Dimensions and homogeneity in mathematical structures

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We present here an exposition of ideas and results around one of the central notions of Mathematical Logic, the notion of categoricity. Our purpose is to show how a research in the purely logical subject developed into a discipline of general mathematical significance with applications in the mainstream of mathematics.

One of the most natural and efficient ways to assess the expressive power of a formal language is to study and classify the structures which can be described in full (up to isomorphism) by means of the language. This property of the structure is called categoricity. In this article by language we always mean a first-order language with equality. More specifically we restrict the language by choosing a set of names Σ for relations and operations which have meaning in a structure

$$\mathbf{M} = (M, \Sigma),$$

where M is the universum of the structure. The complete description of a structure $\mathbf{M}$ in the language, $\mathrm{Th}(\mathbf{M})$ the **theory of $\mathbf{M}$**, is just the set of all sentences formulated in the language Σ which are true in $\mathbf{M}$. Any structure satisfying the theory $Th(\mathbf{M})$ is called **a model** of the theory. Here we consider only the case when Σ is finite or at most countable, thus the theory is at most countable too.

It is well known that the first order description of an $\mathbf{M}$ can be (absolutely) categorical iff M is finite, which is a quite trivial situation, thus the importance of a more subtle definition.

Definition A structure $\mathbf{M}$ is said to be categorical in cardinality λ if there is exactly one, up to isomorphism, structure $\mathbf{M}'$ of cardinality λ satisfying $\mathrm{Th}(\mathbf{M})$.

In other words, if we add to $\mathrm{Th}(\mathbf{M})$ the (non first-order) statement that the cardinality of the universum is λ the description becomes categorical.
Of special interest is the case of uncountable cardinality λ. In his pioneering work [Mo] on categoricity M.Morley proved that the categoricity of a theory in one uncountable λ implies the categoricity in all uncountable cardinalities. We have then a large (of an unrestricted cardinality) structure which has a concise (countable) categorical description. It could be a priori conjectured that such structures must be

very special. Indeed, abstract mathematics being really a pure science is based on objects having a concise and clear description. Thus the structures categorical in uncountable cardinalities must be of principal importance in mathematics as whole. We show below how this very general "logical" approach brings applications in various parts of "positive" mathematics.
To understand the methods and ideas of Model Theory, the corresponding branch of Mathematical Logic, we need some exact definitions. It is of principal importance that studying a structure a logician restricts his attention to specific sets that can be constructed in the structure.

Definition A subset $S \subseteq M^n$ in structure $\mathbf{M}$ is said to be a **definable (using parameters from $X \subseteq M$) subset** if there is a first order formula $\varphi(x_1, \dots, x_n)$ in language Σ with $x_1, \dots, x_n$ free variables such that

$$S = \{\langle a_1, \dots, a_n \rangle \in M^n : \varphi(a_1, \dots, a_n) \text{ holds in } \mathbf{M}\}.$$

If $S \subseteq M^n$ is a definable subset and E an equivalence relation on S which is definable too, then the quotient-set S/E is said to be a **definable set in M**.
Suppose sets S and relations $R_1 \subseteq S^{m_1}$, ...$R_k \subseteq S^{m_k}$ all are definable in $\mathbf{M}$. Then the structure $\mathbf{S}$ on the set S given by the relations $R_1, \dots, R_k$ is said to be definable in $\mathbf{M}$.

The basic examples, from which the theory of categoricity started are:
(1) Trivial structures (Σ contains only the equality);
(2) Abelian divisible torsion-free groups; Abelian groups of prime exponent;
(3) Algebraically closed fields.

