MAF cvičení, 8.12.2020, 9:00

Limity funkcí v nevl. bodech

$\lim\limits_{x\to+\infty} f(x) = a$ znamená:

$\forall \varepsilon>0\ \exists \delta>0\ \forall x \in \overset{(U)}{P}(+\infty,\delta)$:

$f(x) \in U(a,\varepsilon)$ $\quad = \left(\frac{1}{\delta}, +\infty\right)$

zahrnuje i případy $a = \pm\infty$

Nekonečné limity: stále platí aritmetika limit včetně

$+\infty + \infty = +\infty$, $+\infty + a = +\infty$

$(+\infty)\cdot a = \begin{cases} +\infty & (a>0) \\ -\infty & (a<0) \end{cases}$, $\frac{a}{\infty} = 0$

Nedefinováno: $\infty - \infty$, $\infty \cdot 0$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\frac{a}{0}$, $\frac{\infty}{0}$

Cv. 8

① $\lim\limits_{x\to\infty} \frac{a_n x^n + \dots + a_1 x + a_0}{A_m x^m + \dots + A_1 x + A_0} =$ $\quad (a_n \neq 0,\ A_m \neq 0)$

$$= \lim_{x\to\infty} \frac{x^n \left(a_n + \overbrace{\frac{a_{n-1}}{x} + \dots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}}^{\to 0}\right)}{x^m\left(A_m + \underbrace{\frac{A_{m-1}}{x} + \dots + \frac{A_1}{x^{m-1}} + \frac{A_0}{x^m}}_{\to 0}\right)} =$$

$$= \begin{cases} 0 & \text{pro } m>n \\ \pm\infty & \text{pro } m<n \quad \left(\operatorname{sgn} = \operatorname{sgn} \frac{a_n}{A_m}\right) \\ \frac{a_n}{A_m} & \text{pro } m=n \end{cases}$$

② $\lim\limits_{x\to\infty} \frac{2x^2+1}{\sqrt{3x^4-6x^2+5}} =$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left(2 + \underbrace{\frac{1}{x^2}}_{\to 0}\right)}{\cancel{\sqrt{x^4}}\cdot\sqrt{3 \underbrace{- \frac{6}{x^2} + \frac{5}{x^4}}_{\to 0}}} = \frac{2}{\sqrt{3}}$$

(3) $$\lim_{x\to\infty} x\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right) = \lim_{x\to\infty} \frac{x\cdot\left(x^2+1-(x^2-1)\right)}{\sqrt{x^2+1}+\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{2x}{\sqrt{x^2}\left(\sqrt{1+\underset{\downarrow 0}{\frac{1}{x^2}}}+\sqrt{1-\underset{\downarrow 0}{\frac{1}{x^2}}}\right)}$$

$$= \lim_{x\to\infty} \frac{2x}{x\cdot 2} = \underline{\underline{1}}$$

$x>0$

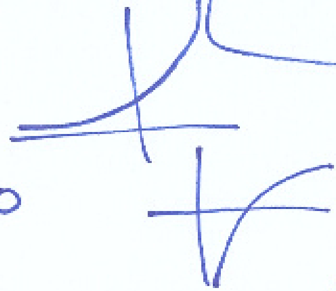
(4) $$\lim_{x\to\infty} x^{\frac{4}{3}}\left(\sqrt[3]{x^2+1}-\sqrt[3]{x^2-1}\right)$$

$$a^3-b^3=(a-b)\cdot(a^2+ab+b^2)$$

$$= \lim_{x\to\infty} \frac{x^{\frac{4}{3}}\cdot\left((x^2+1)-(x^2-1)\right)}{(x^2+1)^{\frac{2}{3}}+(x^4-1)^{\frac{1}{3}}+(x^2-1)^{\frac{2}{3}}} = \lim \frac{\cancel{x^{\frac{4}{3}}}\cdot 2}{\cancel{x^{\frac{4}{3}}}\underbrace{\left(\left(1+\underset{\downarrow 0}{\frac{1}{x^2}}\right)^{\frac{2}{3}}+\left(1-\underset{\downarrow 0}{\frac{1}{x^4}}\right)^{\frac{1}{3}}+\left(1-\underset{\downarrow 0}{\frac{1}{x^2}}\right)^{\frac{2}{3}}\right)}_{\to 3}} = \underline{\underline{\frac{2}{3}}}$$

$$\lim_{x\to+\infty} e^x = +\infty, \quad \lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to+\infty} \ln x = +\infty, \quad \lim_{x\to 0+} \ln x = -\infty$$



L'Hospitalovo pravidlo

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Předpoklady L'Hosp.:

