDash \Sigma$ not $\mathfrak{A} \vDash \varphi$, in particular Σ is satisfiable, similarly if not $\Sigma \vDash \neg\varphi$. ■

9.3 Definition - Complete

An $\mathscr{L}$-theory Σ is **complete** if and only if for every $\mathscr{L}$-sentence φ either $\Sigma \vDash \varphi$ or $\Sigma \vDash \neg\varphi$.

9.4 Definitions - Theory and Elementary Equivalence

1. Let $\mathfrak{A}$ be any $\mathscr{L}$-structure, we write $\mathrm{Th}(\mathfrak{A})$ for the **theory of** $\mathfrak{A} := \{\varphi | \varphi$ an $\mathscr{L}$-sentence and $\mathfrak{A} \vDash \varphi\}$. Clearly $\mathrm{Th}(\mathfrak{A})$ is a complete $\mathscr{L}$-theory, in fact for φ an $\mathscr{L}$-sentence $\varphi \in \mathrm{Th}(\mathfrak{A})$ or $\neg\varphi \in \mathrm{Th}(\mathfrak{A})$.

2. Let $\mathfrak{A}$, $\mathfrak{B}$ be $\mathscr{L}$-structures, we say $\mathfrak{A}$ is **elementarily equivalent** to $\mathfrak{B}$ written $\mathfrak{A} \equiv \mathfrak{B}$ if and only if $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{A})$.

Discussion/Definition - Axiomatizations

Let $\mathfrak{A}$ be a $\mathscr{L}$-structure, a set of $\mathscr{L}$-sentences $T \subseteq \mathrm{Th}(\mathfrak{A})$ is an **axiomatization** of $\mathrm{Th}(\mathfrak{A})$ or are **axioms** for $\mathrm{Th}(\mathfrak{A})$ if and only if for any $\varphi \in \mathrm{Th}(\mathfrak{A})$, $T \vDash \varphi$. An aim of Model Theory is to find intelligible axioms for a given interesting structure.

9.5 Example

Let $\mathscr{L}$ be the empty language and $\mathfrak{A}$ an $\mathscr{L}$-structure with A an infinite set, I <u>claim</u> that $\mathfrak{A}$ is axiomatized by $T := \{\chi_1, \chi_2, ...\}$.

Proof of Claim:

Let $\varphi \in \mathrm{Th}(\mathfrak{A})$. Need $T \vDash \varphi$, assume not, let $\mathfrak{B} \vDash T$ with $\mathfrak{B} \vDash \neg\varphi$, so $\mathrm{dom}(\mathfrak{B})$ infinite. (From an equivalent statement of the Axiom of Choice known as Cardinal Comparability) we know there exists $\pi_1 : A \hookrightarrow B$ or $\pi_2 : B \hookrightarrow A$. In the former case clearly $\pi_1 : \mathfrak{A} \hookrightarrow \mathfrak{B}$, let $\mathfrak{A}$' be the unique substructure of $\mathfrak{B}$ with domain $A' := \pi_1[A]$. Then $\pi_1 : \mathfrak{A} \cong \mathfrak{A}'$. Since A' infinite, $\mathfrak{A}' \leq \mathfrak{B}$ by 8.6.1. Finally as $\mathfrak{B} \vDash \neg\varphi$ we have $\mathfrak{A}' \vDash \neg\varphi$ (by def 8.2), giving $\mathfrak{A} \vDash \neg\varphi$ (by 8.4) $\lightning$. Latter case similar. ■

In future we will use without mention:

9.6 Lemma

Let $\mathfrak{A}, \mathfrak{B}$ be $\mathscr{L}$-structures, if $\mathfrak{A} \preceq \mathfrak{B}, \mathfrak{A} \cong \mathfrak{B}$ or there exists $\pi : \mathfrak{A} \preceq \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$.

Proof: Exercise

10 Important Examples of Theories

10.1 Dense Linear Order Without Endpoints (DLO)

Language $\mathscr{L}_{\mathrm{DLO}}$: One binary relation symbol $\dot{<}$ (note dot).

Axioms DLO:

1. $\forall v_1 \forall v_2 (v_1 \dot{<} v_2 \rightarrow \neg v_2 \dot{<} v_1)$ (Linear Ordering Dichotomy)
2. $\forall v_1 \forall v_2 \forall v_3 ((v_1 \dot{<} v_2 \wedge v_2 \dot{<} v_3) \rightarrow v_1 \dot{<} v_3)$ (Linear Ordering Transitivity)
3. $\forall v_1 \forall v_2 (v_1 \dot{<} v_2 \vee v_1 \dot{<} v_2 \vee v_1 \doteq v_2)$ (Linear Ordering Totality)
4. $\forall v_1 \forall v_2 (v_1 \dot{<} v_2 \rightarrow \exists v_3 (v_1 \dot{<} v_3 \wedge v_3 \dot{<} v_2)$ (Density)
5. $\forall v_1 \exists v_1 \dot{<} v_2$ (No Greatest Element)
6. $\forall v_1 \exists v_2 \dot{<} v_1$ (No Least Element)

Some Models: $\langle \mathbb{Q}; < \rangle, \langle \mathbb{R}; < \rangle$ (note no dot).

Known Properties:

1. Complete.
2. $\aleph_0$-**categorical** (i.e. exactly one model with domain of size $\aleph_0$ up to isomorphism).
3. **Quantifier elimination** (i.e. for every $\mathscr{L}_{\mathrm{DLO}}$-formula $\varphi(\overline{v})$ there is a QF-$\mathscr{L}_{\mathrm{DLO}}$-formula $\psi(\overline{v})$ s.t. $\forall \overline{v}(\varphi(\overline{v}) \leftrightarrow \psi(\overline{v}))$ is a logical consequence of DLO)

10.2 Divisible Torsion-Free Abelian Groups (DTFAG)

Language $\mathscr{L}_{\mathrm{DTFAG}}$: One binary function symbol $\dot{+}$, one unary function symbol $\dot{-}$, one constant symbol $\dot{0}$ (again note dots).

Axioms AG:

1. $\forall v_1 (v_1 \dot{+} (\dot{-} v_1)) \doteq \dot{0})$ (note brackets included for readability only)
2. $\forall v_1 \forall v_2 \forall v_3 ((v_1 \dot{+} (v_2 \dot{+} v_3)) \doteq ((v_1 \dot{+} v_2) \dot{+} v_3))$ (Associativity)
3. $\forall v_1 \forall v_2 (v_1 \dot{+} v_2 \doteq v_2 \dot{+} v_1)$ (Abelianness)

Axiom Schema TF: for each $n \geq 2 : \forall v_1 (((\underbrace{v_1 \dot{+} (v_1 \dot{+} ... \dot{+} v_1)}_{n \text{ times}})...) \doteq \dot{0} \rightarrow v_1 \doteq \dot{0})$

Axiom Schema D: for each $n \geq 2 : \forall v_1 \exists v_2 (v_1 \doteq (\underbrace{v_2 \dot{+} (v_2 \dot{+} ... \dot{+} v_2)}_{n \text{ times}}...))$

Some Models: $\langle \mathbb{Q}; +, -; 0 \rangle, \langle \mathbb{R}; +, -; 0 \rangle, \langle \mathbb{R}_{>0}; \cdot, ^{-1}; 1 \rangle, \langle \{0\}; +, -; 0 \rangle, \langle \mathbb{Q}[i]; +, -; 0 \rangle$.

Known Properties:

1. Not complete; consider $\exists v_1 \forall v_2 (v_1 \doteq v_2)$ or $\forall v_1 (v_1 \doteq \dot{0})$, as these nor their negation is true in every model of DTFAG.
2. *Is* complete if we add $\exists v_1 \forall v_2 (v_1 \doteq v_2)$, then all models are infinite.
3. Not $\aleph_0$-categorical as $\langle \mathbb{Q}; +, -; 0 \rangle \not\cong \langle \mathbb{Q}[i]; +, -; 0 \rangle$ yet both models of DTFAG ($\not\cong$ follows from a difference of dimension when the structures are considered as vector spaces)
4. $2^{\aleph_0}$-categorical
5. Quantifier elimination.

10.3 Algebraically Closed Fields (ACF)

Language $\mathscr{L}_{\mathrm{ACF}}$**:** Binary function symbols $\dot{+}$, $\times$, one unary function symbol $\dot{-}$, two constant symbols $\dot{0}$, $\dot{1}$ (known as language of rings).

Axioms ACF:

1. Axioms for fields with $\neg \dot{0} \doteq \dot{1}$
2. Axiom Schema for Algebraic closure: for each $n \geq 1$, $\forall v_1, ..., \forall v_n \exists v_{n+1} (v_{n+1}^n \dot{+} v_1 \times v_{n+1}^{n-1} \dot{+} ... \dot{+} v_{n-1} \times v_{n+1} \dot{+} v_n \doteq \dot{0})$, where $v_j^m := \underbrace{v_j \times (v_j \times (... \times v_j}_{m \text{ times}})...)$. This means in human readable form for a $\mathscr{L}$-structure $\mathfrak{A}$, $\mathfrak{A} \models$ ACF if and only if $\mathfrak{A}$ is a field and every zero of every monic polynomial over A is in A.

Some Models: $\langle \mathbb{C}; +, \cdot, -; 0, 1 \rangle$, **not** $\langle \mathbb{R}; +, \cdot, -; 0, 1 \rangle$, the countable structure $\langle \mathcal{A}; +, \cdot, -; 0, 1 \rangle$ where $\mathcal{A} \subseteq \mathbb{C}$ consisting of algebraic numbers.