Given these structures one can construct more complicated ones still having the property of uncountable categoricity or very close to this. The most typical ones are algebraic groups over algebraically closed fields. Let $\mathbf{K}$ be such a field and $\mathbf{H}$ an algebraic matrix group over $\mathbf{K}$, more precisely, the $\mathbf{K}$-points of the group. $\mathbf{H}$ can be considered as a structure in the language of groups, i.e. one based on a binary operation or in a much stronger language witnessing the way the group is coordinatized by the field $\mathbf{K}$. In the first case $\mathbf{H}$ is uncountably categorical provided it is simple, but is not such in general. In the case of the stronger language all algebraic groups are categorical, because $\mathbf{K}$ is. In both cases any algebraic group over $\mathbf{K}$ is *ω-stable of finite Morley rank* in the sense of model theory, i.e. there is a nice dimension theory on $\mathbf{H}$ (see (d1)-(d4) below).
After the work of M.Morley [Mo] twenty years of research showed that two notions, *dimension and homogeneity,* are of principal importance for explaining the fenomenon of uncountable categoricity. These notions in fact are very broad and have many versions and applications which are studied in model theoretical *Stability Theory.* The biggest contribution to stability theory has been made by S.Shelah (see e.g. [Sh]). When applied to concrete class of structures the notion of uncountable categoricity becomes of great interest in the corresponding field of mathematics. The theory of simple uncountably categorical groups has developed into a rich discipline with deep

structural theorems resembling the theory of algebraic groups. It is believed that it sheds light onto the theory of finite simple groups and can help to reshape the latter towards less computational and more geometric proofs in case the following conjecture could be proved:

Conjecture *Any simple uncountably categorical group is isomorphic to an algebraic group over an algebraically closed field.*
The conjecture had been formulated independently by G.Cherlin [Ch] and B.Zilber [Z1] and is unsolved up to now.

In [BL] it was shown that in the study of uncountable categoricity and ω-stability a special type of structures is of principal importance.

Definition A structure $\mathbf{M}$ is said to be **minimal** if any subset definable in the structure using parameters is either finite or co-finite (a complement to a finite set).
$\mathbf{M}$ is said to be **strongly minimal** if any model of $\mathrm{Th}(\mathbf{M})$ is minimal.

The examples (1)-(3) given above are all strongly minimal. This fact easily follows from the analysis of the definable subsets in the structures. For (3), for example, the Tarski-Seidenberg Theorem states that any definable subset of M^n is just an algebraic constructible set, i.e. a Boolean combination of Zariski closed subsets (zero-sets of systems of polynomial equations).
In any structure $\mathbf{M}$ for any finite $X \subseteq M$, as a generalization of the classical notion, it is defined the **algebraic closure of** X :
$y \in \mathrm{acl}(X)$ iff there is a finite subset $Y \subseteq M$ which is definable using parameters in X, and $y \in Y$.

For any $\mathbf{M}$ the following properties of the algebraic closure hold for any finite $X \subseteq M$, $y, z \in M$:
(i) $X \subseteq \mathrm{acl}(X)$;
(ii) $\mathrm{acl}(\mathrm{acl}(X)) = \mathrm{acl}(X)$;
If $\mathbf{M}$ is minimal then the classical exchange principle holds:
(iii) $z \in \mathrm{acl}(X, y)$ and $z \notin \mathrm{acl}(X)$ implies $y \in \mathrm{acl}(X, z)$.

The properties imply that in minimal structures the following definitions are correct

Definition A subset $X' \subseteq X$ is said to be a **basis** of X if X' is a minimal subset of X with the property $X \subseteq \mathrm{acl}(X')$.
The **(combinatorial) dimension** of X (denoted $\mathrm{c.d.}(X)$) is the cardinality of a basis of X.
Set X is said to be **independent** if $\mathrm{c.d.}(X) = \mathrm{card}(X)$.

In examples above c.d.(X) is, correspondingly, card(X) in (1), the linear dimension l.d.(X) over the field of rationals or the finite prime field in (2), and in (3) the transcendense degree tr.d.(X).

In minimal structures of infinite dimensions one can dually define **geometric dimension** of definable subsets $S \subseteq M^n$ as

$$\dim S = \max\{\text{c.d.}(x_1, \ldots, x_n) : \langle x_1, \ldots, x_n\rangle \in S\}.$$

In algebraic geometry (i.e. in example (3)) this is just the definition of the dimension of an algebraic variety S.