(A) buď $f(x)\to 0$, $g(x)\to 0$ pro $x\to a$ nebo $g(x)\xrightarrow[x\to a]{}\pm\infty$, & (B) existuje limita vpravo

5, 6, 7, 8, 9

(5) : 2

(6) : $\frac{1}{6}$

(7) : $\frac{1}{2}$

(8) : $e^0 = 1$

(9) : $\frac{1}{e}$

(7) $$\lim_{x\to 0} \frac{1-\cos x^2}{x^2\cdot \sin x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{\cancel{2x}\cdot \sin x^{2} \to 0}{\cancel{2x}\cdot \underbrace{\sin x^2}_{\to 0} + x^2\cdot \cancel{2x}\cdot \cos x^2} \overset{L'H}{=}$$

čit. i jm. $\to 0$

$$= \lim_{x\to 0} \frac{\cancel{2x}\cdot \cos x^{2} \to 1}{\cancel{2x}\cdot \underbrace{\cos x^2}_{\to 1} + \cancel{2x}\cdot \underbrace{\cos x^2}_{\to 1} + x^2\cdot \cancel{2x}\cdot \underbrace{(-\sin x^2)}_{\to 0}} =$$

$$= \lim_{x\to 0} \frac{1}{1+1+\underbrace{\underbrace{x^2}_{\to 0}\cdot \underbrace{\frac{-\sin x^2}{\cos x^2}}_{\text{omez.}}}_{\to 0}} = \frac{1}{2}$$

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nebo: $$\lim_{x\to 0} \underbrace{\frac{1-\cos x^2}{x^4}}_{\to \frac{1}{2}} \cdot \underbrace{\frac{x^2}{\sin x^2}}_{\to 1} = \frac{1}{2}$$

(8) $$\lim_{x\to 0+} x^x = \lim_{x\to 0} e^{x\cdot \ln x} = e^{\lim_{x\to 0+} x\cdot \ln x} = e^0 = 1$$

$$\lim_{x\to 0+} x\cdot \ln x = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x} \to +\infty} \overset{L'H}{=} \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to 0+} -\frac{x^2}{x} = \lim_{x\to 0+}(-x) = 0$$

(5) $\lim\limits_{x\to 0} \dfrac{\overset{\to 0}{\operatorname{tg} x - x}}{\underset{\to 0}{x - \sin x}} \overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{\cos^2 x} - 1}{1 - \cos x} = \lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{\cos^2 x (1-\cos x)} =$

$= \lim\limits_{x\to 0} \dfrac{(1+\cos x)\cancel{(1-\cos x)}}{\cos^2 x \cdot \cancel{(1-\cos x)}} = \underline{\underline{2}}$

$\overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{+2\cdot\cos^{-3}x\cdot\cancel{\sin x}}{\cancel{\sin x}} = \underline{\underline{2}}$

Symboly $O, o, \sim$

Def: $f(x) = o(g(x))$ pro $x \to a$: $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 0$ $\quad (\exists C, \exists \delta)$

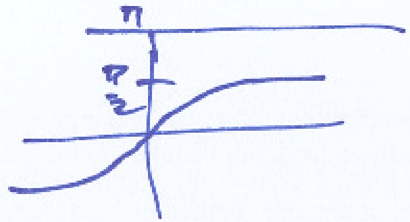
$f(x) = O(g(x))$ — " — : $|f(x)| \le C \cdot |g(x)|$ pro nějaké $C > 0$ a pro $\forall x \in P(a,\delta)$, nějaké δ

Platí: • $f = o(g) \Rightarrow f = O(g)$

• $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = B \in \mathbb{R} \Rightarrow f = O(g)$ (třeba pro $C = 2B$)

$f(x) \sim g(x)$ pro $x \to a$: $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 1$ (fce jsou asymptoticky ekvivalentní)

(10) Dokažte: $\operatorname{arctg} x = O(1), \ x \to \infty$



stačí vzít $C := \pi$

víme $\lim\limits_{x\to\infty} \operatorname{arctg} x = \frac{\pi}{2}$

$\Rightarrow |\operatorname{arctg} x| \le \pi$ pro $\forall x \in \mathbb{R}$

(11) $x^2 e^{-x} = o(x^a), \ x \to \infty$, pro $\forall a < 0$

$$\lim_{x\to\infty} \frac{x^2 e^{-x}}{x^a} = \lim_{x\to\infty} \frac{\overbrace{e^{-x}}^{\to 0}}{\underbrace{x^{a-2}}_{\to 0}} =$$

pro $\forall a<0$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-\overbrace{e^{-x}}^{\to 0}}{(a-2)\cdot \underbrace{x^{a-3}}_{\to 0}} \to \text{takhle ne}$$