Known Properties:

1. Not complete.
2. Is complete if we specify characteristic, i.e. for each prime p, let $\Gamma_p := \underbrace{1 \dot{+} (1 \dot{+} (1 \dot{+} ... \dot{+} 1)}_{p \text{ times}} ...) \doteq \dot{0}$ then $\mathrm{ACF}_p := \mathrm{ACF} \cup \{\Gamma_p\}$ is complete.
3. $\mathrm{ACF}_0 := \mathrm{ACF} \cup \{\neg\Gamma_2, \neg\Gamma_3, \neg\Gamma_5, ...\}$ is complete
4. Quantifier elimination.
5. ACF_0 and ACF_p are not $\aleph_0$-categorical but are κ-categorical for $\kappa > \aleph_0$.

11 The Lowenheim Skolem Theorem

For 11.1 let $\mathscr{L}$ be a countable language, i.e. the set of $\mathscr{L}$-formulas is a countable set, which is equivalent to saying I, J and K are countable.

11.1 Theorem - The Downward Lowenheim Skolem Theorem

Let $\mathfrak{A}$ be any $\mathscr{L}$-structure with $\mathrm{dom}(\mathfrak{A})$ infinite, then there exists $\mathfrak{B} \preceq \mathfrak{A}$ with $\mathrm{dom}(\mathfrak{B})$ countable. Further if $S \subseteq \mathrm{dom}(\mathfrak{A})$ is countable, we may choose $\mathfrak{B}$ s.t. $S \subseteq \mathrm{dom}(\mathfrak{B})$.

Proof:

Fix $\theta \in A$, for each $\mathscr{L}$-formula $\varphi(v_1, ..., v_n)$ and $1 \leq j \leq n$ we define (using AC) a function $f_j : A^{n-1} \to A$ by

$$f_{\varphi,j}(a_1, ..., a_{j-1}, a_{j+1}, ..., a_n) = \begin{cases} a_j & \text{s.t. } \mathfrak{A} \vDash \varphi[a_1, ..., a_j, ..., a_n] \text{ if such an } a_j \text{ exists} \\ \theta & \text{o.w.} \end{cases}$$

Define $\mathfrak{A}* := \langle \mathfrak{A}; \{f_{\varphi,j}\}_{\varphi,j'\text{s as above}} \rangle$ and $\mathscr{L}*$ appropriate for $\mathfrak{A}*$. Notice that $\mathscr{L}*$ is countable since $\mathscr{L}$ was. By Q2 week 3, it follows that there is an $\mathscr{L}*$-structure $\mathfrak{B}* \subseteq \mathfrak{A}*$ with $S \subseteq B*$ and $B*$ countable. Define $\mathfrak{B} := \mathfrak{B} * |\mathscr{L}$, so $\mathfrak{B} \subseteq \mathfrak{A}$ and we also know B is closed under arbitrary $f_{\varphi,j}$. Let $a_1, ..., a_{j-1}, a_{j+1}, ..., a_n \in B$ and $a'_j \in A$ then $a_j := f_{\varphi,j}(a_1, ..., a_{j-1}, a_{j+1}, ..., a_n) \in \mathfrak{B}$ is satisfies condition (ii) of Tarski's Lemma 8.3 by construction of $f_{\varphi,j}$ (**WARNING:** the notation $\mathfrak{A}$ and $\mathfrak{B}$ is reversed in 8.3(i)). Therefore $\mathfrak{B} \preceq \mathfrak{A}$ ■

11.2 Theorem - DLS Theorem Full Version

Let $\aleph_0 \leq |\mathscr{L}| \leq \kappa$. Let $\mathfrak{A}$ be any $\mathscr{L}$-structure with $|\mathrm{dom}(\mathfrak{A})| \geq \kappa$, then if $S \subseteq \mathrm{dom}(\mathfrak{A})$ is s.t. $|S| \leq \kappa$, then there exists $\mathfrak{B} \subseteq \mathfrak{A}$ s.t. $S \subseteq \mathrm{dom}(\mathfrak{B})$ and $|B| = \kappa$. (**Proof** similar to 11.1 and non-examinable) ■

11.3 Theorem - The Upward Lowenheim Skolem Theorem

Let $\aleph_0 \leq |\mathscr{L}| = \kappa_0$. Let $\mathfrak{A}$ be any $\mathscr{L}$-structure with $|\mathrm{dom}(\mathfrak{A})| \geq \aleph_0$, then for any $\kappa \geq \max\{\kappa_0, |A|\}$, then there exists $\mathfrak{B} \succeq \mathfrak{A}$ s.t. $|B| = \kappa$. (**Proof** postponed until we do "method of diagram" and non-examinable)

12 $\aleph_0$-categoricity

Let $\mathscr{L}$ be a countable language.

12.1 Definition - $\aleph_0$-categorical

An $\mathscr{L}$-theory T is called $\aleph_0$-**categorical** if and only if T has exactly one model with domain of size $\aleph_0$ up to isomorphism.

12.1.1 Remark/Definition - Absolutely Categorical

Let T be an $\mathscr{L}$-theory, T is **absolutely categorical** if and only if *all* models of T are isomorphic. We ignore such theories T because T cannot have an infinite model by 11.3.

12.2 Theorem - Vaught's Test

Let T be an $\aleph_0$-categorical $\mathscr{L}$-theory such that T has no finite models, then T is complete.

Proof

Assume T not complete. Let φ be a $\mathscr{L}$-sentence s.t. $T \nvDash \varphi$ and $T \nvDash \neg\varphi$. Note φ and $\neg\varphi$ are satisfiable, otherwise we would have $T \vDash \neg\varphi$ or $T \vDash \neg\neg\varphi$ (i.e. $T \vDash \varphi$) respectively. Let $\mathfrak{A}$ be an $\mathscr{L}$-structure s.t. $\mathfrak{A} \vDash T$, $\mathfrak{A}\neg\varphi$ and $\mathfrak{B}$ be an $\mathscr{L}$-structure s.t. $\mathfrak{B} \vDash T$, $\mathfrak{B} \vDash \neg\neg\varphi$ (i.e. $\mathfrak{B} \vDash \varphi$). Since T has no finite models, $\mathfrak{A}$, $\mathfrak{B}$ are infinite. Therefore by DLS theorem 11.1, there exists countable $\mathscr{L}$-sentences $\mathfrak{A}_0$ and $\mathfrak{B}_0$ s.t. $\mathfrak{A}_0 \preceq \mathfrak{A}$ and $\mathfrak{B}_0 \preceq \mathfrak{B}$. By 8.3 Tarski's Lemma ((ii) part) it follows that $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are models of T, and thus also infinite. Therefore $\mathfrak{A}_0 \cong \mathfrak{B}_0$ since T is $\aleph_0$-categorical, hence $\mathfrak{A}_0 \equiv \mathfrak{B}_0$, hence by 9.6 $\mathfrak{A}_0 \vDash \varphi \wedge \neg\varphi$! $\lightning$ ■

12.3 Theorem

DLO is $\aleph_0$-categorical, and complete.

Proof:

This proof is a typical example of a "back-and-forth construction". Recall DLO axioms:

1. $\forall v_1 \forall v_2 (v_1 \dot{<} v_2 \rightarrow \neg v_2 \dot{<} v_1)$ (Linear Ordering Dichotomy)
2. $\forall v_1 \forall v_2 \forall v_3 ((v_1 \dot{<} v_2 \wedge v_2 \dot{<} v_3) \rightarrow v_1 \dot{<} v_3)$ (Linear Ordering Transitivity)
3. $\forall v_1 \forall v_2 (v_1 \dot{<} v_2 \vee v_2 \dot{<} v_1 \vee v_1 \doteq v_2)$ (Linear Ordering Totality)
4. $\forall v_1 \forall v_2 (v_1 \dot{<} v_2 \rightarrow \exists v_3 (v_1 \dot{<} v_3 \wedge v_3 \dot{<} v_2)$ (Density)
5. $\forall v_1 \exists v_1 \dot{<} v_2$ (No Greatest Element)
6. $\forall v_1 \exists v_2 \dot{<} v_1$ (No Least Element)

Let $\langle A, <_A \rangle \vDash$ DLO. Let $a_0 \in A$, by 5 $\exists a_1 \in A$ s.t. $a_0 < a_1$, then by 5 again $\exists a_2 \in A$ s.t. $a_1 < a_2$, etc, etc ... so we get $a_0 < a_1 < ... < a_n$ for any n. For any $0 \leq i \leq j \leq n$, by 2 $a_i \leq a_j$, suppose $a_i = a_j$ then $a_i \leq a_j$ *and* $a_j \leq a_i$ which contradicts 1. Therefore all a_i's are distinct, hence A is infinite. By 12.2 if remains to show DLO is $\aleph_0$-categorical. Suppose $|A| = \aleph_0$, let $\langle B, <_B \rangle \vDash$ DLO also of size $\aleph_0$. We can write the elements (not necessarily in order) as follows:

$$A = \{a_1, a_2, ..., a_n, ...\}, \quad B = \{b_1, b_2, ..., b_n, ...\}$$

We will construct sequences $a'_1, a'_2, ...$ from A and $b'_1, b'_2, , ...$ from B by induction s.t. for each k for $0 \leq i \leq j \leq k, a'_i <_A a'_j \Leftrightarrow b'_i <_B b'_j$ -(†) and s.t. the map $a'_i \mapsto b'_i$ is bijective.

Base Case: Trivial.

Inductive step: Write $(\dagger)_n$ for the statement "(†) holds for n". Assume $(\dagger)_n$

Case 1: n odd: ("Forth")

Let m be minimal (under the usual ordering of $\mathbb{N}$) s.t. $a_m \notin \{a'_1, ..., a'_n\}$, $a'_{n+1} := a_m$.

Case 1.1 $a_m <_A a'_i$ for $1 \leq i \leq n$: By 1,2 and 3 $\{b'_1, ..., b'_n\}$ has a unique least element b'_r say, w.r.t $<_B$. By 6 we can define b'_{n+1} s.t. $b'_{n+1} <_B b'_r$.

Case 1.2 $a_i <_A a_i m$ for $1 \leq i \leq n$: Similar, use 5 instead of 6.