Minimal structures $\mathbf{M}$ are **homogeneous** in the following sense:
Any bijection of a basis of $\mathbf{M}$ *onto a basis of* $\mathbf{M}$ *can be extended to an automorphism of the structure.*

Moreover, a stronger property holds *Any bijection of a basis of* $\mathbf{M}$ *onto a basis of* $\mathbf{M}'$, *a model of* Th$(\mathbf{M})$, *can be extended to an isomorphism of the two structures. In particular, since* $\dim M + \aleph_0 = \mathrm{card} M$, *strongly minimal structures are categorical in uncountable cardinalities.*

In fact, if $\dim M$ is infinite for a minimal structure M, then it is strongly minimal.

It follows from the combinatorial geometric properties of a strongly minimal structure $\mathbf{M}$, that the geometric dimension in $\mathbf{M}$ satisfies the following nice properties:
(d1) For any definable S in $\mathbf{M}$ $\dim S = 0$ iff S is finite;
(d2) for any S_1, S_2 definable sets in $\mathbf{M}$ $\dim(S_1 \cup S_2) = \max\{\dim S_1, \dim S_2\}$;
(d3) for any S definable in $\mathbf{M}$ and a number d
$\dim S > d$ iff there are infinitely many disjoint definable subsets $S_1, \ldots S_k \ldots \subseteq S$ with $\dim S_i \geq d$ for all $i = 1, \ldots, k, \ldots$;
(d4) if $f : S \to P$ is a definable map from definable S onto definable P such that for all $p \in P$ $\dim f^{-1}(p) = m$ then $\dim S = \dim P + m$.

In general, if $\mathbf{M}$ is a structure such that for any model $\mathbf{M}'$ of Th$(\mathbf{M})$ there is a dimension notion dim on definable sets in $\mathbf{M}'$ with values in ordinal numbers such that (d1)-(d3) hold, $\mathbf{M}$ is said to be **ω-stable**. If dim takes only natural values, $\mathbf{M}$ is said to be **ω-stable of finite Morley rank.** The condition (d4) does not hold for such structures in general but for uncountably categorical ones all (d1)-(d4) hold and dim is finite.
In an ω-stable structure $\mathbf{M}$ any 1-dimensional definable **irreducible** (not a union of two distinct such sets) set with the relations induced on it from $\mathbf{M}$ is a strongly minimal structure. Geometric properties of the strongly minimal structures effect crucially structural properties of $\mathbf{M}$.

An interesting example of an ω-stable structure of finite Morley rank is the **structure of a compact complex manifold**. This is an arbitrary compact complex manifold M with Σ all the n-ary relations given by analytic subsets $R \subseteq M^n$ for all n. Notice that $\dim S$ in such a structure is not necessarily equal to the analytic dimension of S (is always less or equal). The theorem on stability of the structures follows from classical theorems of Remmert, Grauert and others on complex manifolds and is, in fact, a way to make the statement "compact complex manifolds are nearly algebraic" precise.

Having a closure operator like acl on a set M we can consider a **combinatorial geometry**. On the set $M \setminus \mathrm{acl}(\emptyset)$ the binary relation Eq defined as

