MAF cvičení, 8.12.2020, 12:20

Limity fcí v nevlastních bodech

$\lim_{x\to\infty} f(x) = a$ znamená:

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \overset{U}{P}(+\infty, \delta)$:

$f(x) \in U(a, \varepsilon)$ $\quad (\frac{1}{\delta}, \infty)$

zahrnuje i případy $a = \pm\infty$

Nekonečné (nevlastní) limity:
stále platí aritmetika limit
včetně: $+\infty + \infty = +\infty$, $+\infty + a = +\infty$

$$(+\infty)\cdot a = \begin{cases} +\infty & (a>0) \\ -\infty & (a<0) \end{cases}, \quad \frac{a}{\pm\infty} = 0$$

Nedefinováno: $\infty - \infty$, $\infty \cdot 0$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\frac{a}{0}$, $\frac{\infty}{0}$

1

Cv. 8

① $$\lim_{x\to\infty} \frac{a_n x^n + \dots + a_1 x + a_0}{A_m x^m + \dots + A_1 x + A_0} = \quad \begin{pmatrix} a_n \neq 0 \\ A_m \neq 0 \end{pmatrix}$$

$$= \lim_{x\to\infty} \frac{x^n\left(a_n + \overbrace{\frac{a_{n-1}}{x} + \dots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}}^{\to 0}\right)}{x^m\left(A_m + \underbrace{\frac{A_{m-1}}{x} + \dots + \frac{A_1}{x^{m-1}} + \frac{A_0}{x^m}}_{\to 0}\right)} =$$

$$= \begin{cases} 0 & \text{pro } m > n \\ \pm\infty & \text{pro } m < n \quad (\operatorname{sgn} = \operatorname{sgn} \frac{a_n}{A_m}) \\ \frac{a_n}{A_m} & \text{pro } m = n \end{cases}$$

② $$\lim_{x\to\infty} \frac{2x^2+1}{\sqrt{3x^4 - 6x^2 + 5}} =$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left(2 + \frac{1}{x^2}\right)}{\cancel{\sqrt{x^4}}\cdot\sqrt{3 - \frac{6}{x^2} + \frac{5}{x^4}}} = \frac{2}{\sqrt{3}}$$

$\frac{1}{x^2} \to 0$, $\frac{6}{x^2} \to 0$, $\frac{5}{x^4} \to 0$

③ $$\lim_{x\to\infty} x\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right) = \lim_{x\to\infty} \frac{x\left((x^2+1)-(x^2-1)\right)}{\sqrt{x^2+1}+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{2x}{|x|\cdot\left(\sqrt{1+\underbrace{\tfrac{1}{x^2}}_{\to 0}}+\sqrt{1-\underbrace{\tfrac{1}{x^2}}_{\to 0}}\right)} =$$

$$= \lim_{x\to\infty}\frac{2x}{x\cdot 2} = \underline{\underline{1}}$$

$|x| = x$
protože $x>0$

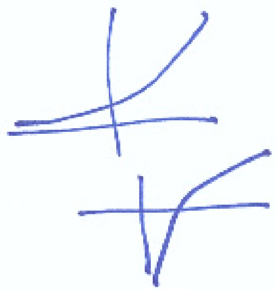
④ $$\lim_{x\to\infty} x^{\frac{4}{3}}\left(\sqrt[3]{x^2+1}-\sqrt[3]{x^2-1}\right) =$$

$a^3-b^3=(a-b)\cdot(a^2+ab+b^2)$

$$= \lim_{x\to\infty}\frac{x^{\frac{4}{3}}\cdot\left((x^2+1)-(x^2-1)\right)}{(x^2+1)^{\frac{2}{3}}+(x^2+1)^{\frac{1}{3}}\cdot(x^2-1)^{\frac{1}{3}}+(x^2-1)^{\frac{2}{3}}} = \lim_{x\to\infty}\frac{\cancel{x^{\frac{4}{3}}}\cdot 2}{\cancel{x^{\frac{4}{3}}}\left(\underbrace{\left(1+\tfrac{1}{x^2}\right)^{\frac{2}{3}}}_{\to 1}+\underbrace{(1+\dots)^{\frac{1}{3}}(1-\dots)^{\frac{1}{3}}}_{\to 1}+\underbrace{(1-\dots)^{\frac{2}{3}}}_{\to 1}\right)} = \underline{\underline{\frac{2}{3}}}$$

$$\lim_{x\to+\infty} e^x = +\infty, \quad \lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to+\infty} \ln x = +\infty, \quad \lim_{x\to 0+} \ln x = -\infty$$