Case 1.3 Not 1.1 & 1.2: Define $S^- := \{p | 1 \leq p \leq n \ \& \ a_p{}' <_A a_m\}$ and $S^+ := \{p | 1 \leq p \leq n \ \& \ a_m <_A a'_p\}$, which are obviously non-empty. Let a'_l be the largest element of $\{a'_i\}_{i \in S^-}$ and $a_s{}'$ the smallest element of $\{a'_i\}_{i \in S^-}$, then $a'_p \leq_A a'_l <_A a_m <_A a'_s \leq_A a'_q$ for all $p \in S^-$ and $q \in S^+$. By the inductive hypothesis $(\dagger)_n$ (and 2) we have $b'_p \leq_B b'_l <_B b'_s \leq_B b'_q$ for all $p \in S^-$ and $q \in S^+$. Finally by 4 we can define b'_{n+1} s.t. $b'_l <_B b'_{n+1} <_B b'_s$.

Case 2: n even: ("Back")

We reverse the roles of A and B, i.e. start by picking m minimal s.t. $b_m \notin \{b'_1, ..., b'_n\}$ etc...

We have clearly shown $(\dagger)_{n+1}$ in each case.
$\therefore$ we have constructed sequences $a'_1, a'_2, ...$ from A and $b'_1, b'_2, , ...$ from B s.t. for $0 \leq i \leq j$ in $\mathbb{N}$, $a'_i <_A a'_j \Leftrightarrow b'_i <_B b'_j$ and s.t. the map $a'_i \mapsto b'_i$ is bijective. It remains to show "we hit" every a_m and b_m i.e. $A = \{a_i\}_{i \in \mathbb{N}}$ and $B = \{b_i\}_{i \in \mathbb{N}}$. This is the purpose of the "back-and-forth construction", as if we just used the "Forth" construction we would ensure $A = \{a_i\}_{i \in \mathbb{N}}$ but not necessarily $B = \{b_i\}_{i \in \mathbb{N}}$. So assume not $A = \{a_i\}_{i \in \mathbb{N}}$, let m be the least element s.t. $a_m \notin \{a_i\}_{i \in \mathbb{N}}$ (should it exist). The base case ensures $m \geq 2$, now choose the least N s.t. $a_1, ..., a_{m-1} \in \{a'_1, ..., a'_N\}$, which is ok since m minimal. If N odd, then in case 1 we chose $a'_{N+1} = a_m$ ↯. If N even then in case 1 we chose a'_{N+1} to correspond to b'_{N+1}, since $N+1$ is even we then chose $a_m = a'_{N+2}$ ↯. Similarly for B. ■

12.4 Definitions - $\mathrm{E}_n(T)$, $\mathrm{F}_n(\mathscr{L})$, $\varphi^{\mathfrak{A}}$

Let $\mathscr{L}$ be a countable language.

(a) For $n \geq 0$, $\mathrm{F}_n(\mathscr{L})$ denotes the set of all $\mathscr{L}$-formulas φ with $\mathrm{FrVar}(\varphi) \subseteq \{v_1, ..., v_n\}$. (**Note:** $\mathrm{F}_0(\mathscr{L})$ denotes the set of all $\mathscr{L}$-sentences.

(b) If $\varphi \in \mathrm{F}_n(\mathscr{L})$ and $\mathfrak{A}$ any $\mathscr{L}$-structure then we define $\varphi^{\mathfrak{A}} := \{\langle a_1, ..., a_n\rangle \in A^n | \mathfrak{A} \vDash \varphi[a_1, ..., a_n]\}$ and say $\varphi^{\mathfrak{A}}$ is the subset of A^n **defined by** φ.

(c) Let T be an $\mathscr{L}$-theory then $\mathrm{E}_n(T)$ denotes the binary relation on $\mathrm{F}_n(\mathscr{L})$ defined by

$$(\varphi, \psi) \in \mathrm{E}_n(T) \Leftrightarrow T \vDash \forall v_1, ..., v_n(\varphi(v_1, ..., v_n) \leftrightarrow \psi(v_1, ..., v_n))$$

Clearly this is an equivilence relation

12.5 Theorem - Ryll - Nardzewski

Let T be a complete theory without finite models. Then T is $\aleph_0$-categorical if and only if for all $n \geq 0$, there are only finitely many $\mathrm{E}_n(T)$-equivalence classes on $\mathrm{F}_n(T)$. **Proof:** see later.

12.6 Remark

(a) The condition above is clearly equivalent to saying that for all $\mathfrak{A} \vDash T$, and all $n \geq 0$, there are only finitely many subsets of A^n that can be defined by a formula in $\mathrm{F}_n(\mathscr{L})$.

(b) The assumption that T is a complete theory is exactly equivalent to the statement that there are exactly two $\mathrm{E}_0(T)$-equivalence classes.

12.7 Examples and Non-Examples

(a) We have proved DLO is $\aleph_0$-categorical, and complete directly. Now DLO has elimination of quantifiers, so we can use R-N theorem to show DLO is $\aleph_0$-categorical by showing there are only finitely many DLO-inequivalent $\mathscr{L}$-quantifier free formulas, which is fairly clear:-

<u>For $n = 1$</u> $v_1 \doteq v_1, \neg v_1 \doteq v_1$

<u>For $n = 2$</u> $v_1 < v_2$, $v_2 < v_1$, $v_1 \doteq v_1$, $\neg v_1 \doteq v_1$, $v_1 \doteq v_2$, $\neg v_1 \doteq v_2$, $(v_1 < v_2 \vee v_1 \doteq v_2)$, $(v_2 < v_1 \vee v_1 \doteq v_2)$.

... etc ... etc

(b) Let $T := \mathrm{Th}(\langle \mathbb{R}; +; 0\rangle =: \mathfrak{R})$, this is not $\aleph_0$-categorical by R-N theorem: **Since:** let $\varphi_n(v_1, v_2) := v_1 \doteq \underbrace{(v_2 \dot{+} (v_2 \dot{+} ... \dot{+} v_2)}_{n \text{ times}} ...)$, then $\varphi_n^{\mathfrak{R}} = \{\langle a_1, a_2\rangle \in \mathbb{R}^2 | \mathfrak{A} \vDash \varphi_n[a_1, a_2]\} = \{\langle a_1, a_2\rangle \in \mathbb{R}^2 | a_1 = n \cdot a_2\}$, then clearly if $n \neq m$ then $\varphi_n^{\mathfrak{R}} \neq \varphi_m^{\mathfrak{R}}$ (e.g. $\langle n, 1\rangle \in \varphi_n^{\mathfrak{R}} \backslash \varphi_m^{\mathfrak{R}}$).

Proof of ($\Leftarrow$) direction of 12.5 R-N Theorem

Let T be complete and assume $|\,\mathrm{F}_n(\mathscr{L}) / \mathrm{E}_n(T)| \in \mathbb{N}, \forall n \geq 0$.

Claim

For any $\mathfrak{A} \vDash T$ and $n \geq 1$, and $a_1, ..., a_n \in A$ there exists $\varphi(v_1, ..., v_n) \in \mathrm{F}_n(\mathscr{L})$ s.t.

(i) $\mathfrak{A} \vDash \varphi[a_1, ..., a_n]$, and

(ii) if $\psi(v_1, ..., v_n)$ is any formula in $\mathrm{F}_n(\mathscr{L})$ s.t. $\langle a_1, ..., a_n\rangle \in \psi^{\mathfrak{A}}$ then $T \vDash \forall v_1 ... v_n(\varphi(v_1, ..., v_n) \rightarrow \psi(v_1, ..., v_n)$ (i.e. $\varphi^{\mathfrak{A}} \subseteq \psi^{\mathfrak{A}}$

Proof of Claim:

Let $\mathfrak{A} \vDash T$, $n \geq 1$ and $a_1, ..., a_n \in A$ be as in statement of claim. Let $\varphi_1, ..., \varphi_N$ be representatives in $\mathrm{F}_n(\mathscr{L})$ of all the $\mathrm{E}_n(T)$ equivalence classes. Let $s := \{i | 1 \leq i \leq N \text{ and } \mathfrak{A} \vDash \varphi_i[a_1, ..., a_n]\}$. (Note N must be even and $|s| = N/2$, **why?**) Clearly $s \neq \emptyset$. Since s is finite we can define $\varphi(v_1, ..., v_n) \in \mathrm{F}_n(\mathscr{L})$ as follows

$$\varphi(v_1, ..., v_n) := \bigwedge_{i \in s} \varphi_i(v_1, ..., v_n)$$

Then $\mathfrak{A}\varphi[a_1, ..., a_n]$, so (i) holds. Now let $\psi(v_1, ..., v_n)$ be s.t. $\mathfrak{A} \vDash \psi[a_1, ..., a_n]$ (i.e. $\langle a_1, ..., a_n\rangle \in \psi^{\mathfrak{A}}$), choose j s.t. $\varphi_j \,\mathrm{E}_n(T) \psi$. Then $\varphi_j^{\mathfrak{A}} = \psi^{\mathfrak{A}}$, so $\langle a_1, ..., a_n\rangle \in \varphi_j^{\mathfrak{A}}$. Thus $j \in s$ and so $\varphi^{\mathfrak{A}} \subseteq \varphi_j^{\mathfrak{A}}$. Hence $\varphi^{\mathfrak{A}} \subseteq \psi^{\mathfrak{A}}$ and hence $\mathfrak{A} \vDash \forall v_1 ... v_n(\varphi(v_1, ..., v_n) \rightarrow \psi(v_1, ..., v_n))$. Now T is complete it thus follows that $T \vDash \forall v_1 ... v_n(\varphi(v_1, ..., v_n) \rightarrow \psi(v_1, ..., v_n))$. ■claim We such a φ of the claim a **principal formula of the n-tuple** $a_1, ..., a_n$. We will now prove $\aleph_0$-categoricity of T by a (generalized) <u>back-and-forth</u> construction.
So let $\mathfrak{A}, \mathfrak{B} \vDash T$ with $|A| = |B| = \aleph_0$; write $A = \{a_1, a_2, ...\}, B = \{b_1, b_2, ...\}$. We want enumerations $a'_1, a'_2, ...$ and $b'_1, b'_2, ...$ of A and B respectively s.t. for all $n \geq 0$

$$(*)_n: \quad \text{for all } \psi \in \mathrm{F}_n\,\mathscr{L}, \quad \mathfrak{A} \vDash \psi[a'_1, ..., a'_n] \quad \Leftrightarrow \quad \mathfrak{B} \vDash \psi[b'_1, ..., b'_n]$$