$$Eq(x_1, x_2) \equiv [\mathrm{acl}(\{x_1\}) = \mathrm{acl}(\{x_2\})]$$

determines an equivalence relation. Take $G(\mathbf{M})$ to be the quotient of $M \setminus \mathrm{acl}(\emptyset)$ by Eq, transfer acl on $G(\mathbf{M})$ in the natural way, then we get the assumptions (i)-(iii) satisfied on $G(\mathbf{M})$ along with
(iv) $\mathrm{acl}(\{x\}) = \{x\}$ for any x.
A structure satisfying (i)-(iv) is said to be a **combinatorial geometry**. A combinatorial geometry is said to be homogeneous if any bijection between two bases of the geometry can be extended to an isomorphism of the geometry. The subsets of the geometry of the form $\mathrm{acl}(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n$ independent are called $n-1$-spaces (points, for $n = 1$, lines, for $n = 2$, e.c.). $G(\mathbf{M})$, obtained from a minimal structure $\mathbf{M}$ as above, is said to be **the geometry of M**. It follows that the geometry of a minimal structure is homogeneous.
It is one of the basic facts of the theory that
Given an uncountably categorical structure $\mathbf{A}$ *there exists a strongly minimal structure* $\mathbf{M}$ *definable in* $\mathbf{A}$, *which "coordinatizes"* $\mathbf{A}$ *in a certain sense, and any two strongly minimal structures definable in* $\mathbf{A}$ *are linked by a definable finite-to-finite binary relation and henceforth have the isomorphic combinatorial geometries.*
The classification of homogeneous combinatorial geometries is of principal importance for the categoricity and stability theory. On the other hand the relevance of the problem to model theory suggested a certain framework for considering this purely combinatorial problem. In 1981 the following was proved by the author:

Theorem 1 (Weak Trichotomy) *Any strongly minimal structure* $\mathbf{M}$ *satisfies one of the following:*
(i) the geometry of $\mathbf{M}$ *is trivial* ($\mathrm{acl}(X) = X$ *for any subset* X *of the geometry);*
(ii) the geometry of $\mathbf{M}$ *is isomorphic to the geometry of a projective or affine space over a finite or countable division ring (for countable languages);*
(iii) there is a pseudoplane definable in $\mathbf{M}$.

Definition A **pseudoplane** is a two-sorted structure (P, L), P is said to be the set of points and L the set of lines, with a binary relation $I \subseteq P \times L$ ($I(x, y)$ is read as "point x belongs to line y") and satisfying the following:
any line contains infinitely many points;
through any point pass infinitely many lines;

any two distinct points belong to at most finitely many lines;
any two distinct lines intersect in at most finitely many points.

This notion is an obvious generalization of that of a projective or affine plane. Over an algebraically closed field K there are many pseudoplanes with P and L 2-dimensional irreducible algebraic varieties, not necessarily closed, and $I \subseteq P \times L$ a binary Zariski closed, in $P \times L$, relation. Then the lines are actually some algebraic curves on P. This suggests certain ideas in studying and classifying pseudoplanes and first of all the relevance of the general notion to algebraic geometry.
In [Z2] the author made a conjecture based on the a priori arguments discussed above:

Trichotomy Conjecture *Any strongly minimal structure either has a geometry of type (i)-(ii) or is definably equivalent to an algebraically closed field* $\mathbf{K}$.

We discuss the conjecture below. But first we want to demonstrate how the ideas work to get the result of [Z3]

Theorem 2 *Any finite homogeneous combinatorial geometry with more than two points on a line and combinatorial dimension* ≥ 7 *is isomorphic to a projective or affine geometry over a finite field.*

The proof of the theorem on the first stage follows the scheme of the Weak Trichotomy Theorem. Thus we come to study a "pseudoplane over the geometry" which is a finite combinatorial object resembling both classical design and the pseudoplane over a field as above. Using dimensions and homogeneity we construct a numerical invariant which is an analogue of the degree of algebraic curves. Then "Bezout Theorem" argument leads to a contradiction showing that only cases (i)-(ii) of the Trichotomy are possible.

