Set Theory 292B: Model-theoretic Forcing and Its Applications

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Abstract

In 1962 Paul Cohen invented set-theoretic forcing to solve the independence problem of continuum hypothesis. It turns out that forcing is quite powerful tool and it has applications in many branches of mathematics. In 1970s Abraham Robinson extended Cohen's forcing to model theory and developed finite forcing and infinite forcing. In this term paper we study Robinson's finite forcing and related applications in model-theoretical settings. It is based on Robinson's work in [1, 2].

1 Preliminaries

Our notations of first-order logic and model theory are standard (refs. [3, 4, 5]). We briefly reviews some relevant ones. Our first-order language has five logic symbols, namely $\wedge$, $\vee$, $\neg$, $=$ and $\exists$. Other symbols (e.g., $\forall$, $\rightarrow$ etc.) are treated as abbreviations. A *basic sentence* is an atomic or negated atomic sentence. We know every formula φ has a logically equivalent prenex form $Q_1 x \ldots Q_n x \psi(x_1, \ldots, x_n)$ where $Q_i = \forall$ or $\exists$. We say φ is *existential* or $\exists_1$ formula, *universal* or $\forall_1$ formula and *inductive* or $\forall_2$ formula if Q_i's, viewed as an regular expression, are of the form $\exists^*$, $\forall^*$ and $\forall^*\exists^*$ respectively. Given a theory T, we write $T_\forall$ (resp. $T_{\forall_2}$) for the set of $\forall$-sentences (resp. $T_{\forall_2}$-sentences) which are logical consequences of T.

Let $\mathscr{L}$ be a first-order language. We adjoin a set C of new constants to $\mathscr{L}$ and denote the new language by $\mathscr{L}[C]$. The size of C is variable according to the context and can be arbitrary large so that we never run out of constant symbols to name objects in discussion. Let $\mathfrak{A}$ be a $\mathscr{L}$-structure with domain A. For notation simplicity we identify A with set of constants such that each of objects in domain A is named by itself in $\mathscr{L}[A]$. Also $(\mathfrak{A}, A)$ denotes the expansion structure of $\mathfrak{A}$ in language $\mathscr{L}[A]$. Often we write $\mathscr{L}[\mathfrak{A}]$ for $\mathscr{L}[A]$ when $\mathfrak{A}$ is implicit from the context.

Definition 1.1 (Diagrams)**.** *Let $\mathfrak{A}$ and $\mathscr{L}[\mathfrak{A}]$ as defined above. A digram $\Delta[\mathfrak{A}]$ is the set of basic sentences of $\mathscr{L}[\mathfrak{A}]$ which are true in $\mathfrak{A}$. Similarly an elementary digram $\mathbb{E}[\mathfrak{A}]$ is the set of sentences of $\mathscr{L}[\mathfrak{A}]$ which are true in $\mathfrak{A}$, i.e., the theory $Th(\mathfrak{A}, A)$ in language $\mathscr{L}[\mathfrak{A}]$.*

An elementary digram is a complete description of $\mathfrak{A}$ in language $\mathscr{L}[\mathfrak{A}]$ and diagram is a partial description using only quantifier-free sentences. Robinson is the first one who systematically used diagrams in the setting of model theory. The following lemma is due to him (refs. [5], page 17, 49).

Lemma 1.1 (Diagram Lemma). *Let $\mathfrak{A}$ and $\mathscr{L}[\mathfrak{A}]$ as defined above. Let $\mathfrak{B}$ be a $\mathscr{L}[\mathfrak{A}]$-structure.*

(1) If $\mathfrak{B} \models \Delta[\mathfrak{A}]$, then $\mathfrak{A}$ can be embedded into $\mathfrak{B}$.

(2) If $\mathfrak{B} \models \mathbb{E}[\mathfrak{A}]$, then $\mathfrak{A}$ can be elementarily embedded into $\mathfrak{B}$.

It follows that given a model $\mathfrak{A}$ with domain A, $\Delta[\mathfrak{A}] \cup T$ is consistent is equivalent to the statement that $\mathfrak{A}$ can be embedded into a model of T.

2 Forcing Condition

Definition 2.1 (Forcing Condition). *Let T be a theory of $\mathscr{L}$. A forcing condition P is a set of basic sentences of $\mathscr{L}[A]$ such that $T \cup P$ is consistent. For a formula φ of $\mathscr{L}[A]$, we define the forcing relation $P \Vdash \varphi$ (read "P forces φ in $\mathscr{L}[A]$ for T", and often abbreviated "P forces φ") inductively on formula structure.*

(1) If φ is atomic then $P \Vdash \varphi$ iff $\varphi \in P$,

(2) $P \Vdash \varphi \vee \psi$ iff $P \Vdash \varphi$ or $P \Vdash \psi$,

(3) $P \Vdash \varphi \wedge \psi$ iff $P \Vdash \varphi$ and $P \Vdash \psi$,

(4) $P \Vdash \neg\varphi$ iff there exists no $Q \supseteq P$ such that $Q \Vdash \varphi$,

(5) $P \Vdash \exists x \varphi(x)$ iff there exists a closed term t of $\mathscr{L}[A]$ such that $P \Vdash \varphi(t)$.

We say that P weakly forces φ, in symbols $P \Vdash^w \varphi$, if $P \Vdash \neg\neg\varphi$.

The following are well-known properties of forcing and weakly forcing.

Lemma 2.1 (Forcing Properties). *Let P, Q be forcing conditions for theory T. Then*

(1) P can not both forces φ and $\neg\varphi$.

(2) If $P \Vdash \varphi$ then $Q \Vdash \varphi$ for $Q \supseteq P$.

(3) If $P \Vdash \varphi$ then $P \Vdash^w \varphi$.

(4) If $P \Vdash^w \neg\varphi$ then $P \Vdash \neg\varphi$.

(5) $P \Vdash^w \neg\neg\varphi$ iff $P \Vdash^w \varphi$.

(6) If $P \Vdash \varphi$ and φ is a basic sentence of $\mathscr{L}[A]$, then $P \cup \{\varphi\}$ is a forcing condition for T.

Forcing is a quite interesting relation. An empty condition can force some formulas.

