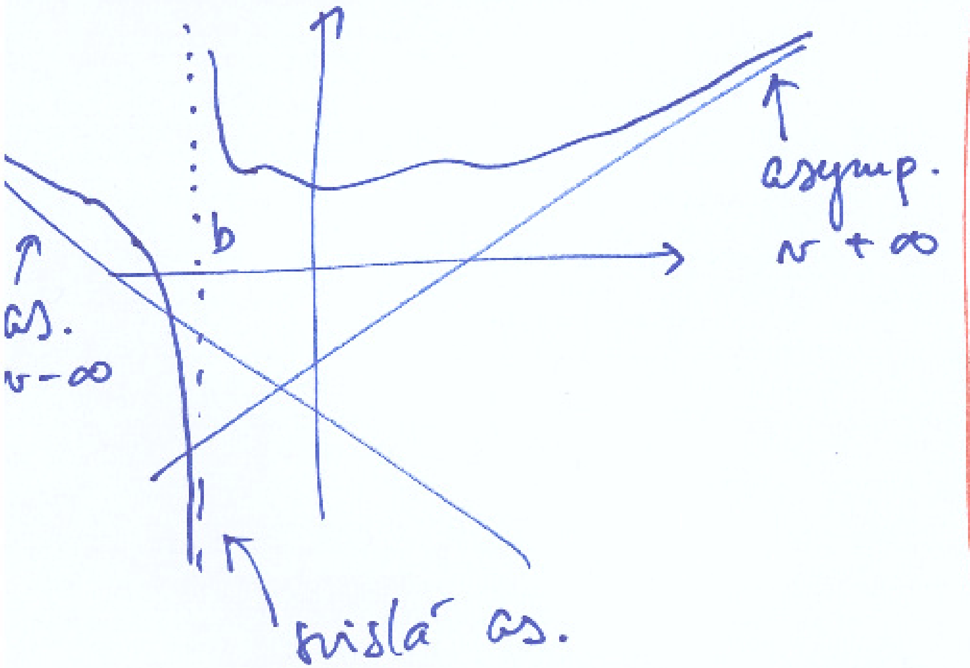
MAF – cvičení, 22.12.2020, 9^{00}

K průběhům fcí: asymptoty



svislá: $\lim\limits_{x \to b\pm} f(x) = \pm\infty$

as. v $+\infty$: je přímka $y = kx + q$

$$\equiv \lim_{x \to +\infty} \big(f(x) - (kx+q) \big) = 0$$

Postup: 1) $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = k \in \mathbb{R}$

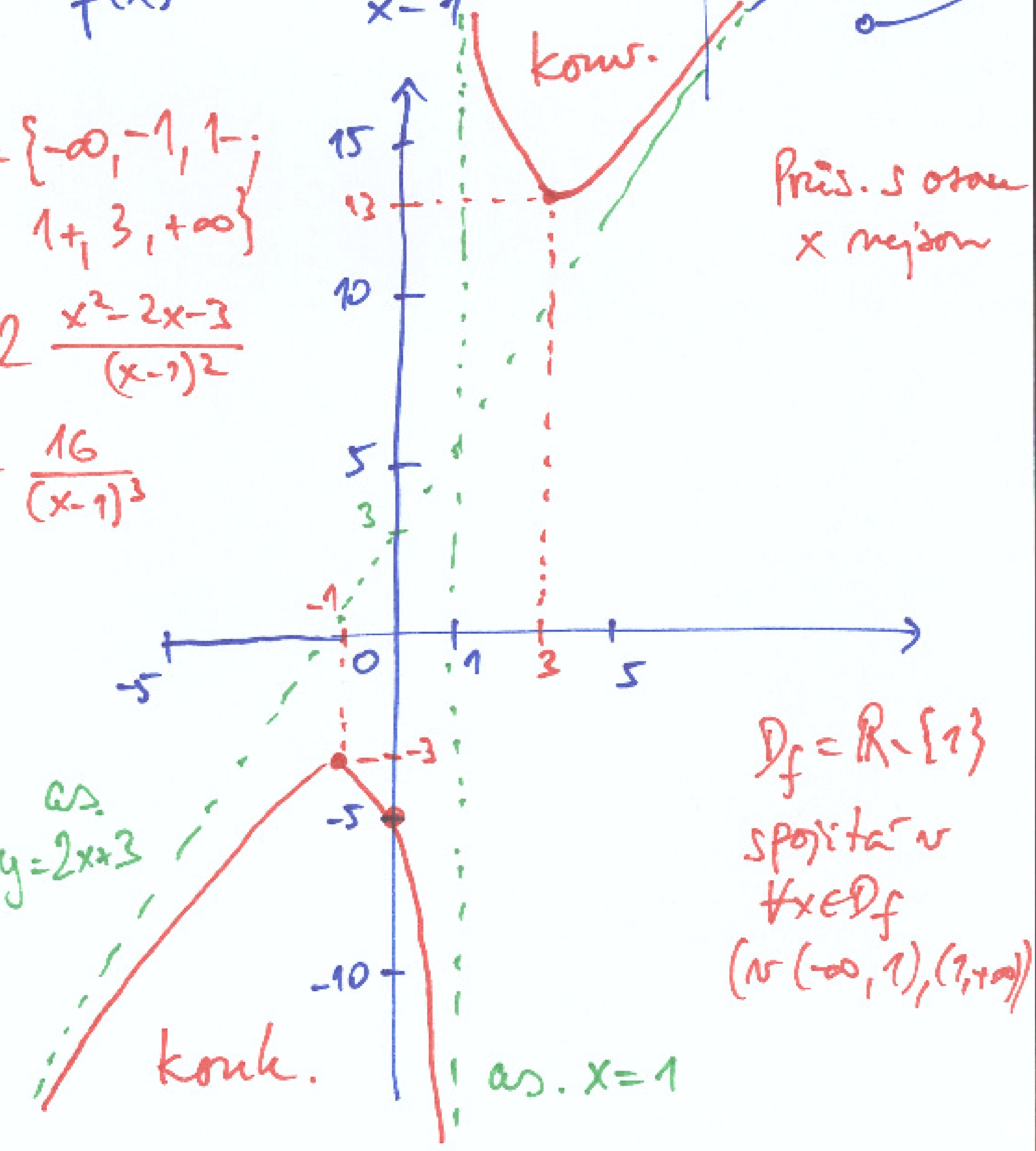
2) $\lim\limits_{x \to +\infty} \big(f(x) - kx \big) = q \in \mathbb{R}$