The Trichotomy Conjecture was refuted in general by E.Hrushovski [H1], [H2]. The ingenious construction the counterexamples were obtained by will be discussed below. On the other hand Hrushovski and the author found additional assumptions on a strongly minimal structure $\mathbf{M}$ under which it satisfies the Trichotomy Conjecture. The assumptions for strongly minimal **Zariski structures** are:

There is a topology on M^n for each n such that
(Z0) Let f_i be a projection ($f_i(x_1, \ldots, x_n) = x_{j(i)}$) or a constant map. Let $f(x) = (f_1(x), \ldots, f_m(x))$. Then $f : M^n \to M^m$ is continuous. Also, the diagonals $x_i = x_j$ and all relations from Σ are closed.
(Z1) Let $F \subseteq M^n$ be a closed irreducible (not a union of two closed) subset and $\pi(F)$ the projection of F to M^m. Then $\pi(F) \subseteq \overline{\pi(F)} \setminus F'$, where $\overline{\pi(F)}$ is the closure of $\pi(F)$ and F' a proper closed subset of $\overline{\pi(F)}$.
(Z2) For F as above and $a \in M^{n-1}$ denote $F(a) = \{b \in M : a\hat{\ }b \in F\}$. Then for some k, for all $a \in M^{n-1}$, $F(a) = M$ or $|F(a)| \leq k$.

(Z3) Let F be a closed irreducible subset of M^n, and let T_{ij} be the diagonal $x_i = x_j$. Then every irreducible component of $F \cap T_{ij}$ has dimension greater than or equal to $\dim(F) - 1$.
It follows from (Z0)-(Z2) rather easily that $\mathbf{M}$ is strongly minimal, moreover the definable subsets of M^n are of restricted complexity with respect to closed sets: they all are Boolean combinations of closed subsets (elimination of quantifiers with respect to closed sets). Also, (Z2) implies that the topology is Noetherian. Nevertheless this is not enough for the proof. (Z3) states, in fact, the property characterizing the smoothness of real or complex manifolds, which is a form of homogeneity for such manifolds.
All "natural" strongly minimal structures are Zariski.

It is interesting to note that

Theorem 3 (Trichotomy for Zariski structures) *Any strongly minimal Zariski structure* $\mathbf{M}$ *satisfies (i) or (ii) of the Trichotomy, or there are an algebraically closed field* $\mathbf{K}$ *definable in* $\mathbf{M}$, *an algebraic smooth curve* C *over* $\mathbf{K}$ *and a surjection* $p : M \to C$ *such that* $|p^{-1}(c)| \leq m$ *for some* m *and any* $c \in C$, *and* p *takes any closed subset of* M^n *onto a Zariski closed subset of* C^n *while pullbacks of Zariski closed subsets of* C^n *are closed in* M^n.

One of the direct applications of the theorem is for compact complex manifolds $\mathbf{M}$. It is not hard to see that any minimal infinite analytic subset $S \subseteq M^n$ (i.e. one that contains no proper infinite analytic) is strongly minimal and after removing finitely many singular points satisfies Zariski assumptions. Thus it satisfies the Trichotomy. In case the analytic dimension of S is 1, this is given by the Riemann Theorem which identifies S as an algebraic curve over $\mathbf{C}$. The Trichotomy states that there are only (i) or (ii) if the alternative holds. Generic complex tori of analytic dimension greater than 1 are examples of (ii). Very recently A.Pillay and T.Scanlon exhibited some examples of compact complex manifolds satisfying (i).

Very important and beautiful applications of the theorem were found by Hrushovski. The applications involve certain, in fact classical, expansions of the languge of fields.

Let $\mathbf{F}$ be a field with an additional operation D called **derivation**. The structure in the language $(+, \cdot, D)$ is said to be a **differential field** if for any $x, y \in F$, $D(x + y) = Dx + Dy$ and $D(xy) = xDy + yDx$. A.Robinson showed that by adding solutions of any system of differential-algebraic equations and negations of such equations, which has a solution in an extension of the field, one comes to a differential field $\tilde{\mathbf{F}}$ which is **differentially closed**, i.e. if a system of differential-algebraic equations and negations of such equations, has a solution in some larger field then the system has a solution in $\tilde{\mathbf{F}}$. (In fact, the Robinson construction is applicable to very general classes of structures, and in general the resulting structures are said to be **existentially closed**. A typical effect of the construction is that the structures are the most homogeneous in the class). Differentially closed fields of characteristic 0 are model-theoretically very nice, they are ω-stable. Characteristic $p > 0$ case is somewhat harder but still analyzable.