Example 2.1. *Consider Peano arithmetic $\mathfrak{N} = \langle \mathbb{N}; S, +, \cdot, < \rangle$. Let $\mathscr{L}(\mathbb{N})$ be the enlarged language. Since $\varnothing$ contains no sentence at all, certainly $\varnothing \nVdash \mathbf{m} < \mathbf{n}$ where $\mathbf{m} < \mathbf{n}$ is an atomic sentences with $\mathbf{m}, \mathbf{n} \in \mathbb{N}$. Therefore $\varnothing \nVdash \exists x(\mathbf{m} < x)$. But $\varnothing \Vdash \neg\neg\exists x(\mathbf{m} < x)$ for any $\mathbf{m} \in \mathbb{N}$. For $\varnothing \nVdash \neg\neg\exists x(\mathbf{m} < x)$ implies*

$$(\exists P)(\forall Q \supseteq P)(\forall \text{ closed term } t)(Q \nVdash \mathbf{m} < t)$$

In particular $P \nVdash \mathbf{m} < t$ for any closed term t. But $P \cup \{\mathbf{m} < S\mathbf{m}\}$ is consistent with $Th(\mathbb{N})$ and $P \cup \{\mathbf{m} < t\} \Vdash \mathbf{m} < S\mathbf{m}$, a contradiction.

Let P be a forcing condition for T. We use $T[P]$ (resp. $T^f[P]$) to denote the set of sentences in $\mathscr{L}$ which are forced (resp. weakly forced) by P. We write T^f for $T^f[\varnothing]$ and call T^f the *forcing companion* of T. We say T^f is the *forcing completion* of T if $T \subseteq T^f$. T is called *forcing complete* if T and T^f are logically equivalent.

3 Forcing Properties

We write $P \Vdash_A \varphi$ to mean that P forces φ in $\mathscr{L}[A]$. The following four lemmas shows that the choice of A is irrelevant as long as A is infinite. None of these lemmas is deep and most proofs are routine but tedious inductions on the structure of formulas. We shall only show the induction step for formulas in the negated form. When we write $P = P(\bar{a})$ and $\varphi = \varphi(\bar{a})$, we mean that P is a condition in $\mathscr{L}[A]$ for T and φ is a sentence of $\mathscr{L}[A]$ where $\bar{a}$ list all constants which are not in $\mathscr{L}$, but occur in either P or φ. By $P(\bar{b})$ and $\varphi(\bar{b})$ we mean the results of replacement of $\bar{a}$ by $\bar{b}$ where $\bar{b}$ is a tuple of constants disjoint with $\bar{a}$.

Lemma 3.1. *Let A be an infinite set of constants, $\bar{a}_0, \bar{a}_1, \bar{a}_2$ be disjoint tuples of constants in A and $\bar{a} = \bar{a}_0\bar{a}_1\bar{a}_2$. Let $P = P(\bar{a})$, $\varphi = \varphi(\bar{a}_0)$. Let $\bar{a}'_1$ be a tuple of constants in A such that $|\bar{a}_1| = |\bar{a}'_1|$, $\bar{a}'_1 \cap \bar{a} = \varnothing$ and let $P' = P(\bar{a}_0, \bar{a}'_1, \bar{a}_2)$. P' is a condition for T and $P \Vdash \varphi$ if and only if $P' \Vdash \varphi$.*

Proof. The first part is obvious. We only show the induction step for $\varphi(\bar{a}) = \neg\psi(\bar{a})$. Suppose for reductio that $P(\bar{a}_0, \bar{a}_1, \bar{a}_2) \Vdash \neg\psi(\bar{a}_0)$, but $P(\bar{a}_0, \bar{a}'_1, \bar{a}_2) \Vdash \neg\psi(\bar{a}_0)$. The second part implies that there exists $Q \supseteq P(\bar{a}_0, \bar{a}'_1, \bar{a}_2)$ such that $Q \Vdash \psi(\bar{a}_0)$. Write Q as $Q(\bar{a}_0, \bar{a}'_1, \bar{a}_2, \bar{a}_3)$ where $\bar{a}_3$ is disjoint with $\bar{a}_0, \bar{a}'_1$ and $\bar{a}_2$. Let $Q' = Q(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3)$. Clearly, Q' is a forcing condition for T and by induction hypothesis $Q' \Vdash \psi(\bar{a}_0)$. Certainly $Q' \supseteq P(\bar{a}_0, \bar{a}_1, \bar{a}_2)$, which contradicts the assumption that $P(\bar{a}_0, \bar{a}_1, \bar{a}_2) \Vdash \neg\psi(\bar{a}_0)$. The converse follows by symmetry. □

Lemma 3.2. *Let A be an infinite set of constants. Let $P = P(\bar{a}, \bar{a}_0)$ and $\varphi = \varphi(\bar{a})$. Let $\bar{b}$ be a tuple of constants not in A such that $|\bar{b}| = |\bar{a}|$ and let $P' = P(\bar{b}, \bar{a}_0)$, $\varphi' = \varphi(\bar{b})$. Let $B = (A \setminus \bar{a}) \cup \bar{b}$. P' is a condition in $\mathscr{L}[B]$ for T and $P \Vdash_A \varphi$ if only if $P' \Vdash_B \varphi'$.*

Proof. The first part is obvious. We only show the induction step for $\varphi(\bar{a}) = \neg\psi(\bar{a})$. Suppose for reductio that $P(\bar{a}, \bar{a}_0) \Vdash_A \neg\psi(\bar{a})$, but $P(\bar{b}, \bar{a}_0) \Vdash_B \neg\psi(\bar{b})$. The second part implies that there exists $Q \supseteq P(\bar{b})$ such that $Q \Vdash_B \psi(\bar{b})$. Write Q as $Q(\bar{b}, \bar{a}_0, \bar{a}_1)$ where $\bar{a}_1$ is disjoint with $\bar{b}$ and $\bar{a}_0$. Let $Q' = Q(\bar{a}, \bar{a}_0, \bar{a}_1)$. Clearly, Q' is a forcing condition for T and by induction hypothesis $Q' \Vdash_A \psi(\bar{a})$. Since $\bar{a}_0, \bar{a}_1 \subseteq A$, $Q' \supseteq P(\bar{a})$, which contradicts the assumption that $P(\bar{a}, \bar{a}_0) \Vdash_A \neg\psi(\bar{a})$. The converse follows by symmetry. □