Lépe: $$\lim_{x\to\infty} \frac{x^{2-a}}{\underbrace{e^x}_{\to +\infty}} \overset{L'H}{=} \lim_{x\to\infty} \frac{(2-a)x^{1-a}}{e^x} \overset{L'H}{=} \underbrace{\ldots}_{k\text{-krát}} = \lim_{x\to\infty} \frac{\text{const.}\cdot \overbrace{x^{2-a-k}}^{\to 0}}{e^x} = 0$$

$2-a-k<0$, $k \in \mathbb{N}$

(14) Najděte $a \in \mathbb{R}$, aby

$$\frac{1+x}{1+x^4} \sim x^a, \quad x \to \infty$$

Hádáme: $a = -3$

$$\lim_{x\to\infty} \frac{x^a}{\frac{1+x}{1+x^4}} = \lim_{x\to\infty} \frac{x^a(1+x^4)}{1+x} =$$

$$= \lim_{x\to\infty} \frac{x^{4+a}\overbrace{\left(1+\frac{1}{x^4}\right)}^{\to 1}}{x\underbrace{\left(1+\frac{1}{x}\right)}_{\to 1}} =$$

$$= \lim_{x\to\infty} x^{3+a} = 1 \Leftrightarrow a+3=0$$

$a = -3$

Sami doma: 12, 13, 15

Limity posloupností

posloupnost: $\{a_n\}_{n=1}^{\infty}$

$\lim_{n\to\infty} a_n = a$: $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$

$\forall n \geq N : a_n \in U(a, \varepsilon)$

Heineho věta:

$$\lim_{x\to+\infty} f(x) = \lim_{n\to\infty} f(n)$$

+ monotonie

$\Rightarrow$ často lze $\lim_{n\to\infty} a_n$ počítat jako $\lim_{x\to+\infty} f(x)$, pokud lze takovou $f(x)$ vytvořit

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

(2) $\lim_{n\to\infty} \frac{a^n}{n!}$, $(a \in \mathbb{R})$

n-tý člen: $\frac{a}{1} \cdot \frac{a}{2} \cdot \ldots \cdot \frac{a}{n}$

vidíme: pro $n > |a|$ je $\left|\frac{a}{n}\right| < 1$

$n_0 := [a+1]$

$$\Rightarrow |a_n| = \underbrace{\left|\frac{a}{1}\right| \cdot \left|\frac{a}{2}\right| \cdots \left|\frac{a}{n_0}\right|}_{\text{const.}} \cdot \underbrace{\left|\frac{a}{n_0+1}\right| \cdots \left|\frac{a}{n}\right|}_{\text{klesá k } 0} \to \underline{\underline{0}}$$

$(n \geq n_0)$

(4) $\lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} \right)$

$$= \lim_{n\to\infty} \left(1 \underbrace{- \frac{1}{2} + \frac{1}{2}}_{0} \underbrace{- \frac{1}{3} +}_{0} \ldots + \frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n+1} \right) = \underline{\underline{1}}$$

(3) $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{x\to\infty} \sqrt[x]{x} = \lim_{x\to\infty} e^{\frac{1}{x}\cdot \ln x}$

$\ldots = 1$

lim sup, lim inf

↓ největší ↓ nejmenší

<u>hromadný bod</u>

= limita podposloupnosti:

Př: $a_n = (-1)^n$: $-1, 1, -1, 1, \dots$

má 2 hrom. body $-1, 1$

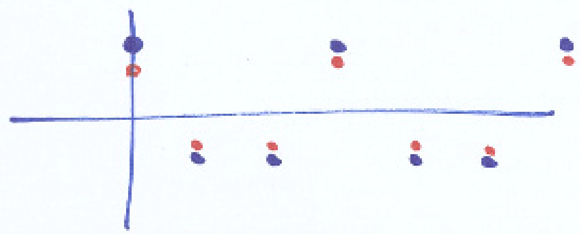
$-1 \to$ = lim inf.

$1 \to$ lim sup

(8) $a_n = \dfrac{n-1}{n+1} \cos \dfrac{2\pi n}{3}$, $\quad \dfrac{n-1}{n+1} \to 1$

n	$\frac{2\pi n}{3}$	$\cos \frac{2\pi n}{3}$
0	0	1
1	$\frac{2\pi}{3}$	$-\frac{1}{2}$
2	$\frac{4\pi}{3}$	$-\frac{1}{2}$
3	2π	1

$\longrightarrow$ má hrom. body $1, -\frac{1}{2}$



červené = a_n

a_n má hr. body $1 = \limsup a_n$

$-\frac{1}{2} = \liminf a_n$