This holds for $n = 0$ since T is complete. Assume $(*)_n$ holds.
<u>Case 1:</u> n is odd
Let a'_{n+1} be a_m, where m is minimal s.t. $a_m \notin \{a'_1, ..., a'_n\}$. Let φ be a principal formaula for $a'_1, ..., a'_{n+1}$, which exists by claim. Let φ^* be the formula $\exists v_{n+1}\varphi$. Now $\mathfrak{A} \vDash \varphi[a'_1, ..., a'_{n+1}]$ (by (i) of claim), so $\mathfrak{A} \vDash \varphi^*[a'_1, ..., a'_n]$. Hence $\mathfrak{A} \vDash \varphi^*[b'_1, ..., b'_n]$ by $(*)_n$. Therefore there is some $b'_{n+1} \in B$ s.t. $\mathfrak{B} \vDash \varphi[b'_1, ..., b'_{n+1}]$ -(†). We show $(*)_{n+1}$ holds for this choice of b'_{n+1}.
<u>$\Rightarrow$:</u> Suppose $\psi(v_1, ..., v_{n+1}) \in \mathrm{F}_{n+1}(\mathscr{L})$ and $\mathfrak{A} \vDash \psi[a'_1, ..., a'_{n+1}]$. Since φ is principal for $a'_1, ..., a'_{n+1}$, $T \vDash \forall v_1 ... v_{n+1}(\varphi \rightarrow \psi)$ (by (ii) of claim). But $\mathfrak{B} \vDash T$, so $\mathfrak{B}\forall v_1 ... v_{n+1}(\varphi \rightarrow \psi)$, and hence by (†) $\mathfrak{B} \vDash \psi[b'_1, ..., b'_n, b'_{n+1}]$ as required. <u>$\Leftarrow$:</u> If $\psi \in \mathrm{F}_{n+1}(\mathscr{L})$ and not $\mathfrak{A} \vDash \psi[a'_1, ..., a'_{n+1}]$, then apply above argument to $\neg\psi$.
$\therefore$ $a'_1, ..., a'_{n+1}, b'_1, ..., b'_{n+1}$ satisfy $(*)_{n+1}$.
<u>Case 2:</u> n even
Let b'_{n+1} be b_m, where m is minimal s.t. $b_m \notin \{b'_1, ..., b'_n\}$ and use above argument with roles of $\mathfrak{A}, \mathfrak{B}$ reversed.

We have $(*)_n$ for every n, so use this for atomic formulas ψ and 4.6 and we get that $a'_i \mapsto b'_i$ is an embedding. Finally we use the same argument as in 12.3 to get the correspondence is bijective, hence $\mathfrak{A} \cong \mathfrak{B}$, therefore T is $\aleph_0$-categorical. $\blacksquare(\Leftarrow)$.

12.8 Exercise - Just convince yourself

Suppose that $\mathscr{L}$ has only finitely many relation symbols, finitely many constant symbols and no function symbols. Let T be a complete $\mathscr{L}$-theory with no finite models and assume that T eliminates quantifiers. Then T is $\aleph_0$-categorical.

12.9 Example

Let $\mathfrak{g} := \langle G; \cdot; ^{-1}; 1\rangle$ be a group s.t. $|G| = \aleph_0$, let $\mathrm{Th}(\mathfrak{g})$ be $\aleph_0$-categorical (i.e. if $\mathfrak{A}$ is a countably infinite group s.t. $\mathfrak{A} \equiv \mathfrak{g}$, then $\mathfrak{A} \cong \mathfrak{g}$). Then $\mathfrak{g}$ has **bounded order**, i.e. there exists $N \geq 1$ s.t. for all $g \in G$, $g^N = 1$.

Proof:

Let $m \geq 1$, let v_1^m denote the term $(\underbrace{v_1 \cdot (v_1 \cdot \ldots \cdot v_1}_{m\text{-times}})\ldots)$, and let $\varphi_m(v_1, v_2) := v_1^m \doteq v_2$. By R-N theorem there are only *finitely many* distinct subsets of $\mathfrak{g}^2$ amoungst the sets $\varphi_1^{\mathfrak{g}}, \varphi_2^{\mathfrak{g}}, \ldots$. Thus there are $m_1, m_2 \geq 2$ (with WLOG $m_2 < m_1$) s.t. $\varphi_{m_1}^{\mathfrak{g}} = \varphi_{m_2}^{\mathfrak{g}}$,
hence for all $g, h \in G$, $g^{m_1} = h$ if and only if $g^{m_2} = h$,
i.e. for all $g \in G$, $g^{m_1} = g^{m_2}$,
i.e. for all $g \in G$, $g^{m_1 - m_2} = 1$ $\blacksquare$

13 Types

Let $\mathscr{L}$ be any language, and T a <u>complete</u> theory in $\mathscr{L}$.

13.1 Definition - n-type

Let $n \geq 0$, then a subset $p \subseteq \mathrm{F}_n(\mathscr{L})$ is called an **n-type (over T)** if and only if the following three conditions hold

(i) if $\varphi \in p$ then $T \vDash \exists v_1, ..., \exists v_n \varphi$ (i.e. for all $\mathfrak{A} \vDash T$, $\varphi^{\mathfrak{A}}$ is a non-empty subset of A^n.);

(ii) if $\varphi \in p$ and $\psi \in p$ then $(\varphi \wedge \psi) \in p$;

(iii) for any $\varphi \in \mathrm{F}_n(\mathscr{L})$, $\varphi \in p$ or[5] $\neg\varphi \in p$

Remark

(i) Because of (iii) above, n-types are sometimes called **complete n-types**

(ii) There is only one 0-type over T, namely $\{\varphi : \varphi \text{ a sentence and } T \vDash \varphi\} \subseteq \mathrm{F}_0(\mathscr{L})$.

Proof of (ii): Exercise

13.2 Theorem

Let $\mathscr{L}$ be countable. Let $n \geq 1$, $p \subseteq \mathrm{F}_n(\mathscr{L})$ and suppose p satisfies 13.1 (i) and (ii), then there exists a countably infinite model $\mathfrak{A} \vDash T$ and $a_1, ..., a_n \in A$ such that for all $\varphi \in p$, $\mathfrak{A} \vDash \varphi[a_1, ..., a_n]$.

Proof:

Add new $c_1, ..., c_n$ to $\mathscr{L}$ to get $\mathscr{L}'$.

$$\Sigma := T \cup \{\varphi(c_1, ..., c_n) | \varphi(v_1, ..., v_n) \in p\}$$

where $\varphi(c_1, ..., c_n) := \varphi^{c_1,...,c_n}_{1,...,n}$.
*Claim: Σ is finitely satisfiable
*Proof of Claim:
Let $\Sigma_0 \subseteq_{\text{finite}} \Sigma$, then $\Sigma_0 \subseteq T \cup \{\varphi_1(c_1, ..., c_n), ..., \varphi_l(c_1, ..., c_n)\}$ for some $\varphi_1, ..., \varphi_l \in p$. By repeated use of 13.1 (ii), $\bigwedge_{1 \leq i \leq l} \varphi_i := \psi$, hence by 13.2 (i), $T \vDash \exists v_1 ... \exists v_n \psi$. Let $\mathfrak{B}$ be an $\mathscr{L}$-structure s.t. $\mathfrak{B} \vDash T$ and $\mathfrak{B}$ countable (we can do this by DLS theorem 11.1). Then for some $b_1, ..., b_n \in \mathfrak{B}$, $\mathfrak{B} \vDash \psi[b_1, ..., b_n]$, let $\mathfrak{B}' := \langle \mathfrak{B}, b_1, ..., b_n \rangle$. Then $\mathfrak{B}' \vDash \Sigma_0$. ■claim
By claim and compactness theorem there exists a model $\mathfrak{A}' \vDash \Sigma$ with $\mathfrak{A}'$ countably infinite (by DLS 11.1). Then $\mathfrak{A}'$ has the form $\langle \mathfrak{A}, a_1, ..., a_n \rangle$ where $\mathfrak{A} \vDash T$, $a_1, ..., a_n \in A$ and $a_i = c_i^{\mathfrak{A}'}$ for $i = 1, ..., n$. Clearly $\mathfrak{A} \vDash \varphi[a_1, ..., a_n]$ for all $\varphi \in p$ by definition of Σ. ■

13.3 Definition - $\mathrm{tp}_n(\overline{a}; \mathfrak{A})$

Let $n \geq 1$, $\mathfrak{A} \vDash T$ and $\overline{a} := \langle a_1, ..., a_n \rangle \in A^n$, define $\mathrm{tp}_n(\overline{a}; \mathfrak{A}) := \{\varphi \in \mathrm{F}_n(\mathscr{L}) : \mathfrak{A} \vDash \varphi[\overline{a}]\}$.

13.3.1 Remarks

(i) (a) $\mathrm{tp}_n(\overline{a}; \mathfrak{A})$ is an n-type (over T). (b) In fact, all n-types (over T) are of this form.

(ii) If $p \subseteq \mathrm{F}_n(\mathscr{L})$ and satisfies *just* 13.1 (i) & (ii) then there is some n-type q say s.t. $p \subseteq q$.