Lemma 3.3. *Let A, B be infinite sets of constants such that $B \supseteq A$ and $B \setminus A$ is infinite. Let $P = P(\bar{a})$ and $\varphi = \varphi(\bar{a})$. Then P is a condition in $\mathscr{L}[B]$ for T and $P \Vdash_A \varphi$ if only if $P \Vdash_B \varphi$.*

Proof. The first part is obvious. We only show the induction step for $\varphi(\bar{a}) = \neg\psi(\bar{a})$. Suppose for reductio that $P(\bar{a}) \Vdash_A \neg\psi(\bar{a})$, but $P(\bar{a}) \Vdash_B \neg\psi(\bar{a})$. The second part implies that there exists $Q \supseteq P(\bar{a})$, $Q \Vdash_B \psi(\bar{a})$. Write Q as $Q(\bar{a}, \bar{a}_0, \bar{b})$ where $\bar{a}_0 \subseteq A$, $\bar{b} \subseteq B \setminus A$ and $\bar{a}_0, \bar{b}$ both are disjoint with $\bar{a}$. Since $\bar{a}$ is finite and B is infinite, we can select a tuple $\bar{a}' \subseteq A$ such that $\bar{a}' \cap \bar{a} = \varnothing$ and $|\bar{a}'| = |\bar{b}|$. Let $Q' = Q(\bar{a}, \bar{a}_0, \bar{a}')$. Clearly Q' is a condition for T. By Lemma 3.1, $Q' \Vdash_B \psi(\bar{a})$. By induction hypothesis $Q' \Vdash_A \psi(\bar{a})$. But $P(\bar{a}) \subseteq Q'$, resulting in a contradiction. Conversely, suppose $P(\bar{a}, \bar{a}_0) \Vdash_B \neg\psi(\bar{a})$, but $P(\bar{a}, \bar{a}_0) \nVdash_A \neg\psi(\bar{a})$. The second part implies that there exists

$Q \supseteq P(\bar{a}, \bar{a}_0)$ such that $Q \Vdash_A \psi(\bar{a})$. By the induction hypothesis, $Q \Vdash_B \psi(\bar{a})$, which contradicts the assumption that $P(\bar{a}, \bar{a}_0) \Vdash_B \neg\psi(\bar{a})$. $\square$

Lemma 3.4. *Let A, C be infinite sets of constants such that $C \supseteq A$ and $C \setminus A$ is infinite. Let $P = P(\bar{a})$ and $\varphi = \varphi(\bar{a})$ where $\bar{a} \subseteq A$. Let $\bar{b} \subseteq C \setminus A$ such that $|\bar{b}| = |\bar{a}|$ and let $P' = P(\bar{b})$, $\varphi' = \varphi(\bar{b})$. Then P' is a condition in $\mathscr{L}[C]$ for T and $P \Vdash_C \varphi$ if only if $P' \Vdash_C \varphi'$.*

Proof. Let $B = (A \setminus \bar{a}) \cup \bar{b}$. Then $B \subseteq C$. By Lemma 3.3, $P(\bar{a})$ and $P(\bar{b})$ both are conditions in $\mathscr{L}[C]$, $P(\bar{a}) \Vdash_C \varphi(\bar{a}) \Leftrightarrow P(\bar{a}) \Vdash_A \varphi(\bar{a})$ and $P(\bar{b}) \Vdash_C \varphi(\bar{b}) \Leftrightarrow P(\bar{b}) \Vdash_B \varphi(\bar{b})$. By Lemma 3.2, $P(\bar{b}) \Vdash_B \varphi(\bar{b}) \Leftrightarrow P(\bar{a}) \Vdash_A \varphi(\bar{a})$. Hence $P(\bar{a}) \Vdash_C \varphi(\bar{a}) \Leftrightarrow P(\bar{b}) \Vdash_C \varphi(\bar{b})$. $\square$

Lemma 3.5. *Let φ be a sentence in $\mathscr{L}$. $T^f[P] \models \varphi$ implies $\varphi \in T^f[P]$. In particular, both $T^f[P]$ and $T[P]$ are consistent.*

Proof. By completeness it suffice to show that if $T^f[P] \vdash \varphi$ then $\varphi \in T^f[P]$. We shall show by induction on proof length a stronger result that for all $\varphi(\bar{x})$ with free variables $\bar{x}$, if $T^f[P] \vdash \varphi(\bar{x})$ then $\varphi(\bar{t}) \in T^f[P]$ where $\bar{t}$ is a sequence of closed terms of $\mathscr{L}[A]$. We leave out verification of all logical axioms, which is routine but tedious work. Consider rule of modus ponen:

$$[\ \varphi(\bar{x}) \rightarrow \psi(\bar{x}),\ \varphi(\bar{x})\ ] \implies \psi(\bar{x})$$

By induction hypothesis $\varphi(\bar{t}) \rightarrow \psi(\bar{t})$, $\varphi(\bar{t}) \in T^f[P]$. By definition we have

$$(\forall Q \supseteq P)(\exists Q' \supseteq Q)(Q' \Vdash \neg\varphi(\bar{t}) \text{ or } Q' \Vdash \psi(\bar{t})) \tag{3.1}$$

$$(\forall Q \supseteq P)(\exists Q' \supseteq Q)(Q' \Vdash \varphi(\bar{t})) \tag{3.2}$$

For a condition $Q \supseteq P$ we claim that there exists a condition $Q' \supseteq Q$ such that $Q' \Vdash \psi(\bar{t})$. For by (3.1), there exists a condition $Q' \supseteq Q$ such that $Q' \Vdash \neg\varphi(\bar{t})$. However by (3.2) there exists a condition $Q'' \supseteq Q'$ such that $Q'' \Vdash \varphi(\bar{t})$. Then by Lemma 2.1(2) Q'' forces both $\varphi(\bar{t})$ and its negation, wich contradicts Lemma 2.1(1). So we have

$$(\forall Q \supseteq P)(\exists Q' \supseteq Q)(Q' \Vdash \psi(\bar{t})), \tag{3.3}$$

that is, $P \Vdash^w \psi(\bar{t})$. Now consider the rule of generalization.