On the other hand in the differentially closed fields some very interesting structures are definable. It was shown by Yu.Manin that *the Mordell-Lang Conjecture for function fields* is expressible in terms of differential fields. The conjecture concerns the number of K-rational points on algebraic curves X of genus greater than 1 for K a finite extension of $\mathbf{Q}$ (the number field case) or K a finite extension of $\mathbf{C}(t)$ (the function field case). It is well known that such a curve X can be embedded into an Abelian variety A (with a group structure on it, the Jacobian of X) such that A has no proper algebraic subgroup containing X. The Mordell-Lang Conjecture generalizes the statement that *if A has no proper connected algebraic subgroup then for any finitely generated subgroup Γ and algebraic subvariety X of A the intersection $\Gamma \cap X$ is finite.* In case Γ is the group of K-rational points of A (which is known to be finitely generated in main cases) the intersection contains the K-rational points of X. Manin showed that for K a function field a group $\tilde{\Gamma}$, slightly bigger than Γ, can be represented as the set of solutions of a certain differential equation. In other words $\tilde{\Gamma}$ is definable in the language of differential fields.
Hrushovski noticed that once one knows that $\tilde{\Gamma}$ in the language of differentially closed fields has strongly minimal subsets of type (ii) only, then the finiteness of $\tilde{\Gamma} \cap X$ follows from the known model-theoretical classification of such structures. The hypothesis would be satisfied if strongly minimal structures in differentially closed fields satisfy the Trichotomy Conjecture. This was shown to be true in [HS] by proving that all the structures are Zariski.
Though proofs of the Mordell-Lang Conjecture both in function and in number fields cases for characteristic 0 have been found before, the model-theoretical approach is of much interest. It gave the complete solution in positive characteristic and, what is more important, brought new geometric view of the problem.

Analogous effective application of the Trichotomy Theorem gives a proof [H4] of *Manin-Mumford Conjecture* which states the finiteness of $\Gamma \cap X$ for Γ the torsion subgroup of A. This time Hrushovski uses **difference fields** which are structures of type $(F, +, \cdot, \sigma)$ with $(F, +, \cdot)$ a field and σ an automorphism of the field. Using the Robinson construction one comes again to **existentially closed difference fields** the theory of which turns out to be model-theoretically nice, though not stable (L.van den Dries, A.Macintyre, C.Wood, E.Hrushovski, Z.Chatzidakis). Hrushovski noticed that like above Manin-Mumford Conjecture is expressible in the theory and is true provided $\tilde{\Gamma}$ is of type (ii). As a matter of fact Trichotomy holds for strongly minimal subsets of existentially closed difference fields and it leads to the final result.

A comprehensive survey of the links between model theory and Mordell-Lang and Manin-Mumford conjectures is given in [P].

The last two examples expose a method that Model Theory can give to the general mathematics: to solve a problem in a specific field of mathematics try to find an "analyzable" structure in the language of which the problem is expressible. Model Theory has powerful criteria of identifying "analyzable" structures: stability and its specifications are part of them. Also, Model Theory has developed methods of constructing various types of "analyzable" structures.

On the other hand model-theoretically nice structures are rather rare on mathematical landscape, as the Trichotomy suggests.
The next issue we discuss here is, as we suppose, a powerful source of "analyzable" structures deeply connected with classical mathematics.
First of all we want to make an a priori conjecture that structures of analytic origin could be model-theoretically nice. There is the classical dimension theory for such structures, and in a certain sense a form of homogeneity, smoothness. The Zariski structures context also points in this direction.
On the other hand the stability notions might turn out to be too narrow in analytic context. E.g. one of the most interesting analytic structures $(\mathbf{C}, +, \cdot, \exp)$ is obviously unstable since the arithmetic is definable in it. But we can generalize the notion of strong minimality for this purpose.

