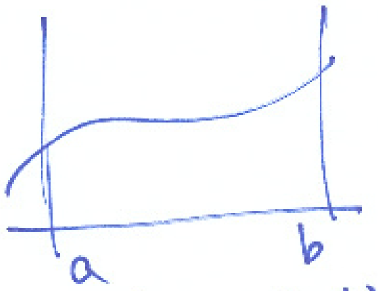
Cvičení MAF, 20.10.2020, 9:00

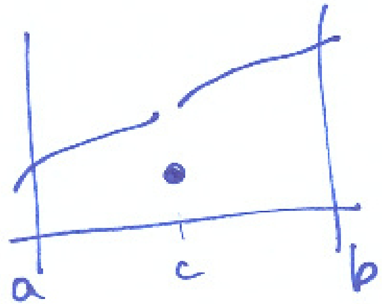
Limity funkcí

① f spojitá v okolí bodu x_0

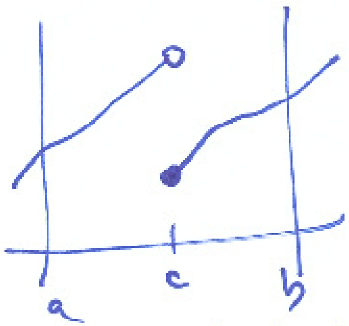
$$\Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$$



spojitá v (a,b)



c = bod nespojitosti



c bod nespoj.

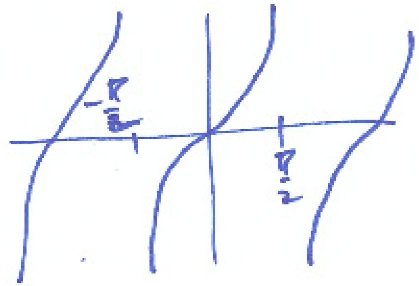
V praxi: víme, že většina používaných fcí je spojitá v $\forall$ intervalu obsaženém v D_f

Př: polynomy spojité v $\mathbb{R}$

$\sqrt[k]{x}$ spoj. v celém D_f $\left\langle \begin{array}{l} \mathbb{R} \\ \langle 0,+\infty) \end{array} \right.$

sin, cos spoj. v $\mathbb{R}$

tg je spoj. v $\forall$ int. tvaru $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k \in \mathbb{Z}$



Př: $\lim_{x \to 1} (x+1) =$ polynom $\Downarrow$ spojitá v $\mathbb{R}$ $\Downarrow$ dosadíme

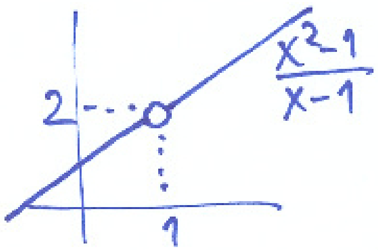
$$= 1+1 = \underline{\underline{2}}$$

(2) pokud $f(x) = g(x)$ pro $\forall x \in P(x_0)$

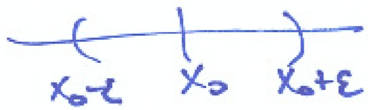
$$\Rightarrow \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$$

Př: $$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} =$$

$$= \lim_{x \to 1} (x+1) = \underline{\underline{2}}$$



$$U_\varepsilon(x_0) = (x_0-\varepsilon, x_0+\varepsilon)$$



okolí bodu x_0

$$P_\varepsilon(x_0) = U_\varepsilon(x_0) \setminus \{x_0\}$$

$$= (x_0-\varepsilon, x_0) \cup (x_0, x_0+\varepsilon)$$

(3) Aritmetika limit

(Věta o limitě $+, -, \cdot, :$)

$$\lim_{x \to x_0} (f(x)+g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

$$-$$

$$\cdot$$

$$:$$

(pokud je operace vpravo definována)

Př: 2a, 2b, 3 z Cv3

$$\underline{\underline{1}} \qquad \underline{\underline{\frac{2}{3}}} \qquad \underline{\underline{-\frac{3}{8}}}$$

③ $$\lim_{x\to 2}\left(\frac{1}{x^2-2x}-\frac{x}{x^2-4}\right)=\lim_{x\to 2}\left(\frac{1}{x(x-2)}-\frac{x}{(x-2)(x+2)}\right)=$$

$$=\lim_{x\to 2}\frac{x+2-x^2}{x(x-2)(x+2)}=\lim_{x\to 2}\frac{\cancel{(x-2)}(-x-1)}{x\cancel{(x-2)}(x+2)}=\underline{\underline{\frac{-3}{8}}}$$

dělení polynomů

$$(-x^2+x+2):(x-2)=-x-1$$
$$\underline{-(-x^2+2x)}$$
$$-x+2$$

Př 4, 5, 6, 7, 8, 9

④ indukcí $\lim_{x\to 0}\frac{(1+x)(1+2x)\dots(1+nx)-1}{x}$ $(n\in\mathbb{N})$

1. ind. krok: $n=1$: $\lim_{x\to 0}\frac{1+x-1}{x}=\lim_{x\to 0}\frac{x}{x}=\lim_{x\to 0}1=\underline{\underline{1}}$

bez indukce: $(1+2+\dots+n)$ $\quad x\cdot(\dots)\to 0$ $\quad x^2(\dots)\to 0$ $\quad\dots\to 0$

$$\lim_{x\to 0}\frac{\cancel{1}+(\cancel{x}+2\cancel{x}+\dots+n\cancel{x})+x^2(\dots)+x^3(\dots)+\dots+x^n n!\cancel{-1}}{\cancel{x}}=1+2+\dots+n=\underline{\underline{\frac{n(n+1)}{2}}}$$