$$\varphi(\bar{x}) \implies \forall \bar{x} \varphi(\bar{x})$$

Suppose, for reductio, that $P \not\Vdash^w \forall \bar{x} \varphi(\bar{x})$, that is, $P \not\Vdash^w \neg\exists \bar{x} \neg\varphi(\bar{x})$. By definition

$$(\exists Q \supseteq P)(\exists \text{ closed terms } \bar{t})(\forall Q' \supset Q)(Q' \not\Vdash \varphi(\bar{t})) \tag{3.4}$$

But the induction hypothesis implies that for all closed terms $\bar{t}$, $P \Vdash^w \varphi(\bar{t})$, which in turn is read

$$(\forall Q \supseteq P)(\forall \text{ closed terms } \bar{t})(\exists Q' \supset Q)(Q' \Vdash \varphi(\bar{t})) \tag{3.5}$$

This contradicts (3.4). The consistency of $T^f[P]$ follows from Lemma 2.1(1) and then the consistency of $T[P]$ follows from Lemma 2.1(3). $\square$

Lemma 3.6. *Let $P = P(\bar{a})$ be a condition in $\mathscr{L}[A]$ for T and $\varphi = \varphi(\bar{a})$ be a sentence of $\mathscr{L}[A]$ where $\bar{a}$ list all constants which are not in $\mathscr{L}$, but occur in either P or φ. Let $P' = P(\bar{t})$, $\varphi' = \varphi(\bar{t})$ be the results of replacement of $\bar{a}$ by closed terms $\bar{t}$ of $\mathscr{L}[A]$. If P' is a condition for T, then $P' \Vdash^w \varphi'(\bar{t})$.*

Proof. Suppose that $P' \not\Vdash^w \varphi'$, then there exists $Q \supseteq P$ such that $Q \Vdash \neg\varphi(\bar{t})$. Let $P'' = P(\bar{c})$ and $\varphi'' = \varphi(\bar{c})$ where $\bar{c}$ is a set of fresh constants not in Q or $\varphi'(\bar{t})$. By Lemma 3.4 P'' is a condition and $P'' \Vdash \varphi(\bar{c})$. It is easily seen that $P'' \cup Q \cup \{\bar{c} = \bar{t}\}$ is consistent with T and so is a condition. However $P'' \cup Q \cup \{\bar{c} = \bar{t}\}$ forces

$$\exists\bar{x}\exists\bar{y}(\neg\varphi(\bar{x}) \wedge \varphi(\bar{y}) \wedge \bar{x} = \bar{y})$$

which contradicts Lemma 3.5. ☐

Recall that given a model $\mathfrak{A}$, $\Delta[\mathfrak{A}] \cup T$ is consistent is equivalent to the statement that $\mathfrak{A}$ can be embedded into a model of T, which in turn is equivalent to the statement that every finite $P \subseteq \Delta[\mathfrak{A}]$ is a condition for T.

Lemma 3.7. *Given a model $\mathfrak{A}$ with domain A such that $\Delta[\mathfrak{A}] \cup T$ is consistent. Then for all existential formula $\varphi(\bar{x})$ and all closed terms $\bar{t}$ in $\mathscr{L}[\mathfrak{A}]$:*

$$\mathfrak{A} \models \varphi(\bar{t}) \text{ if and only if } P \Vdash \varphi(\bar{t}) \text{ for some finite } P \subseteq \Delta[\mathfrak{A}]$$

Proof. We only show "→" direction by induction on the structure of formulas. Similar arguments can complete the reverse direction.

(1) $\varphi(\bar{t})$ is an atomic or negated atomic formula. It suffices to let $P = \{\varphi(\bar{t})\}$.

(2) $\varphi(\bar{t}) = \varphi_1(\bar{t}) \vee \varphi_2(\bar{t})$. Suppose that $\mathfrak{A} \models \varphi_1(\bar{t}) \vee \varphi_2(\bar{t})$. Then either $\mathfrak{A} \models \varphi_1(\bar{t})$ or $\mathfrak{A} \models \varphi_2(\bar{t})$. Without loss let us assume $\mathfrak{A} \models \varphi_1(\bar{t})$. By induction hypothesis $P \Vdash \varphi_1(\bar{t})$ for some finite $P \subseteq \Delta[\mathfrak{A}]$. Clearly we have $P \Vdash \varphi_1(\bar{t}) \vee \varphi_2(\bar{t})$.

(3) $\varphi(\bar{t}) = \varphi_1(\bar{t}) \wedge \varphi_2(\bar{t})$. Suppose that $\mathfrak{A} \models \varphi_1(\bar{t}) \wedge \varphi_2(\bar{t})$. Then $\mathfrak{A} \models \varphi_1(\bar{t})$ and $\mathfrak{A} \models \varphi_2(\bar{t})$. By induction hypothesis $P \Vdash \varphi_1(\bar{t})$ for some finite $P \subseteq \Delta[\mathfrak{A}]$ and $Q \Vdash \varphi_1(\bar{t})$ for some finite $Q \subseteq \Delta[\mathfrak{A}]$. Clearly $P \cup Q \subseteq \Delta[\mathfrak{A}]$ is finite and a condition for T. Moreover, $P \cup Q \Vdash \varphi_1(\bar{t})$ and $P \cup Q \Vdash \varphi_2(\bar{t})$. So $P \cup Q \Vdash \varphi_1(\bar{t}) \wedge \varphi_2(\bar{t})$.

(4) $\varphi(\bar{t}) = \exists\bar{x}\varphi(\bar{x}, \bar{t})$. Suppose that $\mathfrak{A} \models \exists\bar{x}\varphi(\bar{x}, \bar{t})$. Then $\mathfrak{A} \models \varphi_1(\bar{t}_0, \bar{t})$ for some closed terms $\bar{t}_0$. By induction hypothesis $P \Vdash \varphi_1(\bar{t}_0, \bar{t})$ for some finite $P \subseteq \Delta[\mathfrak{A}]$. It follows immediately that $P \Vdash \exists\bar{x}\varphi(\bar{x}, \bar{t})$.