(iii) Suppose $\mathfrak{A}, \mathfrak{B} \vDash T$ and $\pi : \mathfrak{A} \preceq \mathfrak{B}$. Then for $\overline{a} \in A^n, \mathrm{tp}_n(\overline{a}; \mathfrak{A}) = \mathrm{tp}_n(\overline{a}; \mathfrak{B})$.

(iv) Suppose $\mathfrak{A} \vDash T$ and $\mathfrak{A} \cong \mathfrak{B}$. Then $\mathfrak{B} \vDash T$ and for any $\overline{a} \in A^n$, $\mathrm{tp}_n(\overline{a}; \mathfrak{A}) = \mathrm{tp}_n(\pi(\overline{a}); \mathfrak{B})$.

[5] this "or" is exclusive by (i) and (ii)

Proof:

(i)(a) Easy check
(i)(b) First check that if p, q are n-types and $p \subseteq q$ then $p = q$, then apply 13.2. **(ii)** By 13.2
(iii) Direct from the definition of "$\preceq$".
(iv) From 8.4. ■

13.4 Definitions - Realization, $\mathrm{S}_n(T)$

Let $n \geq 1$ and p be an n-type (over T) and let $\mathfrak{A} \vDash T$.

(i) Let $\overline{a} \in A^n$. Then we say that $\overline{a}$ **realizes** p **in** $\mathfrak{A}$, if and only if $\mathfrak{A} \vDash \varphi[\overline{a}]$ for all $\varphi \in p$.[6]

(ii) We say that p is **realized** in $\mathfrak{A}$ or $\mathfrak{A}$ **realizes** p if and only if for some $\overline{a} \in A^n$, $\overline{a}$ realizes p in $\mathfrak{A}$,

(iii) if not we say that p is **ommitted** in $\mathfrak{A}$ or $\mathfrak{A}$ **omits** p.

(iv) We denote by $\mathrm{S}_n(T)$ the set of all n-types (over T).

13.5 Remark

If $\mathfrak{A} \vDash T$ and $\mathfrak{A} \cong \mathfrak{B}$ then for all $n \geq 1$, $\mathfrak{A}$ and $\mathfrak{B}$ realize exactly the same n-types (over T). **Proof:** Follows from 13.3.1 (iii) ■

13.6 Definition - Principal

Let $n \geq 1$ and let $p \in \mathrm{S}_n(T)$. Then p is called **principal** if and only if there is some $\varphi \in p$ such that for all $\psi \in p$, $T \vDash \forall v_1 ... v_n (\varphi \rightarrow \psi)$. Such a formula φ (which is unique modulo $\mathrm{E}_n(T)$) is called a **principal formula** for p.

13.7 Exercise - On mid term exam 2009

If $p \in S_n(T)$ is a principal n-type (over T) then for all $\mathfrak{A} \models T$, p is realized in $\mathfrak{A}$.

13.8 Theorem - Omitting Types Theorem

($\mathscr{L}$ countable) Let $n \geq 1$ and $p \in S_n(T)$ be a *non-principal* n-type (over T). Then there exists a countably infinite model of T that omits p. ■non-examinable

13.9 Theorem - Existence of non-principal types

Let $n \geq 1$. There exists a non-principal n-type (over T) if and only if there are infinitely many $\mathrm{E}_n(T)$ equivalence classes in $F_n(\mathscr{L})$.

13.10 Exercise

Let $\mathscr{L}$ be any language and let $\varphi(v_1, ..., v_n) \in \mathrm{F}_n(\mathscr{L})$, let $\gamma_1, ..., \gamma_n$ be constant symbols <u>not</u> in $\mathscr{L}$.

(i) Assume that Σ is an $\mathscr{L}$-theory and that $\Sigma \vDash \varphi(\gamma_1, ..., \gamma_n)$. Then $\Sigma \vDash \forall v_1 ... v_n \varphi(v_1, ..., v_n)$.

(ii) Now assume that Σ is an $\mathscr{L}$-theory, that ψ is an $\mathscr{L}$-sentence and that $\Sigma \cup \{\varphi(\gamma_1, ..., \gamma_n)\} \vDash \psi$. Then $\Sigma \cup \{\exists v_1 ... v_n \varphi(v_1, ... v_n)\} \vDash \psi$.

[6] equivalently; $p = \mathrm{tp}_n(\overline{a}; \mathfrak{A})$, or equivalently; if $\overline{a} \in \varphi^{\mathfrak{A}}$ for all $\varphi \in p$.

Proof of 13.9

($\Rightarrow$): Exercise (never needed).
($\Leftarrow$): Assume there are infinitely many $E_n(T)$ equivalence classes of formulas in $F_n(\mathscr{L})$. Now we call a formula $\varphi \in F_n(\mathscr{L})$ a *principal formula* (over T) if

(i) $T \models \exists v_1 \dots \exists v_n \varphi(v_1, \dots, v_n)$

(ii) for all $\chi \in F_n(\mathscr{L})$, either $T \models \forall v_1 \dots \forall v_n(\varphi \to \chi)$ or $T \models \forall v_1 \dots \forall v_n(\varphi \to \neg\chi)$ (i.e. for all $\mathfrak{A} \models T$, all $\chi \in F_n(\mathscr{L})$, either $\varphi^{\mathfrak{A}} \subseteq \chi^{\mathfrak{A}}$ or $\varphi^{\mathfrak{A}} \subseteq A^n \setminus \chi^{\mathfrak{A}}$).

Then one easily shows:

(1) If p is a principal n-type (over T) and φ is a principal formula for p, φ is a principal formula (over T).

(2) If φ is a principal formula (over T) and we define

$$p := \{\chi \in F_n(\mathscr{L}) : T \models \forall v_1 \dots \forall v_n(\varphi \to \chi)\}$$

then p is a principal n-type (over T).

(3) If φ is a principal formula (over T) and $\chi \in F_n(\mathscr{L})$ satisfies $T \models \forall v_1 \dots \forall v_n(\chi \to \varphi)$, then either $T \models \neg\exists v_1 \dots \exists v_n \chi$ or $T \models \forall v_1 \dots \forall v_n(\chi \leftrightarrow \varphi)$.

(4) If φ, ψ are principal formulas (over T) then either $\varphi E_n(T)\psi$ or $T \models \neg\exists v_1 \dots \exists v_n(\varphi \leftrightarrow \psi)$. (I.e. for all $\mathfrak{A} \models T$, either $\varphi^{\mathfrak{A}} = \psi^{\mathfrak{A}}$ or $\varphi^{\mathfrak{A}} \cap \psi^{\mathfrak{A}} = \emptyset$.)

Claim (Under the assumption that $F_n(\mathscr{L})$ has infinitely many equivalence classes)
If $\varphi_1, \dots, \varphi_n \in F_n(\mathscr{L})$ are principal formulas (over T) then

$$T \models \exists v_1 \dots \exists v_n(\neg\varphi_1 \wedge \neg\varphi_2 \wedge \dots \wedge \neg\varphi_m).$$

Proof of Claim:
Suppose false. Then since T is complete,

$$T \models \forall v_1 \dots \forall v_n(\varphi_1 \vee \varphi_2 \vee \dots \vee \varphi_m) \quad (*)$$

Now let χ be any formula in $F_n(\mathscr{L})$. Then either $T \models \neg\exists v_1 \dots \exists v_n \chi$, or else by (*), there is some j, with $1 \le j \le m$ such that

$$T \models \exists v_1 \dots \exists v_n(\chi \wedge \varphi_j) \quad (\dagger).$$

(Using completeness of T always.)
But then, applying the definition that φ_j is principal we have

$$T \models \forall v_1 \dots \forall v_n(\varphi_j \to \chi). \quad (\ddagger)$$

Let $1 \le j_1 < \dots < j_r \le m$ be all the js such that (†) holds.
Then $T \models \forall v_1 \dots \forall v_n((\bigvee_{i=1}^r \varphi_{j_i}) \to \chi)$.
We also have $T \models \forall v_1 \dots \forall v_n(\chi \to (\bigvee_{i=1}^r \varphi_{j_i}))$, otherwise, using completeness of T,

$$T \models \exists v_1 \dots \exists v_n(\chi \wedge \neg(\bigvee_{i=1}^r \varphi_{j_i})),$$

$$\text{i.e. } T \models \exists v_1 \dots \exists v_n(\chi \wedge \neg\varphi_{j_1} \wedge \dots \wedge \neg\varphi_{j_r}).$$

But then by (*), there would be some $j \in \{1, \dots, m\}$ such that $T \models \exists v_1 \dots \exists v_n(\chi \wedge \varphi_j)$, and this contradicts the choice of $j_1, \dots, j_r$ as being the only js such that (†) hols.

Therefore $T \models \forall v_1 \dots \forall v_n(\chi \leftrightarrow (\bigvee_{i=1}^r \varphi_{j_i}))$ so χ (which was an arbitrary element of $F_n(\mathscr{L})$) is $E_n(T)$-equivalent to one of 2^m formulas.

This contradicts there being infinitely many $E_n(T)$-equivalence classes. ■Claim.

Given the claim, we construct a non-principal type as follows.

Let p be the set of all formulas $\psi \in F_n(\mathscr{L})$ with the property that for some $m \ge 1$ (depending on ψ) and some principle formulas $\varphi_1, \dots, \varphi_m$ (over T) we know $T \models \forall v_1 \dots \forall v_n((\bigwedge_{i=1}^m \neg\varphi_i) \to \psi)$.

Note:
(**) if $\varphi \in F_n(\mathscr{L})$ is a principle formula (over T), then $\neg\varphi \in p$.

Notice that p satisfies 13.1 (i), (ii) in the definition if an n-type. (13.1(ii) is easy, 13.1(i) is by the claim.)

Hence by 13.3.1(ii), there exists an n-type q (over T) such that $p \subseteq q$.

I claim that q is non-principal.