$$f(x) = \frac{2x^2 + x + 5}{x - 1}$$

$V_f = \{-\infty, -1, 1-, 1+, 3, +\infty\}$

$f'(x) = 2\,\dfrac{x^2 - 2x - 3}{(x-1)^2}$

$f''(x) = \dfrac{16}{(x-1)^3}$



$D_f = \mathbb{R} \setminus \{1\}$

spojitá v $\forall x \in D_f$

(v $(-\infty, 1), (1, +\infty)$)

Taylorovy polynomy

polynomiální aproximace fcí

dána $f(x)$, dán bod x_0, stupeň $n \in \mathbb{N}$

$$\to T^n_{f,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^k$$

Platí: $f(x) = T^n_{f,x_0}(x) + \underbrace{}_{o((x-x_0)^n)} \quad (x \to x_0)$

$\boxed{x_0 = 0}$

$e^x: \quad 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + o(x^n)$

$\sin x: \quad x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$ $\quad 2n+2$

$\cos x: \quad 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n})$ $\quad 2n+1$

$\sinh x: \quad x + \frac{x^3}{6} + \frac{x^5}{120} + \ldots$

$\cosh x: \quad 1 + \frac{x^2}{2} + \frac{x^4}{24} + \ldots$

$\ln(1+x) \overset{=f(x)}{=} ?$ Spočítejme:

$k=0: \quad \frac{f(0)}{0!} \cdot 1 = \frac{\ln 1}{1} = 0$

$k=1: \quad f'(x) = (\ln(1+x))' = \frac{1}{1+x}$

$f'(0) = \frac{1}{1+0} = 1$

$\Rightarrow \frac{1}{1!} x = x$

$k=2: \quad f''(x) = \frac{-1}{(1+x)^2}, \; f''(0) = -1$

$\Rightarrow \frac{-1}{2} x^2$

$k=3: \quad f'''(x) = \frac{2}{(1+x)^3}, \; f'''(0) = 2$

$\Rightarrow \frac{2}{3 \cdot 2 \cdot 1} x^3 = \frac{1}{3} x^3$

$k=4: \quad f^{(4)} = \frac{-3 \cdot 2}{(1+x)^4}, \; f^{(4)}(0) = -3 \cdot 2$

$\Rightarrow \frac{-3 \cdot 2}{4 \cdot 3 \cdot 2 \cdot 1} \Rightarrow -\frac{1}{4} x^4$

$\Rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1} \frac{x^n}{n} + o(x^n)$,

7)

$(1+x)^a: \quad 1 + ax + \binom{a}{2}x^2 + \binom{a}{3}x^3 + \dots + \binom{a}{n}x^n + o(x^n)$

$(a \in \mathbb{R})$

kde definujeme pro $a \in \mathbb{R}$: $\binom{a}{k} = \dfrac{a\cdot(a-1)\cdot\ldots\cdot(a-k+1)}{k!}$
$(k \in \mathbb{N}_0)$

$\left(\text{pro } a = n \in \mathbb{N}: \quad \binom{a}{k} = \binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}\right)$

Odvození dalších Tayl. polynomů (pro danou fci a stupeň, obvykle $x_0=0$)