☐

Lemma 3.8. *Let φ be a universal sentence of $\mathscr{L}[\mathfrak{A}]$. Then for all condition P, $P \Vdash \varphi$ if and only if $T \cup P \models \varphi$.*

Proof. Note that φ is universal if and only if $\neg\varphi$ is existential. So it suffices to show that $P \Vdash \neg\varphi$ if and only if $T \cup P \models \neg\varphi$ any existential formula φ. Let $\mathfrak{A}$ be a model of $T \cup P$ with domain A. Obviously $P \subseteq \Delta[\mathfrak{A}]$. If $T \cup P \models \neg\varphi$, then $\mathfrak{A} \not\models \varphi$ and hence by Lemma 3.7 for any finite $Q \subseteq \Delta[\mathfrak{A}]$, $Q \not\Vdash \varphi$. Then certainly for any condition $Q \supseteq P$, $Q \not\Vdash \varphi$. It follows that $P \Vdash \neg\varphi$. Conversely, if $P \Vdash \neg\varphi$ then for any $Q \supseteq P$, $Q \not\Vdash \varphi$. We claim that for any finite $Q \subseteq \Delta[\mathfrak{A}]$, $Q \not\Vdash \varphi$. For there exists such Q, $Q \Vdash \varphi$, then $P \cup Q \subseteq \Delta[\mathfrak{A}]$ is a condition and forces φ, which contradicts the assumption. So by Lemma 3.7 again, $\mathfrak{A} \models \neg\varphi$. Since $\mathfrak{A}$ is an arbitrary model of $T \cup P$ we have $T \cup P \models \neg\varphi$. ☐

Lemma 3.9. *Let P be a finite basic sentences in $\mathscr{L}[\mathfrak{A}]$. P is a forcing condition for T if and only if P is a forcing condition for $T_\forall$. And for any sentences φ in $\mathscr{L}[\mathfrak{A}]$, $P \Vdash \varphi$ for T if and only if $P \Vdash \varphi$ for $T_\forall$. In particular, $T[P] = T_\forall[P]$.*

Proof. We only need to show that if P is consistent with $T_\forall$ then P is consistent with T. The converse is obvious. Since P is finite, we can write as $\varphi(\bar{a})$ the conjunction of all basic sentences in P where $\bar{a}$ are constants not in $\mathscr{L}$. If P is not consistent with T, then $T \models \neg\varphi(\bar{a})$. Since $\bar{a}$ don't appear in T, $T \models \forall\bar{x}\neg\varphi(\bar{x})$. Then $\forall\bar{x}\neg\varphi(\bar{x})$ is in $T_\forall$ and hence $T_\forall \models \neg\varphi(\bar{a})$. So P is not consistent with $T_\forall$. The rest follows immediately since forcing relation is completely determined by forcing conditions. ☐

Lemma 3.10. *Let $P(\bar{a})$ be a forcing condition and $\varphi(\bar{a})$ be a sentence in $\mathscr{L}[\mathfrak{A}]$ where $\bar{a}$ list all constants not in $\mathscr{L}$. Let $P(\bar{x})$, $\varphi(\bar{x})$ the results of substituting variables $\bar{x}$ for $\bar{a}$. If $P \Vdash \varphi$ then the following sentences is in T^f:*

$$\forall\bar{x}(\bigwedge P(\bar{x}) \to \varphi(\bar{x}))$$

where $\bigwedge P(\bar{x})$ is a conjunction of all sentences in $P(\bar{x})$.

Proof. We need to show that $\varnothing \Vdash^w \neg\exists\bar{x}\neg(\neg\bigwedge P(\bar{x}) \vee \varphi(\bar{x}))$. By Lemma 2.1(3) it suffices to show that $\varnothing \Vdash \neg\exists\bar{x}\neg(\neg\bigwedge P(\bar{x}) \vee \varphi(\bar{x}))$. Suppose the contrary, then there exists a condition Q and a set of closed terms $\bar{t}$ such that

$$Q \Vdash \neg(\neg\bigwedge P(\bar{t}) \vee \varphi(\bar{t}))$$

which further implies that

$$(\forall Q' \supseteq Q)\ Q' \nVdash (\neg\bigwedge P(\bar{t})) \text{ and } (\forall Q' \supseteq Q)\ Q' \nVdash \varphi(\bar{t}) \tag{3.6}$$

The first part of (3.6) implies $\bar{Q} \supseteq Q'$ such that $\bar{Q} \Vdash \bigwedge P(\bar{t})$. Since $\bigwedge P(\bar{t})$ is universal, by Lemma 3.8 any sentence in $P(\bar{t})$ is a logical consequence of $\bar{Q} \cup T$. So $\bar{Q} \cup P(\bar{t})$ is a condition for T. By Lemma 3.6 $P(\bar{t}) \Vdash \varphi(\bar{t})$ and hence $\bar{Q} \cup P(\bar{t}) \Vdash \varphi(\bar{t})$. But $\bar{Q} \cup P(\bar{t}) \supseteq Q$, which contradicts to the second part of (3.6). ☐

4 Generic Sequences and Generic Structures

Definition 4.1 (Generic Sequence)**.** *Let $\mathscr{L}[A]$ be a countable language. Let $\mathbb{P} = \{P_i : i < \omega\}$ be a sequence of forcing conditions for T. We say $\mathbb{P}$ is T-generic sequence if the following two conditions hold:*

(1) $\mathbb{P}$ is complete; i.e., for any atomic sentence φ in $\mathscr{L}[A]$, exactly one of φ and $\neg\varphi$ is in $\bigcup\mathbb{P}$.

(2) For any sentences φ in $\mathscr{L}[A]$, exactly one of φ and $\neg\varphi$ is forced by some $P \in \mathbb{P}$.

Some authors use the term *generic set.* A set $\mathbb{G}$ of conditions is *T-generic* if $\mathbb{G} = \bigcup\mathbb{P}$ for some T-generic sequence $\mathbb{P}$. A fundamental result of forcing is that generic sequence always exists for any consistent theory T assuming $\mathscr{L}[A]$ is countable.