For if not, there is some $\varphi \in q$ that is principal for q. But by (1), φ is a principal formula. But then by (**), $\neg\varphi \in p \subseteq q$, therefore $\neg\varphi \in q$, therefore $(\varphi \wedge \neg\varphi) \in q$ – contradiction. ■

Summary of Proof

1. Note that the definition of a principal formula implies that principal formulas *generate* principal n-types.
2. $p'(\overline{v}) := \{\neg\varphi(\overline{v}) | \varphi \text{ a principal formula over } T\}$.
3. It follows from $|\,\mathrm{F}_n(T)/\,\mathrm{E}_n(T)| = \infty$ that $p'(\overline{v})$ has the *finite intersection property* (modulo T),
4. hence $p'(\overline{v})$ can be extended to an n-type $p(\overline{v})$ say,
5. and $p(\overline{v})$ does not contain any principal formulas.
6. It then follows that $p'(\overline{v})$ is a non-principal n-type (over T).

Proof of ⇒ direction of 12.5 R-N Theorem

Assume $|\,\mathrm{F}_n(T)/\,\mathrm{E}_n(T)| = \infty$. Then by 13.9 there exists a non-principal n-type (over T) p_0 say. Hence by 13.8 there exists a countably infinite model $\mathfrak{A} \vDash T$ s.t. $\mathfrak{A}$ omits p_0. Moreover by 13.2 there exists a countably infinite model $\mathfrak{A}' \vDash T$ that realizes p_0, thus $\mathfrak{A}$ and $\mathfrak{A}'$ do *not* realize all the same n-types (over T). So by 13.5 $\mathfrak{A} \ncong \mathfrak{A}'$ and both countable.

$\therefore$ T is *not* $\aleph_0$-categorical ■12.5 R-N

13.11 Example

$\mathscr{L} = \emptyset$, $T = \mathcal{T}\langle(\langle\mathbb{N}\rangle)$.

What is $F_2(\mathscr{L})/E_2(T)$?

Clearly for any $c \neq d \in \mathbb{N}$, and $a \neq b \in \mathbb{N}$, there is an automorphism $\pi : \mathfrak{N} \to \mathfrak{N}$ such that $\pi(c) = a$ and $\pi(d) = b$. From this we get that the only formulas in $F_2(\mathscr{L})$ up to $E_2(T)$-equivalence are

$v_1 \doteq v_2$, $\neg v_1 \doteq v_2$, $v_1 \doteq v_1$, $\neg v_1 \doteq v_1$.

$v_1 \doteq v_2$ and $\neg v_1 \doteq v_2$ are principal formulas. Therefore there are just 2 2-types, both principal.

13.12 remark

It is now clear by combining R-N theorem and 13.9 that; every n-type (over T) is principal if and only if T is $\aleph_0$-categorical.

14 The Method of Diagrams

Fix a language $\mathscr{L}$, and let $\mathfrak{A}$ be an $\mathscr{L}$-structure.

14.1 Definitions

(i) $\mathscr{L}(\mathfrak{A})$ denotes the language where we add to $\mathscr{L}$ a new constant symbol c_a for each $a \in A$.

(ii) $\mathfrak{A}^+$ denotes the $\mathscr{L}(\mathfrak{A})$-structure which is the same as $\mathfrak{A}$ as an $\mathscr{L}$-structure, and where the constant symbol c_a is interpreted as a (for each $a \in A$).
Thus $\mathfrak{A}^+ = \langle \mathfrak{A}; \{a : a \in A\} \rangle$.

(iii) The **diagram of** $\mathfrak{A}$, denoted by $\mathrm{Diag}(\mathfrak{A})$ is, by definition, the set of all quantifier-free sentences of $\mathscr{L}(\mathfrak{A})$ that are true in $\mathfrak{A}^+$.
$\mathrm{Diag}(\mathfrak{A}) := \{\varphi : \varphi \text{ a QF-sentence of } \mathscr{L}(\mathfrak{A}) \text{ such that } \mathfrak{A}^+ \models \varphi\}$.

14.2 Remark

A QF-sentence of $\mathscr{L}(\mathfrak{A})$ may (clearly) be written in the form $\varphi(c_{a_1}, \ldots, c_{a_n})$ where $\varphi(v_1, \ldots, v_n)$ is a QF-formula of $\mathscr{L}$.

So $\varphi(c_{a_1}, \ldots, c_{a_n}) \in \mathrm{Diag}(\mathfrak{A})$ means precisely that $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$ (and φ is QF).

14.3 Notation

If we have two languages $\mathscr{L}$, $\mathscr{L}'$ with $\mathscr{L} \subseteq \mathscr{L}'$ (in the obvious sense) and $\mathfrak{A}'$ is an $\mathscr{L}'$-structure then we may consider $\mathfrak{A}'$ as an $\mathscr{L}$-structure, $\mathfrak{A}$. We write $\mathfrak{A} = \mathfrak{A}' \restriction \mathscr{L}$ and call $\mathfrak{A}$ the **reduct** of $\mathfrak{A}'$ to $\mathscr{L}$.
We also say that $\mathfrak{A}'$ is an **expansion** of $\mathfrak{A}$ to $\mathscr{L}'$.

14.4 Theorem - The Method of Diagrams

Let $\mathfrak{A}$ be an $\mathscr{L}$-structure and $\mathfrak{B}'$ any $\mathscr{L}(\mathfrak{A})$-structure such that $\mathfrak{B}' \models \mathrm{Diag}(\mathfrak{A})$. Then the function $\pi : A \to B'$ defined by $\pi(a) = c_a^{\mathfrak{B}'}$ is an embedding from $\mathfrak{A}$ to $\mathfrak{B}' \restriction \mathscr{L}$.

Proof

We use 4.6.
So let $\varphi(v_1, \ldots, v_n)$ be an atomic formula of $\mathscr{L}$ and $a_1, \ldots, a_n \in A$. Suppose that $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$.

Then by 14.2, $\varphi(c_{a_1}, \ldots, c_{a_n}) \in \mathrm{Diag}(\mathfrak{A})$. Therefore $\mathfrak{B}' \models \varphi(c_{a_1}, \ldots, c_{a_n})$. I.e. $\mathfrak{B}' \restriction \mathscr{L} \models \varphi[c_{a_1}^{\mathfrak{B}'}, \ldots, c_{a_n}^{\mathfrak{B}'}]$.
I.e. $\mathfrak{B}' \restriction \mathscr{L} \models \varphi[\pi(a_1), \ldots, \pi(a_n)]$.
Similarly in "$\Leftarrow$" direction by considering $\neg\varphi(v_1, \ldots, v_n)$.
Hence the result by 4.6. ■

14.5 Definition - Complete Diagram

The **complete diagram** of an $\mathscr{L}$-structure $\mathfrak{A}$, denoted $\mathrm{CDiag}(\mathfrak{A})$, is the set

$$\{\varphi : \varphi \text{ is an } \mathscr{L}(\mathfrak{A})\text{-sentence sucht that} \mathfrak{A}^+ \models \varphi\}.$$

Then just as in the proof of 14.4 (replacing atomic $\varphi(v_1, \ldots, v_n)$ by arbitrary $\varphi(v_1, \ldots, v_n)$) one shows

14.6 Theorem - The Method of (Complete) Diagrams

Let $\mathfrak{A}$ be an $\mathscr{L}$-structure and $\mathfrak{B}'$ an $\mathscr{L}(\mathfrak{A})$-structure such that $\mathfrak{B}' \models \mathrm{CDiag}(\mathfrak{A})$. Then the function $\pi : A \to B'$ defined by $\pi(a) = c_a^{\mathfrak{B}'}$ is an elementary embedding from $\mathfrak{A}$ to $\mathfrak{B}' \restriction \mathscr{L}$. ■

14.7 Definition - Existential/Universal

An $\mathscr{L}$-formula $\varphi(v_1, ..., v_n)$ is called **existential** or **universal** if and only if it has the form $\exists v_{n+1} ... \exists v_{n+r} \psi(v_1, ..., v_{n+r})$ or $\forall v_{n+1} ... \forall v_{n+r} \psi(v_1, ..., v_{n+r})$ respectively where ψ is a QF-formula of $\mathscr{L}$.

14.8 Lemma

Let $\pi : \mathfrak{A} \to \mathfrak{B}$ be an embedding of $\mathscr{L}$-structures and $\varphi(v_1, ..., v_n)$ a formula that is **logically equivilent** to an existential formula (by logically equivilent to a formula $\gamma(v_1, ..., v_n)$ say we mean $\vDash \forall v_1...v_n(\gamma(v_1, ..., v_n) \leftrightarrow \varphi(v_1, ..., v_n)))$. Then for all $a_1, ..., a_n \in A$ if $\mathfrak{A} \vDash \varphi[a_1, ..., a_n]$ then $\mathfrak{B} \vDash \varphi[\pi(a_1), ..., \pi(a_n)]$.

Proof:

Suppose $\mathfrak{A} \vDash \varphi[a_1, ..., a_n]$, then by logical equivilence it follows that $\mathfrak{A} \vDash \gamma[a_1, ..., a_n]$. We can write γ as $\exists v_{n+1}...v_{n+r}\psi(v_1, ..., v_{n+r})$ where ψ a $\mathscr{L}$-quantifier free formulaso for some $a_{n+1}, ..., a_{n+r} \in A, \mathfrak{A} \vDash \psi[a_1, ..., a_{n+r}]$. Thus by 4.11 $\mathfrak{B} \vDash \psi[\pi(a_1), ..., \pi(a_{n+r})]$, so $\mathfrak{B} \vDash \exists v_{n+1}...v_{n+r}\psi[\pi(a_1), ..., \pi(a_n)]$ i.e. $\mathfrak{B} \vDash \gamma[\pi(a_1), ..., \pi(a_n)]$
$\therefore \mathfrak{B} \vDash \varphi[\pi(a_1), ..., \pi(a_n)]$ ∎

14.9 Remark

(1) The same result holds if $\varphi(v_1, \ldots, v_n)$ is universal if we reverse the implication.
(*Reason* (easy exercise[7]): If $\rho(v_1, \ldots, v_n)$ is existential, then $\neg\rho(v_1, \ldots, v_n)$ is logically equivalent to a universal formula.)