L'Hospitalovo pravidlo

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Předpoklady:

Ⓐ buď $f(x)\to 0$, $g(x)\to 0$ pro $x\to a$ nebo $g(x)\to\pm\infty$ pro $x\to a$ & Ⓑ existuje limita vpravo

Cv. 8: 5, 6, 7, 8, 9

⑤: 2

⑥: $\frac{1}{6}$

⑦: $\frac{1}{2}$

⑧: 1

⑨: $\frac{1}{e}$

⑤ $$\lim_{x\to 0} \frac{\overbrace{tg\,x - x}^{\to 0}}{\underbrace{x - \sin x}_{\to 0}} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{\cos^2 x} - 1}{1-\cos x} = \lim_{x\to 0} \frac{1-\cos^2 x}{(1-\cos x)\cdot\cos^2 x} =$$

$$= \lim_{x\to 0} \frac{\cancel{(1-\cos x)}(1+\cos x)}{\cancel{(1-\cos x)}\cos^2 x} = \frac{2}{1} = \underline{\underline{2}}$$

$$L'H: \lim_{x\to 0} \frac{+2\cos^{-3}x\cdot\cancel{\sin x}}{\cancel{\sin x}} = \underline{\underline{2}}$$

⑥ $$\lim_{x\to 0} \frac{x(e^x+1)-2(e^x-1)}{x^3} \overset{L'H}{=} \lim_{x\to 0} \frac{1\cdot(e^x+1)+x\cdot e^x - 2e^x}{3x^2} =$$

$$= \lim_{x\to 0} \frac{x\cdot e^x - e^x + 1}{3x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{(x+1)e^x - e^x}{6x} = \lim_{x\to 0} \frac{\cancel{x}e^x}{6\cancel{x}} = \underline{\underline{\frac{1}{6}}}$$

⑦ $$\lim_{x\to 0} \frac{\overbrace{1-\cos x^2}^{\to 0}}{\underbrace{x^2\cdot\sin x^2}_{\to 0}} \overset{L'H}{=} \lim_{x\to 0} \frac{\cancel{2x}\cdot\overbrace{\sin x^2}^{\to 0}}{\cancel{2x}\cdot\underbrace{\sin x^2}_{\to 0} + \underbrace{x^2}_{\to 0}\cdot\cancel{2x}\cdot\cos x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{\cancel{2x}\cdot\cos x^2}{\cancel{2x}\cdot\cos x^2 + \cancel{2x}\cdot\cos x^2 + x^2\cdot\cancel{2x}\cdot(-\sin x^2)} =$$

$$= \frac{1}{1+1+0} = \underline{\underline{\frac{1}{2}}}$$

Jinak: $$\lim_{x\to 0} \underbrace{\frac{1-\cos x^2}{x^4}}_{\frac{1}{2}} \cdot \underbrace{\frac{x^2}{\sin x^2}}_{1} = \underline{\underline{\frac{1}{2}}}$$

(8) $\lim\limits_{x\to 0+} x^x = \lim\limits_{x\to 0+} e^{x\cdot \ln x} = e^{\lim\limits_{x\to 0+} x\cdot\ln x} = e^0 = \underline{\underline{1}}$

$\Uparrow$

$$\left[\lim_{x\to 0+} x\cdot \ln x = \lim_{x\to 0+} \frac{\ln x}{\underbrace{\frac{1}{x}}_{\to +\infty}} \overset{L'H}{=} \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} \frac{-x^2}{x} = \lim_{x\to 0+}(-x) = 0\right]$$

$\cancel{\lim\limits_{x\to 1} \frac{\ln x}{x-1}}$

Symboly o, O, $\sim$

Def: $f(x) = o(g(x)),\ x\to a \iff \lim\limits_{x\to a} \frac{f(x)}{g(x)} = 0$

$f(x) = O(g(x)),\ x\to a \iff |f(x)| \le C\cdot|g(x)|$ pro nějaké $C>0$ a pro $\forall x \in P(a,\delta)$ pro nějaké $\delta>0$

Platí:
- $f = o(g) \Rightarrow f = O(g)$
- $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = B \in \mathbb{R} \Rightarrow f = O(g)$ (třeba pro $C = 2B \neq 0$)

$f(x) \sim g(x),\ x\to a \iff \lim\limits_{x\to a} \frac{f(x)}{g(x)} = 1$ (fce jsou asymptoticky ekvivalentní)