Theorem 4.1. *Assuming $\mathscr{L}[A]$ is countable, for any theory T in $\mathscr{L}$, there exists a T-generic sequence $\mathbb{P}$.*

Proof. Since $\mathscr{L}[A]$ is countable, we can enumerate all sentences in a sequence $\varphi_0, \varphi_1, \ldots$. Let $P_0 = \varnothing$. Assume P_i is already constructed. If $P_i \Vdash \neg\varphi_i$, then let

$$P_{i+1} = \begin{cases} P_i \cup \{\neg\varphi_i\} & \text{if } \varphi_i \text{ is atomic,} \\ P_i & \text{otherwise.} \end{cases}$$

By Lemma 2.1(6) P_{i+1} is condition. If $P_{i+1} \nVdash \neg\varphi_i$, then there must be a condition $Q \supseteq P_i$ such that $Q \Vdash \varphi_i$. Let $P_{i+1} = Q$. Finally let $\mathbb{P} = \{P_i : i < \omega\}$. It is easily seen that $\langle P_i : i < \omega \rangle$ is increasing and for any sentence φ of $\mathscr{L}[A]$ exactly one of φ and $\neg\varphi$ is forced by some P_i. In case φ is atomic, this means that exactly one of φ and $\neg\varphi$ is in $\bigcup \mathbb{P}$. Hence $\mathbb{P}$ is generic. □

Definition 4.2 (Generic Structure)**.** *Let $\mathfrak{A}$ be a $\mathscr{L}$-structure. $\mathfrak{A}$ is T-generic if the following two conditions are satisfied.*

(1) $T \cup \Delta[\mathfrak{A}]$ is consistent.

(2) For every sentence φ of $\mathscr{L}[\mathfrak{A}]$, $\mathfrak{A} \models \varphi$ if and only if there exists a finite $P \subseteq \Delta[\mathfrak{A}]$ such that $P \Vdash \varphi$.

The second condition says that a model is T-generic if it realizes exactly those sentences that are forced by some finite subset of its diagram. Note that there is no requirement that $\mathfrak{A}$ is countable, but a countable T-generic structure always exists assuming $\mathscr{L}$ is countable.

Theorem 4.2 (Generic Structure)**.** *Let $\mathscr{L}$ be a countable language. For any T-generic sequence $\mathbb{P}$, there exists a countable T-generic structure $\mathfrak{A}$ and for every sentence φ of $\mathscr{L}[\mathfrak{A}]$, $\mathfrak{A} \models \varphi$ if and only if $P \Vdash \varphi$ for some finite $P \subseteq \Delta[\mathfrak{A}]$.*

Proof. Let A be a countably infinite set of constants. Surely $\mathscr{L}[A]$ is countable. By Theorem 4.1 there exists a T-generic sequence $\mathbb{P}$. Let $\mathfrak{A}$ be canonical term model over $\bigcup \mathbb{P}$ (refs. [5], page 18,19). Obviously $\Delta[\mathfrak{A}] = \bigcup \mathbb{P}$. For the second part, first note that for any finite $P \subseteq \Delta[\mathfrak{A}]$ there exists $P_i \in \mathbb{P}$ such that $P \subseteq P_i$ and for any $P_i \in \mathbb{P}$, $P_i \subseteq \Delta[\mathfrak{A}]$. So it suffices to show that for each sentence φ of $\mathscr{L}[\mathfrak{A}]$, $\mathfrak{A} \models \varphi$ if and only if $P_i \Vdash \varphi$ for some $P_i \in \mathbb{P}$. The proof is by induction. The induction steps on atomic or negated atomic sentence, sentences built up using connectives $\wedge$, $\vee$ and $\exists$ are essentially the same as those in the proof of Lemma 3.7. Now consider the case that $\varphi = \neg\psi$. By the induction hypothesis and the definition of generic sequence, $\mathfrak{A} \models \neg\psi$ if and only if $P_i \nVdash \psi$ for any $P_i \in \mathbb{P}$, and if and only if $P_j \Vdash \neg\psi$ for some $P_j \in \mathbb{P}$. □

The converse of the above theorem is also true. If $\mathfrak{A}$ is a countable T-generic structure, then $\mathbb{P} = \{\text{finite } P : P \subseteq \Delta[\mathfrak{A}]\}$ is a T-generic sequence.

Corollary 4.1. *A countable $\mathscr{L}$-structure $\mathfrak{A}$ is T-generic if and only if there exists a T-generic sequence $\mathbb{P}$ in $\mathscr{L}[\mathfrak{A}]$ such that $\Delta[\mathfrak{A}] = \bigcup \mathbb{P}$.*

Recall that the T^f is the forcing companion of T. For any $\varphi \in T^f$, $\neg\neg\varphi$ is forced by any condition of T. The following is immediate.

Lemma 4.1. *Every T-generic structure is a model of T^f.*

5 Forcing for Model Completions

Definition 5.1 (Model Complete)**.** *Let T be a first-order theory. T is model-complete if any embedding between two models of T is elementary.*

Theorem 5.1 (Robinson's test)**.** *A first-order theory T is model-complete iff every formula is equivalent modulo T to an existential formula.*

Note that completeness and model-completeness are two different properties of a theory. Generally neither one implies the other. For example, the theory $Th(\mathfrak{N})$ of Peano arithmetic $\mathfrak{N} = \langle \mathbb{N}; 0, S, +, \cdot, < \rangle$ is a complete theory. But it is well-known that not all sentences are equivalent to existential sentences modulo $Th(\mathfrak{N})$, so $Th(\mathfrak{N})$ is not model-complete by Robinson's test. One the other hand, the theory of algebraically closed field is model-complete. Actually it admits quantifier elimination, i.e., every formula is equivalent to a quantifier-free formula. But the theory is not complete. In particular, sentences like "I'm of characteristic p" for any prime p are not its logical consequences. However there is close connection between completeness and model-completeness. The following lemma justify the name of "model-completeness". Actually this is the original formulation of model-completeness by Robinson. It follows immediately from Robinson's digram lemma (Lemma 1.1).

Lemma 5.1 (Model Complete)**.** *T is model-complete iff for any model $\mathfrak{A}$ of T, $T \cup \Delta[\mathfrak{A}]$ is a complete theory in $\mathscr{L}[\mathfrak{A}]$.*

Definition 5.2 (Model Completion)**.** *Let T, T^* be two theory of $\mathscr{L}$ such that $T \subseteq T^*$. We say T^* is the model-completion of T if for any model $\mathfrak{A}$ of T, $T^* \cup \Delta[\mathfrak{A}]$ is a complete theory in $\mathscr{L}[\mathfrak{A}]$.*

Lemma 5.2. *Let T be a first-order theory in $\mathscr{L}$.*

(1) If T^* is the model completion of T, then any model of T can be embedded into a model of T^*.

(2) If T^* is the model completion of T, then T^* is model-complete.

(3) If T is model-complete, then T is the model completion of itself.

(4) If T_1^* and T_2^* both are model completion of T, then T_1^* and T_2^* are equivalent.