(2) We do not have "if and only if" in 14.8 for φ existential.
E.g. consider the embedding $\mathrm{id} : \langle \mathbb{Z}; + \rangle \hookrightarrow \langle \mathbb{Q}; + \rangle$. Let $\varphi(v_1)$ be the existential formula

$$\exists v_2\, v_1 \doteq (v_2 + v_2).$$

Then $\langle \mathbb{Q}; + \rangle \models \varphi[\mathrm{id}_{\mathbb{Z}}(1)]$. But it is not true that $\langle \mathbb{Z}; + \rangle \models \varphi[1]$.
Another example $\pi : \underbrace{\langle \{1\} \rangle}_{\mathfrak{A}} \hookrightarrow \underbrace{\langle \{1, 2\} \rangle}_{\mathfrak{B}}$ ($\mathscr{L}$ empty):
Let $\varphi(v_1)$ be $\exists v_2 \neg v_1 \doteq v_2$. Then $\mathfrak{B} \models \varphi[\pi(1)]$ but not $\mathfrak{A} \models \varphi[1]$.

14.10 Lemma

Let $\mathfrak{A}, \mathfrak{B}$ be any $\mathscr{L}$-structures then TFAE

1. for every existential sentence α of $\mathscr{L}$, if $\mathfrak{B} \vDash \alpha$ then $\mathfrak{A} \vDash \alpha$.

2. there exists a $\mathscr{L}$-structure $\mathfrak{A}^*$ and embeddings $\pi_1 : \mathfrak{B} \hookrightarrow \mathfrak{A}^*$ and $\pi_2 : \mathfrak{A} \preceq \mathfrak{A}^*$.

Remark: We cannot have, in general, an embedding from $\mathfrak{B}$ to $\mathfrak{A}$, since, e.g., $\mathrm{Card}(B)$ might be $> \mathrm{Card}(A)$.

Proof

(2)⇒(1): Suppose π_1, π_2, $\mathfrak{A}^*$ exist as in (2). Let α be any existential sentence of $\mathscr{L}$ such that $\mathfrak{B} \models \alpha$. Then $\mathfrak{A}^* \models \alpha$ by 14.8. But since $\pi_2 : \mathfrak{A} \preceq \mathfrak{A}^*$ we have $\mathfrak{A} \equiv \mathfrak{A}^*$. Therefore $\mathfrak{A} \models \alpha$.

(1)⇒(2): Assume (1). We use the method of diagrams.
Consider the languages $\mathscr{L}(\mathfrak{A})$ and $\mathscr{L}(\mathfrak{B})$, where we assume that the constants c_a (for $a \in A$) and c_b (for $b \in B$) do not overlap. Let $\mathscr{L}'$ denote the language consisting of $\mathscr{L}$ and both these sets of new constant symbols.

Let $\Sigma := \mathrm{Diag}(\mathfrak{B}) \cup \mathrm{CDiag}(\mathfrak{A})$. (So Σ is a set of $\mathscr{L}'$-sentences.)

If Σ has a model, $\mathfrak{A}'$ say, then by 14.4, 14.6 we may take $\mathfrak{A}^*$ to be $\mathfrak{A}' \restriction \mathscr{L}$. So we must show that Σ is satisfiable. We use the Compactness Theorem. So we only have to show that Σ is finitely satisfiable.

So let $\Sigma_0 \subseteq_{\mathrm{fin}} \Sigma$. Then there are $\varphi_1, \ldots, \varphi_l \in \mathrm{Diag}(\mathfrak{B})$ such that $\Sigma_0 \subseteq \{\varphi_1, \ldots, \varphi_l\} \cup \mathrm{CDiag}(\mathfrak{A})$. Let $\varphi = \bigwedge_{i=1}^{l} \varphi_i$. Then $\varphi \in \mathrm{Diag}(\mathfrak{B})$, and we must show that $\mathrm{CDiag}(\mathfrak{A}) \cup \{\varphi\}$ has a model.

Suppose not. Then $\mathrm{CDiag}(\mathfrak{A}) \models \neg\varphi$. Write φ as $\psi(c_{b_1}, \ldots, c_{b_n})$ where $b_1, \ldots, b_n \in B$ and $\psi(v_1, \ldots, v_n)$ is a QF-formula of $\mathscr{L}$. Then $\mathrm{CDiag}(\mathfrak{A}) \models \neg\psi(c_{b_1}, \ldots, c_{b_n})$. But the constant symbols $c_{b_1}, \ldots, c_{b_n}$ do not occur in any sentence in $\mathrm{CDiag}(\mathfrak{A})$. Therefore (by a previous exercise)

$$\mathrm{CDiag}(\mathfrak{A}) \models \forall v_1 \ldots \forall v_n \neg\psi(v_1, \ldots, v_n).$$

Therefore $\mathrm{CDiag}(\mathfrak{A}) \models \neg\exists v_1 \ldots \exists v_n \psi(v_1, \ldots, v_n)$. (*)
But $\psi(c_{b_1}, \ldots, c_{b_n}) \in \mathrm{Diag}(\mathfrak{B})$, so $\mathfrak{B} \models \psi[b_1, \ldots, b_n]$ (by 14.2).

[7] for solution see proof of 15.3.1

Therefore $\mathfrak{B} \models \exists v_1 \ldots \exists v_n \psi(v_1, \ldots, v_n)$. Let α be the $\mathscr{L}$-sentence $\exists v_1 \ldots \exists v_n \psi(v_1, \ldots, v_n)$. Then $\mathfrak{B} \models \alpha$. But by (*), since $\mathfrak{A}^+ \models \mathrm{CDiag}(A)$, we have $\mathfrak{A}^+ \models \neg\alpha$. But this is an $\mathscr{L}$-sentence, therefore $\mathfrak{A} \models \neg\alpha$. This contradicts (1) of the lemma hypothesis. ■

15 Preservation Theorems

Fix a language $\mathscr{L}$.

15.1 Lemma

Let $\mathscr{C}$ be a class of $\mathscr{L}$-sentences and Δ a collection of $\mathscr{L}$-sentences that is closed under disjunction. Then $\mathscr{C}$ is Δ-axiomatizable if and only if

(a) $\mathscr{C}$ is axiomatizable and

(b) whenever $\mathfrak{B} \in \mathscr{C}$ and $\mathfrak{A}$ is any $\mathscr{L}$-structure s.t. $\mathrm{Th}(\mathfrak{B}) \cap \Delta \subseteq \mathrm{Th}(\mathfrak{A})$ then $\mathfrak{A} \in C$.

Proof

"$\Rightarrow$": Suppose $\mathcal{C}$ is Δ-axiomatizable. Say $T \subseteq \Delta$ and for all $\mathfrak{A}$, $\mathfrak{A} \in \mathcal{C} \Leftrightarrow \mathfrak{A} \models T$. Obviously (a) holds. For (b), suppose $\mathfrak{B} \in \mathcal{C}$ and $\mathcal{T}\langle(\mathfrak{B}) \cap \Delta \subseteq \mathcal{T}\langle(\mathfrak{A})$. Then $\mathfrak{B} \models T$, so $T \subseteq \mathcal{T}\langle(\mathfrak{B}) \cap \Delta$. Hence $T \subseteq \mathcal{T}\langle(\mathfrak{A})$, i.e. $\mathfrak{A} \models T$, as required.

"$\Leftarrow$": Suppose (a) and (b) hold. By (a) we may choose some set T of $\mathscr{L}$-sentences such that for all $\mathfrak{A}$, $\mathfrak{A} \in \mathcal{C} \Leftrightarrow \mathfrak{A} \models T$.

Let $\Sigma\{\varphi : \varphi \in \Delta \text{ and } T \models \varphi\}$. Obviously $\Sigma \subseteq \Delta$ and every $\mathfrak{A} \in \mathcal{C}$ satisfies $\mathfrak{A} \models \Sigma$.

So it is sufficient to show that for all $\mathscr{L}$-structures $\mathfrak{A}$, if $\mathfrak{A} \models \Sigma$ then $\mathfrak{A} \in \mathcal{C}$.

So fix any $\mathfrak{A} \models \Sigma$.

Let $H = \{\neg\chi : \chi \in \Delta \text{ and } \mathfrak{A} \models \neg\chi\}$.

Claim: $T \cup H$ is satisfiable.

Proof of claim:

By the Compactness Theorem, it is sufficient to consider $\neg\chi_1, \dots, \neg\chi_N \in H$ and show that $T \cup \{\neg\chi_1, \dots, \neg\chi_N\}$ has a model.

Suppose not. Then $T \models \underbrace{(\bigvee_{i=1}^{N} \chi_i)}_{\chi}$.

Since $\chi_1, \dots, \chi_N \in \Delta$ and Δ is closed under disjunction, we get that $\chi \in \Delta$. Therefore $\chi \in \Sigma$, by definition of Σ. Further $\mathfrak{A} \models \neg\chi$ since $\mathfrak{A} \models \neg\chi_i$ for $i = 1, \dots, N$ (by definition of H). (NB. $\models (\neg \bigvee_{i=1}^{N} \chi_i \leftrightarrow \bigwedge_{i=1}^{N} \neg\chi_i)$) This is a contradiction since $\mathfrak{A} \models \Sigma$. $\dashv$ claim.

Using the claim, let $\mathfrak{B}$ be an $\mathscr{L}$-structure such that $\mathfrak{B} \models T \cup H$. Then $\mathfrak{B} \models T$ so $\mathfrak{B} \in \mathcal{C}$. We want to show $\mathfrak{A} \in \mathcal{C}$. So it is sufficient (by (b)) to show that $\mathcal{T}\langle(\mathfrak{B}) \cap \Delta \subseteq \mathcal{T}\langle(\mathfrak{A})$.