(5) $$\lim_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1}$$

~~$(x^{50}-1)^2 = x^{100}-2\cdot x^{50}+1$~~

musíme čit. i jm. vydělit $(x-1)$

Trik: $(x^n-1) = (x-1)(x^{n-1}+x^{n-2}+\ldots+x^2+x+1)$

$$= \lim_{x\to 1} \frac{x^{100}-1-2x+2}{x^{50}-1-2x+2} = \lim_{x\to 1} \frac{\left[(x^{99}+x^{98}+\ldots+x^2+x+1)-2\right]\cancel{(x-1)}}{\left[(x^{49}+x^{48}+\ldots+x^2+x+1)-2\right]\cancel{(x-1)}} = \frac{100-2}{50-2}$$

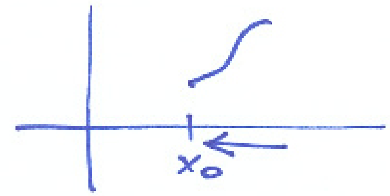
$$= \underline{\underline{\frac{49}{24}}}$$

Př (10.) $$\lim_{x\to 0} \frac{\frac{2}{x^2}+1}{\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}} \cdot \frac{x^2}{x^2} = \lim_{x\to 0} \frac{2+x^2}{\sqrt{3-6x^2+5x^4}} \underset{\substack{\text{dosadí}\\ x=0}}{=} \underline{\underline{\frac{2}{\sqrt{3}}}}$$

(11.) $$\lim_{x\to 0+} \frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{\frac{1}{x^2}-1}}{x} \cdot \frac{x}{x} = \lim_{x\to 0+} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2} = \%$$

limita zprava:

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in \underline{(x_0, x_0+\delta)} : f(x) \in (a-\varepsilon, a+\varepsilon)$$



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11. – pokrač.

$$(a-b)(a+b) = a^2 - b^2$$

$$= \lim_{x \to 0+} \frac{\left(\sqrt{1+x^2} - \sqrt{1-x^2}\right) \cdot \left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)}{x^2 \cdot \left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)} = \lim_{x \to 0+} \frac{1+x^2 - (1-x^2)}{x^2 \left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)} =$$

$$= \lim_{x \to 0+} \frac{2\cancel{x^2}}{\cancel{x^2}\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)} = \underline{\underline{1}}$$

jinak:

$$\lim_{x \to 0+} \frac{\left(\sqrt{\frac{1}{x^2}+1} - \sqrt{\frac{1}{x^2}-1}\right)\left(\sqrt{\frac{1}{x^2}+1} + \sqrt{\frac{1}{x^2}-1}\right)}{x \left(\sqrt{\frac{1}{x^2}+1} + \sqrt{\frac{1}{x^2}-1}\right)} = \lim \frac{\frac{1}{x^2}+1-\left(\frac{1}{x^2}-1\right)}{x\left(\sqrt{\ } + \sqrt{\ }\right)} =$$

$$= \ldots = 1$$

! $\sqrt{x^2} = x$ pouze pro $x \geq 0$, obecně $\sqrt{x^2} = |x|$

Sami: totéž pro $x \to 0-$

Př. 13a, 13b, 14

$\downarrow$ $\frac{1}{4}$, $\downarrow$ $\frac{1}{2}$, $\downarrow$ 0

13b
$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \lim_{x\to 0}\frac{\cancel{x}+\cancel{1}-\cancel{1}}{\cancel{x}(\sqrt{x+1}+1)} = \frac{1}{2}$$

15
$$\lim_{x\to 0}\frac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\sqrt[3]{x^4}} = \lim_{x\to 0}\frac{\left(\sqrt[3]{27+x}-\sqrt[3]{27-x}\right)\left(\sqrt[3]{27+x}^2+\sqrt[3]{27+x}\sqrt[3]{27-x}+\sqrt[3]{27-x}^2\right)}{\left(x+2\sqrt[3]{x^4}\right)\left(\text{———}\ {''}\ \text{———}\right)}$$

$$\boxed{a^3-b^3=(a-b)(a^2+ab+b^2)}$$

$$=\lim_{x\to 0}\frac{(\cancel{27}+x)-(\cancel{27}-x)}{\left(x+2\sqrt[3]{x^4}\right)}\cdot\frac{1}{\sqrt[3]{27}^2+\sqrt[3]{27}^2+\sqrt[3]{27}^2} =$$

$$=\lim_{x\to 0}\frac{2\cancel{x}}{\cancel{x}\underbrace{(1+2\underset{\downarrow\,0}{x^{\frac{1}{3}}})}_{\to 1}}\cdot\frac{1}{27} = \frac{2}{27}$$

Výsledky

6 $\frac{mn(n-m)}{2}$ 12 1

7 $\frac{n(n+1)}{2}$

8 $\frac{n(n+1)}{2}$

6 9 $\frac{m-n}{2}$