Proof. (1)-(3) are trivial consequences of Robinson's diagram lemma (Lemma 1.1). (4) is deep, but we will show in the next section a more general result which implies it (see Theorem 6.2). □

The following theorem establishes the connection between T-generic structures and structures of model-complete theories.

Theorem 5.2. *A model $\mathfrak{A}$ of T^f is T-generic if and only if $T^f \cup \Delta[\mathfrak{A}]$ is complete.*

Proof. (⇒) Let φ be a formula in language $\mathscr{L}[\mathfrak{A}]$. Without loss of generality, assume that $\mathfrak{A} \models \varphi$. By definition, there exists $P \subseteq \Delta[\mathfrak{A}]$ such that $P \Vdash \varphi$. Write $P = P(\bar{a})$, $\varphi = \varphi(\bar{a})$ where $\bar{a}$ list all constants not in $\mathscr{L}$. Quoting Lemma 3.10 we know that $\forall \bar{x}(\bigwedge P_i(\bar{x}) \to \varphi(\bar{x}))$ is in T^f. So $T^f \models \bigwedge P(\bar{a}) \to \varphi(\bar{a})$. It follows that $T^f \cup \Delta[\mathfrak{A}] \models \varphi(\bar{a})$ as $\Delta[\mathfrak{A}] \models \bigwedge P_i(\bar{a})$. Hence $T^f \cup \Delta[\mathfrak{A}]$ is complete.
(⇐) Assume that $T^f \cup \Delta[\mathfrak{A}]$ is complete in $\mathscr{L}[\mathfrak{A}]$. To show that $\mathfrak{A}$ is T-generic we shall check the two conditions. The consistency of $\Delta[\mathfrak{A}] \cup T$ is going to be shown in the next section (see Theorem 6.1). To prove the second condition, as before due to Lemma 3.7, it suffices to do the induction on the case that $\varphi = \neg\psi$. Assume that $\mathfrak{A} \models \neg\psi$. By completeness $T^f \cup \Delta[\mathfrak{A}] \models \neg\psi$, and by compactness $T^f \cup P \models \neg\psi$ for some finite $P \subseteq \Delta[\mathfrak{A}]$. We claim that $P \Vdash \neg\psi$. If not, there exists $Q \supseteq P$ such that $Q \Vdash \psi$. By Lemma 3.6

$$T^f \models \forall \bar{x}(\bigwedge Q(\bar{x}) \to \psi(\bar{x}))$$

where $Q(\bar{x})$, $\psi(\bar{x})$ are formed by replacing constants $\bar{a}$ not in $\mathscr{L}$ by variables $\bar{x}$. It is immediate that

$$T^f[Q] \models \forall \bar{x}(\bigwedge Q(\bar{x}) \to \psi(\bar{x}))$$

Since $T^f[Q] \models \bigwedge Q(\bar{a})$, $T^f[Q] \models \psi(\bar{a})$. But $T^f \cup P \subseteq T^f[P] \subseteq T^f[Q]$ and hence $T^f[Q] \models \neg\psi(\bar{a})$. A contradiction. Now assume that $P \Vdash \neg\psi$ for some finite $P \subseteq \Delta[\mathfrak{A}]$. Then for any $Q \supseteq P$, $Q \nVdash \psi$. We claim that there is no $Q' \subseteq \Delta[\mathfrak{A}]$ such that $Q' \Vdash \psi$. For $Q' \cup P \supseteq P$ and $Q' \cup P \Vdash \psi$, a contradiction. So by induction hypothesis $\mathfrak{A} \not\models \psi$ and hence $\mathfrak{A} \models \varphi$. □

Corollary 5.1. *T^f is model-complete if and only if every model of T^f is T-generic.*

Let $\mathbf{K}$ be a class of $\mathscr{L}$-structures. We say $\mathbf{K}$ is *inductive* if $\mathbf{K}$ is closed under union of chains.

Theorem 5.3. *The class of T-generic structures is closed under unions of chains.*

Proof. Let $\langle \mathfrak{A}_\alpha : \alpha < \lambda \rangle$ be an increasing chains of T-generic structures where λ is a limit ordinal. Let $\mathfrak{A} = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$. By Lemma 4.1 every T-generic structure is a model of T^f. By Theorem 5.2 $T^f \cup \Delta[\mathfrak{A}_\alpha]$ is a complete theory in $\mathscr{L}[\mathfrak{A}_\alpha]$. So $\mathfrak{A}_\alpha \preccurlyeq \mathfrak{A}_{\alpha+1}$ ($\mathfrak{A}_\alpha$ is elementary substructure of $\mathfrak{A}_{\alpha+1}$) for $\alpha < \lambda$. By Tarski-Vaught theorem on unions of elementary chains (refs. [5], page 49), $\mathfrak{A}_\alpha \preccurlyeq \mathfrak{A}$ for each $\alpha < \lambda$. Also $\Delta[\mathfrak{A}] \cup T$ is consistent as $\Delta[\mathfrak{A}_\alpha] \cup T$ is consistent for each $\alpha < \lambda$. Let φ be a sentence of $\mathscr{L}[\mathfrak{A}]$. Suppose that $\mathfrak{A} \models \varphi$. Since only finite number of constants $\bar{a}$ of A are in φ, there exists $\alpha < \lambda$ such that $\bar{a} \subseteq \mathfrak{A}_\alpha$. So $\mathfrak{A}_\alpha \models \varphi$ for some $\alpha < \lambda$. By the assumption, there exists $P \subseteq \Delta[\mathfrak{A}_\alpha]$. Clearly $P \subseteq \Delta_[\mathfrak{A}]$ and hence $\mathfrak{A}$ is T-generic. The converse follows from the similar argument. □

6 Forcing for Model Companions

Definition 6.1 (Mutual Model Consistency)**.** *Let T_1, T_2 be first-order theories. T_1 is model-consistent with T_2 if any model of T_2 is consistent with T_1; i.e., any model of T_2 can be embedded into a model of T_1. T_1, T_2 are mutually model-consistent if T_1 is model-consistent with T_2 and T_2 is model-consistent with T_1.*

Lemma 6.1. *T_1 is model-consistent with T_2 if and only if $(T_1)_\forall \subseteq (T_2)_\forall$. T_1, T_2 are mutually model-consistent if and only if $(T_1)_\forall = (T_2)_\forall$.*