So let $\chi \in \mathcal{T}\langle(\mathfrak{B}) \cap \Delta$. If $\chi \notin \mathcal{T}\langle(\mathfrak{A})$ then $\neg\chi \in \mathcal{T}\langle(\mathfrak{A})$, i.e. $\mathfrak{A} \models \neg\chi$. So since $\chi \in \Delta$, we have $\neg\chi \in H$. But $\mathfrak{B} \models H$, so $\mathfrak{B} \models \neg\chi$ – contradiction since $\chi \in \mathcal{T}\langle(\mathfrak{B})$.

15.2 Definition - Axiomatization

let $\mathscr{C}$ be a class of $\mathscr{L}$-structures, $\mathscr{C}$ is **axiomatizable** if and only if there exists a set T of $\mathscr{L}$-sentences s.t. $\mathscr{C}$ is precisely the class of models of T, i.e. for all $\mathscr{L}$-structures $\mathfrak{A}$, $\mathfrak{A} \in \mathscr{C}$ if and only if $\mathfrak{A} \vDash T$. $\mathscr{C}$ is **finitely** axiomatizable if and only if there exists a finite set T of $\mathscr{L}$-sentences s.t. for all $\mathscr{L}$-structures $\mathfrak{A}$, $\mathfrak{A} \in \mathscr{C}$ if and only if $\mathfrak{A} \vDash T$. $\mathscr{C}$ is **existentially/universally** axiomatizable if and only if there exists set T of existential/universal $\mathscr{L}$-sentences respectively s.t. for all $\mathscr{L}$-structures $\mathfrak{A}$, $\mathfrak{A} \in \mathscr{C}$ if and only if $\mathfrak{A} \vDash T$.

15.3 Theorem

let $\mathscr{C}$ be a axiomatizable class of $\mathscr{L}$-structures then TFAE

(i) $\mathscr{C}$ is existentially axiomatizable

(ii) $\mathfrak{B} \in \mathscr{C}$ and $\mathfrak{A}$ a $\mathscr{L}$-structure s.t. $\exists \pi : \mathfrak{B} \hookrightarrow \mathfrak{A}$ then $\mathfrak{A} \in \mathscr{C}$

Proof

<u>(i) $\Rightarrow$ (ii):</u> Let T be a set of existential $\mathscr{L}$-sentences axiomatizing $\mathscr{C}$. Let $\mathfrak{B} \in \mathscr{C}$, $\mathfrak{A}$ be an $\mathscr{L}$-structure s.t. there is a π s.t. and $\pi : \mathfrak{B} \hookrightarrow \mathfrak{A}$. Let $\varphi \in T$, so $\mathfrak{B} \vDash \varphi$, hence $\mathfrak{A} \vDash \varphi$ by 14.8. Thus $\mathfrak{A} \vDash T$, therefore $\mathfrak{A} \in \mathscr{C}$.
<u>(ii) $\Rightarrow$ (i):</u> Let $\Delta = \{\varphi |$ for some existential $\mathscr{L}$-sentence $\psi, \vDash (\varphi \leftrightarrow \psi)\}$. **Claim:** Δ is closed under disjunction.

Proof of Claim: exercise

If $\mathscr{C}$ is Δ-axiomatizable then by definition of Δ it is existentially axiomatizable and we are given $\mathscr{C}$ is axiomatizable, thus it is sufficient to verify (b) in 15.1. So suppose $\mathfrak{B} \in \mathscr{C}$ and $\mathfrak{A}$ is any $\mathscr{L}$-structure s.t. $\mathrm{Th}(\mathfrak{B}) \cap \Delta \subseteq \mathrm{Th}(\mathfrak{A})$. $\mathrm{Th}(\mathfrak{B}) \cap \Delta \subseteq \mathrm{Th}(\mathfrak{A})$ implies (1) of 14.10, thus there exists a $\mathscr{L}$-structure $\mathfrak{A}^*$ and embeddings $\pi_1 : \mathfrak{B} \hookrightarrow \mathfrak{A}^*$ and $\pi_2 : \mathfrak{A} \preceq \mathfrak{A}^*$. Apply (ii) (what we are assuming) to $\mathfrak{B}$ and $\mathfrak{A}^*$ to get $\mathfrak{A}^* \in \mathscr{C}$, now since π_2 is elementary we have $\mathfrak{A} \equiv \mathfrak{A}^*$, finally since $\mathscr{C}$ is axiomatizable we get $\mathfrak{A} \in \mathscr{C}$ as required for 15.1. ■

15.3.1 Theorem - 15.3 for "Universally"

let $\mathscr{C}$ be an axiomatizable class of $\mathscr{L}$-structures then TFAE

(i) $\mathscr{C}$ is universally axiomatizable

(ii) $\mathfrak{A} \in \mathscr{C}$ and $\mathfrak{B}$ a $\mathscr{L}$-structure s.t. $\exists \pi : \mathfrak{A} \hookrightarrow \mathfrak{B}$ then $\mathfrak{B} \in \mathscr{C}$

Proof - On Mid Term Exam 09

15.4 Corollary

Let $\varphi(v_1, \ldots, v_n)$ be any formula of $\mathscr{L}$ with the property that it is preserved under embeddings, i.e. whenever $\pi : \mathfrak{B} \hookrightarrow \mathfrak{A}$ is an embedding of $\mathscr{L}$-structures, and $b_1, \ldots, b_n \in B$ and $\mathfrak{B} \models \varphi[b_1, \ldots, b_n]$, then $\mathfrak{A} \models \varphi[\pi(b_1), \ldots, \pi(b_n)]$. Then φ is logically equivalent to an existential formula.

Proof:

Let $c_1, \ldots, c_n$ be new constant symbols and let $\mathscr{L}'$ be the corresponding extension of $\mathscr{L}$. Let T be the $\mathscr{L}'$-theory $\{\varphi(c_1, \ldots, c_n)\}$, and let $\mathcal{C} := \{\mathfrak{A}' : \mathfrak{A}' \text{ an } \mathscr{L}'\text{-structure such that } \mathfrak{A}' \models T\}$. (May assume $\mathcal{C} \neq \emptyset$, i.e. T *is* a theory. Otherwise φ is equivalent to $\exists v_1 \neg v_1 \doteq v_1$.)

Note that $\mathcal{C}$ is an axiomatizable class. We verify 15.3(ii).
So let $\mathfrak{B}' \in \mathcal{C}$ and $\pi : \mathfrak{B}' \hookrightarrow \mathfrak{A}'$ be an embedding of $\mathscr{L}'$-structures. We want to show $\mathfrak{A}' \in \mathcal{C}$.

Let $\mathfrak{B} = \mathfrak{B}' \restriction \mathscr{L}$, $\mathfrak{A} = \mathfrak{A}' \restriction \mathscr{L}$. Now $\mathfrak{B}' \models \varphi(c_1, \ldots, c_n)$, so $\mathfrak{B} \models \varphi[c_1^{\mathfrak{B}'}, \ldots, c_n^{\mathfrak{B}'}]$. Therefore by hypothesis, $\mathfrak{A} \models \varphi[\pi(c_1^{\mathfrak{B}'}), \ldots, \pi(c_n^{\mathfrak{B}'})]$ (since π is an embedding of $\mathscr{L}$-structures). But since π is also an embedding of $\mathscr{L}'$-structures, we have $\pi(c_1^{\mathfrak{B}'}) = c_1^{\mathfrak{A}'}, \ldots, \pi(c_n^{\mathfrak{B}'}) = c_n^{\mathfrak{A}'}$.
Therefore $\mathfrak{A} \models \varphi[c_1^{\mathfrak{A}'}, \ldots, c_n^{\mathfrak{A}'}]$. I.e. $\mathfrak{A}' \models \varphi(c_1, \ldots, c_n)$. Therefore $\mathfrak{A}' \models T$ so $\mathfrak{A}' \in \mathcal{C}$ and 15.3(ii) is established.
Therefore by 15.3, T is existentially axiomatizable. Let Σ be a set of existential sentences such that for all

$$\mathscr{L}'\text{-structures } \mathfrak{A}', \mathfrak{A}' \models \Sigma \text{ iff } \mathfrak{A}' \in \mathcal{C} \text{ iff } \mathfrak{A}' \models \varphi(c_1, \ldots, c_n) \quad (*).$$

Then $\Sigma \models \varphi(c_1, \ldots, c_n)$ and therefore (exercise)$\Sigma_0 \subseteq \Sigma$, such that $\Sigma_0 \models \varphi(c_1, \ldots, c_n)$.

Let $\psi(c_1, \ldots, c_n)$ be the conjunction of the formulas in Σ_0 (where $\psi(v_1, \ldots, v_n)$ is an $\mathscr{L}$-formula). Then $\psi(c_1, \ldots, c_n)$ is logically equivalent to an existential sentence of $\mathscr{L}'$ (e.g. $\exists v_r \psi_1(v_r, \ldots) \wedge \exists v_r \psi_2(v_r, \ldots)$ is logically equivalent to $\exists v_r \exists v_{r'} (\psi_1(v_r, \ldots) \wedge \psi_2(v_{r'}, \ldots))$ where $r' \neq r$ is chosen to be freely substitutable for v_r in ψ_2) and $\psi(c_1, \ldots, c_n) \models \varphi(c_1, \ldots, c_n)$.

By (*), $\varphi(c_1, \ldots, c_n) \models \psi(c_1, \ldots, c_n)$. Therefore $\models (\varphi(c_1, \ldots, c_n) \leftrightarrow \psi(c_1, \ldots, c_n))$. Therefore by a previous remark,

$$\models \forall v_1 \ldots \forall v_n (\varphi(v_1, \ldots, v_n) \leftrightarrow \psi(v_1, \ldots, v_n)),$$

as required. ■