Definition An uncountable structure $\mathbf{M}$ is said to be **quasi-minimal** if any definable subset of M is either countable or a complement to a countable.
It is not hard to see that for a quasi-minimal homogeneous structure one has by analogy the correct notions of combinatorial and geometric dimensions.

Conjecture. $(\mathbf{C}, +, \cdot, \exp)$ is quasi-minimal and homogeneous.

This goal was pursued by the author with no essential success until recently a deep connection between this context and the Hrushovski construction of counterexamples to the Trichotomy Conjecture has been found.

One of the most interesting examples of "anti-algebraic" structures obtained by Hrushovski's construction is the set carrying two field structures, with no definable isomorphism between the fields, in such a way that together this is a strongly minimal structure [H3]. The two fields can be of different characteristics, if one likes. Such a situation is impossible in algebraic geometry over algebraically closed field since any two field structures definable in an object of algebraic geometry are birationally isomorphic. The basic idea of Hrushovski was that in order to fuse two structures in one preserving stability one should ensure that the fusion is as free as possible. This principle can be made more precise.

Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the two field structures of the same cardinality which we want to put on one set. We can do that by constructing a map $f : F_1 \to F_2$ which identifies elements of the two fields. The Hrushovski construction of f assumes that for any distinct $x_1, \ldots, x_n \in F_1$

$$\text{tr.d.}(x_1, \ldots, x_n) + \text{tr.d.}(f(x_1), \ldots, f(x_n)) - n \geq 0.$$

In other words, the more $x_1, \ldots, x_n$ are dependent in $\mathbf{F}_1$ the less $f(x_1), \ldots, f(x_n)$ are dependent in $\mathbf{F}_2$. The kind of freeness given by the Hrushovski inequality is the maximal possible in the presence of a good dimension theory.
Hrushovski considers the category of two-sorted structures $(\mathbf{F}_1, \mathbf{F}_2, f)$ satisfying the inequality with appropriate notions of embedding and extension of structures of the class. Correspondingly, existentially closed structures of the class turn out to be ω-stable. This is not the final stage if we want to have a strongly minimal structure on each of the sorts. But for our purposes we can stop here.
The construction is applicable in more general situations, $\mathbf{F}_1$ and $\mathbf{F}_2$ may be any strongly minimal structures and f need not be a bijection. For example, we may try to apply the construction for two fields and f being an homomorphism of the additive group of $\mathbf{F}_1$ onto the multiplicative group of $\mathbf{F}_2$, in this case we have to assume $\mathbf{F}_1$ is of characteristic 0. The Hrushovski inequality then takes the form

$$\text{tr.d.}(x_1, \ldots, x_n) + \text{tr.d.}(f(x_1), \ldots, f(x_n)) - \text{l.d.}(x_1, \ldots, x_n) \geq 0$$

where l.d. is the linear dimension over $\mathbf{Q}$. Assume $\mathbf{F}_1$ and $\mathbf{F}_2$ are copies of complex numbers. Then for exp in place of f the inequality holds provided the Schanuel Conjecture holds. I.e. the model-theoretical inequality in this case is a special case of Schanuel Conjecture which states that for any complex $x_1, \ldots, x_n$

$$\text{tr.d.}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) - \text{l.d.}(x_1, \ldots, x_n) \geq 0$$

(usually it is stated equivalently under the assumption that $x_1, \ldots, x_n$ are linearly independent over $\mathbf{Q}$). Schanuel-type conjectures are known for other classical transcendental functions. If true the corresponding inequalities seem to be a common feature of classical analytic structures. Thus the Hrushovski counterexamples rather than being mathematical pathologies show that model-theoretical stability ideology might be applicable in understanding classical analytic structures.
We undertook an attempt to make systematic use of the approach in the study of structure $(\mathbf{C}, +, \cdot, \exp)$ and its **reducts**, i.e. structures on the same universum in a weaker language. The best results so far are obtained for structures of the form