Pozn.: f lichá/sudá $\Leftrightarrow$ její T.p. obsahuje jen liché/sudé mocniny x

__Př__: T.p. pro $\operatorname{tg} x$ stupně 5:

$\operatorname{tg} x = \dfrac{\sin x}{\cos x}$, $\quad \operatorname{tg} x = ax + bx^3 + cx^5 + o(x^5)$

$\Rightarrow \sin x = \operatorname{tg} x \cdot \cos x$

$\left(x - \dfrac{x^3}{6} + \dfrac{x^5}{120} + o(x^5)\right) = \left(ax + bx^3 + cx^5 + o(x^5)\right)\cdot\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} + o(x^5)\right)$

$x - \dfrac{x^3}{6} + \dfrac{x^5}{120} = ax + \left(b - \dfrac{a}{2}\right)x^3 + \left(\dfrac{a}{24} - \dfrac{b}{2} + c\right)x^5 + o(x^5)$

$ax \cdot o(x^5) = a\cdot o(x^6) = o(x^6)$

ale $x^6 = o(x^5)$ $(x \to 0)$

$\Rightarrow \underline{a = 1}$

$b - \dfrac{1}{2} = -\dfrac{1}{6} \Rightarrow b = \dfrac{1}{3}$

$\Rightarrow c = \dfrac{1}{6} - \dfrac{1}{24} + \dfrac{1}{120} = \dfrac{2}{15}$

$$\boxed{\operatorname{tg} x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^5)}$$

Př: T.p. $\cos(\sin x)$, 5. stupně v 0:

$$\cos(\sin x) = \cos\left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right) =$$

$$= 1 - \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)^2 + \frac{1}{24}\left(x + o(x^3)\right)^4 =$$

$$= 1 - \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^5)$$

Př: T.p. $\arcsin x$, 5. stupně v 0:

1) z definice, 2) přes inverzní fci: $\arcsin(\sin y) = y$ $\quad \left(y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$\arcsin x$: lichá

u x^1: $\arcsin' = \frac{1}{\sqrt{1-x^2}}$

$\arcsin'(0) = \underline{\underline{1}}$

$\Rightarrow \arcsin = x + ax^3 + bx^5 + o(x^5)$

$\Rightarrow \underline{\underline{y}} = \arcsin(\sin y) = \sin y + a\cdot(\sin y)^3 + b(\sin y)^5 + o((\sin y)^5)$

$$= \left(y - \frac{y^3}{6} + \frac{y^5}{120} + o(y^5)\right) + a\left(y - \frac{y^3}{6} + o(y^3)\right)^3 + b\left(y + o(y)\right)^5 =$$

$$= y - \frac{y^3}{6} + \frac{y^5}{120} + a\left(y^3 - 3y^2\cdot\frac{y^3}{6}\right) + by^5 + o(y^5)$$

$$= y - \left(a - \frac{1}{6}\right)y^3 + \left(\frac{1}{120} - \frac{a}{2} + b\right)y^5 + o(y^5)$$

$a - \frac{1}{6} = 0$, $\underline{\underline{a = \frac{1}{6}}}$

$\frac{1}{120} - \frac{1}{12} + b = 0$

$\Rightarrow \underline{b = \frac{3}{40}}$

$\Rightarrow x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + o(x^5)$

Př: T.p. st. 5 (v 0) fce $f(x) = \sin(2\sin x) - 2\sin x$

a spočtěte $\lim\limits_{x\to 0} \frac{f(x)}{x^3}$

$$\sin(2\sin x) - 2\sin x = \sin\left(2x - \frac{x^3}{3} + \frac{x^5}{60}\right) - 2x + \frac{x^3}{3} - \frac{x^5}{60} + o(x^5) =$$

$$= \left(2x - \frac{x^3}{3} + \frac{x^5}{60}\right) - \frac{1}{6}\left(2x - \frac{x^3}{3}\right)^3 + \frac{1}{120}(2x)^5 - 2x + \frac{x^3}{3} - \frac{x^5}{60} + o(x^5) =$$

$$= -\frac{4}{3}x^3 + \frac{14}{15}x^5 + o(x^5) \qquad \text{(dopočítat sami!)}$$

Pak $\lim\limits_{x\to 0} \frac{f(x)}{x} = \lim\limits_{x\to 0} \frac{-\frac{4}{3}\cancel{x^3} + \frac{14}{15}\cancel{x^5}^{\,2} + o(\cancel{x^5}^{\,x^2})}{\cancel{x^3}} = \boxed{-\frac{4}{3}}$

DÚ-5: $\lim\limits_{x\to 0} \frac{g(x)}{\operatorname{arctg}^3 x} = \lim\limits_{x\to 0} \frac{g(x)}{x^3} \cdot \underbrace{\frac{x^3}{\operatorname{arctg}^3 x}}_{= \left(\frac{x}{\operatorname{arctg} x}\right)^3}$