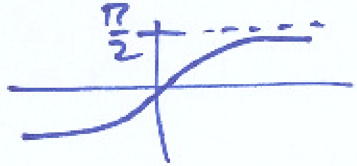
$\asymp$

Cv. 8 (10) dokažte, že

$$\operatorname{arctg} x = O(1), \quad x \to \infty$$

víme: $\lim\limits_{x\to\infty} \frac{\operatorname{arctg} x}{1} = \frac{\pi}{2}$

$\Rightarrow$ stačí brát $C = \frac{\pi}{2}$



(11) $x^2 e^{-x} = o(x^a), \ x \to \infty, \ \forall a<0$

$$\lim_{x\to\infty} \frac{x^2 e^{-x}}{x^a} \left(\overset{?}{=} 0\right) = \lim_{x\to\infty} \frac{\overbrace{x^{2-a}}^{\to\infty}}{\underbrace{e^x}_{\to\infty}} = $$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{(2-a)\cdot x^{1-a}}{e^x} \overset{L'H}{=} \dots = \lim_{x\to\infty} \frac{\text{const.}\ \overbrace{x^{2-a-k}}^{\to 0}}{e^x} = \underline{\underline{0}}$$

L'H k-krát

$k \in \mathbb{N}: 2-a-k<0$

Sami doma:

12, 13, 15

(14) Najděte $a \in \mathbb{R}$, aby

$$\frac{1+x}{1+x^4} \sim x^a, \quad x \to \infty$$

Tipy: -4, -3 ✓,

$$\lim_{x\to\infty} \frac{x^a}{\frac{1+x}{1+x^4}} = \lim_{x\to\infty} \frac{x^a(1+x^4)}{1+x} \ (*) =$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{a x^{a-1} + (4+a) x^{3+a}}{1} =$$

$$= \lim_{x\to\infty} x^{a+3}\Big((4+a) + \underbrace{\frac{a}{x^4}}_{\to 0}\Big)$$

$$= 1 \iff \underline{\underline{a=-3}}$$

jinak: $(*) = \lim\limits_{x\to\infty} \frac{x^{a+4}\left(1+\frac{1}{x^4}\right)}{x\left(1+\frac{1}{x}\right)} =$ ($\to 1$)

$$= \lim_{x\to\infty} x^{a+3} = 1 \iff \underline{\underline{a=-3}}$$

Limity posloupností:

posloupnost: $\{a_n\}_{n=1}^{\infty}$

$\lim_{n\to\infty} a_n = a$: $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$

$\forall n \in \mathbb{N}, n \geq N : a_n \in U(a, \varepsilon)$

Heineho věta:

$$\lim_{x\to+\infty} f(x) = \lim_{n\to\infty} f(n)$$

+ monotonie

$\Rightarrow$ mnoho limit posl. lze počítat jako limity fcí

Cv. 9 (2) $\lim_{n\to\infty} \frac{a^n}{n!}$ $(a \in \mathbb{R})$

n-tý člen: $a_n = \frac{a}{1} \cdot \frac{a}{2} \cdot \frac{a}{3} \cdot \ldots \cdot \frac{a}{n}$

stačí najít n_0, že $\frac{|a|}{n_0} \leq \frac{1}{2}$

$n_0 = [2|a|+1] \Rightarrow$

$n \geq n_0$ $\quad a_n = \frac{a^{n_0}}{n_0!} \cdot \underbrace{\underbrace{\frac{a}{n_0+1}}_{\leq \frac{1}{2}} \cdot \underbrace{\frac{a}{n_0+2}}_{\leq \frac{1}{2}} \cdot \ldots \cdot \underbrace{\frac{a}{n}}_{\leq \frac{1}{2}}}_{\left(\frac{1}{2}\right)^{n-n_0} \to 0}$

$\Rightarrow \lim_{n\to\infty} \frac{a^n}{n!} = 0$

(4) $\lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n\cdot(n+1)} \right) = \lim_{n\to\infty} \left(\frac{1}{1} \underbrace{- \frac{1}{2} + \frac{1}{2}}_{0} \underbrace{- \frac{1}{3} + \ldots + \frac{1}{n}}_{0} - \frac{1}{n+1} \right)$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ $\quad = \lim_{n\to\infty} \left(1 - \underbrace{\frac{1}{n+1}}_{\to 0} \right) = \underline{\underline{1}}$

6) $\boxed{\text{Sami: } 3}$