$$\mathbf{C}_r(K) = (\mathbf{C}, +, \cdot, x^r)_{r \in K}$$

where K is a countable subfield of $\mathbf{C}$ and by x^r we mean the transcendental multi-valued operations "raising to a complex power r" given as

$$y = x^r \text{ iff } \exists t \exp(t) = x \,\&\, \exp(r \cdot t) = y.$$

By definition the language of the structure contains the above binary relations '$y = x^r$' for each $r \in K$.
Going along the lines of the Hrushovski construction described above we construct an abstract analogue of $\mathbf{C}_r$

Theorem 4 *For any countable field K of characteristic 0 and $\mathbf{F}$ an algebraically closed field of characteristic 0 and of infinite transcendence degree there is a structure $\mathbf{F}_r(K)$ of the form*

$$(F, +, \cdot, x^r)_{r \in K}$$

with predicates of the form 'y = x^r' satisfying abstract algebraic properties of raising to powers. The structure satisfies the corresponding Schanuel Conjecture. The structure is quasi-minimal, homogeneous and existentially closed. The correctness of the notion of combinatorial dimension in the structure directly follows from the Schanuel Conjecture. The geometric dimension in the structure is a good analogue of the classical dimension of analytic sets. For any finitely generated subgroup Γ of $(\mathbf{F}^*)^n$ there is a structure of the form $\mathbf{F}_r(K)$ on $\mathbf{F}$ such that Γ is definable in the structure. I.e. the Mordell-Lang Conjecture both for number and function fields in case $A = (\mathbf{F}^*)^n$ is expressible in the structure.

The theorem gives only some restricted understanding of the structure. The structure is much better "analyzable" provided the followng holds

Conjecture Let $W \subseteq \mathbf{C}^n$ be an irreducible algebraic variety. Then there is finite family $\tau(W)$ of proper tori (algebraic subvariety of $\mathbf{C}^n$ given by multiplicative identities) in $\mathbf{C}^n$ such that for any torus $T \subseteq \mathbf{C}^n$ given an irreducible component $S \subseteq W \cap T$ with $\dim S > \dim W + \dim T - n$ there is $Tor \in \tau(W)$ such that $S \subseteq Tor$.

Notice that for $\dim T = 0$ the conjecture just states that the torsion subgroup of $(\mathbf{C}^*)^n$ intersects with any proper subvariety $W \subseteq \mathbf{C}^n$ in a finite set, which is a special case of the Manin-Mumford Conjecture.

Theorem 5 *If the conjecture on intersection with tori is true, $\mathbf{F}_r(K)$ is always superstable. The definable subsets in the structure are of restricted complexity (the quantifier-elemination to the level of $\exists$-formulas holds).*

Theorem 6 *Assume the conjecture on intersection with tori and the Schanuel conjecture are true. Then $\mathbf{C}_r(K)$ satisfies both the theorems above iff it is existentially closed. This is the case when $K \subseteq \mathbf{R}$.*

The last statement of the theorem involves a "Bezout Theorem" for systems of **quasi-algebraic** equations, i.e. ones of the form

$$\sum_{k_1,\dots k_n} a_{k_1,\dots k_n} x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$$

for $a_{k_1,\dots k_n} \in \mathbf{C}$ and $k_1, \dots, k_n \in K \subseteq \mathbf{C}$. The "non-degenerate" case of the theorem for real powers is considered in [Kh, Chapter 6]. We have managed to prove the general case for real powers under the conjectures above. The case of arbitrary complex powers is much less understood but from what was said above about $\mathbf{F}_r(K)$ we can conjecture that a corresponding "Bezout Theorem" holds for all quasi-algebraic systems.

As was mentioned above the ideology of the last three theorems is applicable for other classical transcendental functions: exponentiation, elliptic, theta, Abelian and maybe some others. In our present understanding we then have to consider corresponding

Schanuel-type conjectures along with the same conjecture on intersection with tori.

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