Proof. If T_1 is model-consistent with T_2, then models of T_2 can be embedded into a model of T_1. Without loss of generality, we can assume that any model of T_2 is a substructure for some model of T_1. So any substructure of models of T_2 is also a substructure of some model of T_1. By Los-Tarski theorem (refs. [5], page 142-143) for a first-order theory T, the models of $T_\forall$ are precisely the substructures of models of T. Then $(T_2)_\forall \models (T_1)_\forall$ and hence $(T_1)_\forall \subseteq (T_2)_\forall$. Conversely, if $(T_1)_\forall \subseteq (T_2)_\forall$, any substructure of models of T_2 is a substructure of some model of T_1. In particular, any model of T_2 is a substructure of some model of T_1. The second statement then follows immediately. □

Lemma 6.2. *For any theory T, $(T^f)_\forall \subseteq T_\forall$.*

Proof. Let $\varphi \in (T^f)_\forall$. Without loss assume that $\varphi = \neg\psi$ where ψ is an existential sentence. By Lemma 3.5, $\varphi \in T^f$, i.e., $\varnothing \Vdash \varphi$. By Lemma 3.8 $T \models \varphi$ and hence $\varphi \in T_\forall$. □

Lemma 6.3. *For any theory T, $T_{\forall_2} \subseteq T^f$. In particular, $T_\forall \subseteq (T^f)_\forall$.*

Proof. Let $\varphi \in T_{\forall_2}$. Assume $\varphi = \neg\exists\bar{x}\neg\psi(\bar{x})$ where $\psi(\bar{x})$ is an existential formula. Suppose $\varphi \notin T^f$, that is, $\varnothing \nVdash \neg\exists\bar{x}\neg\psi(\bar{x})$. Then there exists condition P and closed terms of $\mathscr{L}[\mathfrak{A}]$ such that $P \Vdash \neg\psi(\bar{t})$. Now $\psi(\bar{t})$ is universal. By Lemma 3.8, $T \cup P \models \neg\psi(\bar{t})$, which contradicts to the assumption that $T \models \forall\bar{x}\psi(\bar{x})$. □

Corollary 6.1. *If T is an inductive theory, i.e., a set of inductive sentences, then $T \subseteq T^f$.*

Theorem 6.1. *T and T^f are mutually model-consistent.*

Proof. By Lemma 6.2, 6.3 and 6.1. □

Corollary 6.2. *Let T_1, T_2 be two first-order theories. T_1 and T_2 are mutually model consistent if and only if $T_1^f = T_2^f$.*

Proof. (⇒) If T_1 and T_2 are mutually model consistent, then $(T_1)_\forall = (T_2)_\forall$. By Lemma 3.9 $T_1^f = ((T_1)_\forall)^f = ((T_2)_\forall)^f = T_2^f$.
(⇐) By Theorem 6.1 $(T_1)_\forall = ((T_1)^f)_\forall = ((T_2)^f)_\forall = (T_2)_\forall$ □

Corollary 6.3. *Let T_1, T_2 be two first-order theories. If T_1 and T_2 are mutually model consistent, then a structure $\mathfrak{A}$ is T_1-generic if and only if it is T_2-generic.*

Proof. By Lemma 3.9 and the assumption that $(T_1)_\forall = (T_2)_\forall$, any condition P for T_1 is also one for T_2 and for any φ in $\mathscr{L}(A)$, $P \Vdash \varphi$ for T_1 if and only if $P \Vdash \varphi$ for T_2. □

Definition 6.2 (Model Companion)**.** *Given a theory T of $\mathscr{L}$, a theory T' of $\mathscr{L}$ is a model companion of T if (i) T and T' are mutually model-consistent and (ii) T' is model-complete.*

By Lemma 5.2(1)-(3), any model completion of T is a model companion of T. For any theory T, its model companion is unique modulo equivalence.

Theorem 6.2. *If T_1 and T_2 both are model companion of T, then T_1 and T_2 are logically equivalent. In particular, if T_1 and T_2 both are model completion of T, then T_1 and T_2 are logically equivalent.*

Proof. By symmetry, it suffices to show that any models of T_1 is a model of T_2. Let $\mathfrak{A}_0$ be a model of T_1. Since T_1 and T_2 are mutually model-consistent, we can iteratively enlarge models to get the following inclusion chain of countable models:

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \dots$$

where $\mathfrak{A}_i$ is model of T_1 if i is even and is model of T_2 otherwise. Since T_1 and T_2 are both model-complete, both

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_2 \subseteq \mathfrak{A}_4 \subseteq \dots \subseteq \mathfrak{A}_{2n} \subseteq \dots$$

and

$$\mathfrak{A}_1 \subseteq \mathfrak{A}_3 \subseteq \mathfrak{A}_5 \subseteq \dots \subseteq \mathfrak{A}_{2n+1} \subseteq \dots$$

are elementary chains. Let $\mathfrak{A}_\omega = \bigcup_{i<\omega} \mathfrak{A}_i$. By Tarski-Vaught theorem, $\mathfrak{A}_\omega$ is an elementary extension of $\mathfrak{A}_i$ for each $i < \omega$. In particular, $\mathfrak{A}_0$ is a model of T_2 as $\mathfrak{A}_\omega$ is. □

Corollary 6.4. *If a theory T has a model companion T', then T' is logically equivalent to its forcing companion T^f.*

Proof. Since T' is model-complete, by Lemma 6.3, T' is equivalent to a set of existential sentences. Certainly, T' is an inductive theory and by Lemma 6.1, $T' \subseteq T^f$. By Lemma 5.2, T^f is model-complete and hence T^f is a model companion of T. Quoting Theorem 6.2 we complete the proof. □

Theorem 6.3. *A theory T is model-complete if and only if every model of T is T-generic.*

Proof. (⇒) If T is model-complete then it is a model companion of itself. So T^f and T are logically equivalent. It follows from Corollary 5.1 that every model of T is T-generic.
(⇐) If every model of T is T-generic, by Lemma 4.1, every model of T is a model of T^f. Then by Theorem 5.2, for every model $\mathfrak{A}$ of T, $T \cup \Delta[\mathfrak{A}]$ is complete in $\mathscr{L}[\mathfrak{A}]$. That is, T is model-complete. □

Corollary 6.5. *Every model-complete theory is forcing-complete.*

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