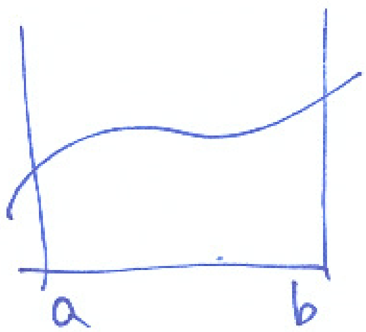
Cvičení MAF, 20.10.2020, 12:20

Limity funkcí

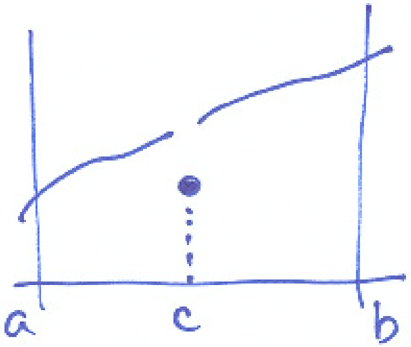
① f spojitá v bodě x_0

$$\Leftrightarrow \forall \varepsilon>0\ \exists \delta>0 : \forall x \in U_\delta(x_0) : f(x) \in U_\varepsilon(f(x_0))$$

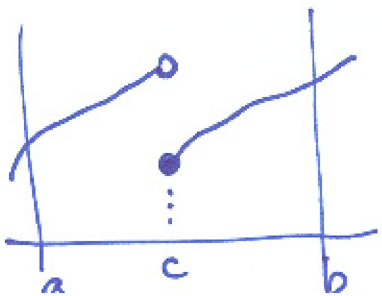
$$\Leftrightarrow \lim_{x\to x_0} f(x) = f(x_0)$$

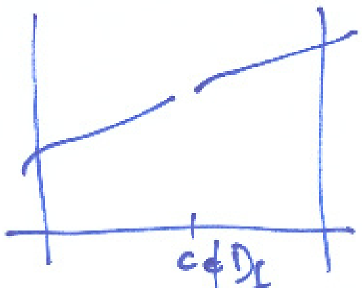


spojitá v (a,b)



c = bod nespojitosti





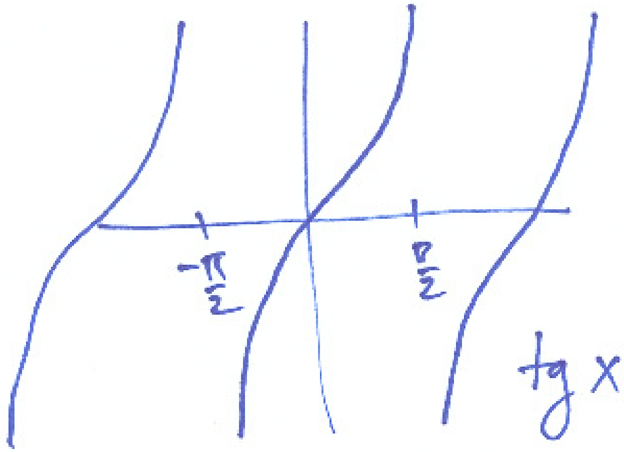
V praxi: víme, že většina používaných fcí je spojitá v $\forall$ intervalu obsaženém v D_f

Př: polynomy – spoj. v $\mathbb{R}$

$\sqrt[k]{x}$: spoj. v D_f $<$ $\mathbb{R}$ / $\langle 0,+\infty)$

sin, cos, exp, log v D_f

tg je spoj. v $\forall$ int. $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$ $\forall k\in\mathbb{Z}$



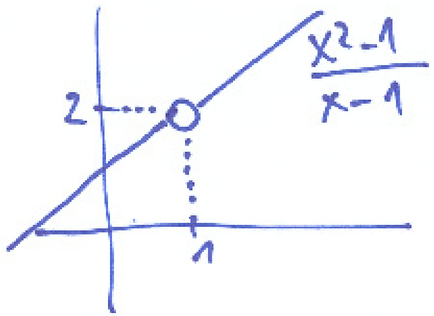
Př: $\lim\limits_{x\to 1}(x+1) = 1+1 = \underline{\underline{2}}$

polynom
$\Downarrow$
spojitá v $\mathbb{R}$
$\Downarrow$
dosadíme x_0 do f

② pokud $f(x) = g(x)$ pro $\forall x \in P(x_0)$

$$\Rightarrow \lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$$

Př: $$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x\to 1}(x+1) = \underline{\underline{2}}$$



③ Aritmetika limit

(Věta o limitě $+, -, \cdot, :$)

$$\lim_{x\to x_0}\left(f(x)+g(x)\right) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$$

$-$ $\qquad$ $-$

$\cdot$ $\qquad$ $\cdot$

$:$ $\qquad$ $:$

(pokud je operace vpravo definovaná)

Př: 2a, 2b, 3

$\underline{\underline{1}}$ $\qquad$ $\underline{\underline{\frac{2}{3}}}$ $\qquad$ $\underline{\underline{-\frac{3}{8}}}$

dosadíme

(2b) $$\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1} = \lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(2x+1)} = \underline{\underline{\frac{2}{3}}}$$

dělení polynomů:

$$(2x^2-x-1):(x-1) = 2x+1$$
$$\frac{-(2x^2-2x)}{x-1}$$

Př: 4, 5, 6, 7, 8, 9

④ $$\lim_{x\to 0} \frac{(1+x)(1+2x)\dots(1+nx)-1}{x} = \quad (n\in\mathbb{N})$$

$$(1+2+\dots+n) + x(\dots) + x^2(\dots) + n!x^{n-1}$$

$$= \lim_{x\to 0} \frac{\cancel{1} + \cancel{x}(1+2+\dots+n) + x^{\cancel{2}}(\dots) + x^3(\dots) + \dots + n!x^n - \cancel{1}}{\cancel{x}} =$$

$$= 1+2+\dots+n = \underline{\underline{\frac{n(n+1)}{2}}}$$

dosadíme $x=0$

⑤ $$\lim_{x\to 1} \frac{x^{100}-2x+1}{x^{50}-2x+1} = \lim_{x\to 1} \frac{x^{100}-x-x+1}{x^{50}-x-x+1} = \lim_{x\to 1}$$

$$x^{100}-x = (x-1)\cdot\,?$$

$$\boxed{(x^n-1) = (x-1)\cdot(x^{n-1}+x^{n-2}+\dots+x^2+x+1)}$$

$$x^{100}-x = x(x^{99}-1) =$$
$$= x\cdot(x-1)\cdot(x^{98}+x^{97}+\dots+x+1)$$

$$\Rightarrow \lim \dots = \lim_{x\to 1} \frac{\cancel{(x-1)}[x(x^{98}+\dots+x+1)-1]}{\cancel{(x-1)}[x(x^{48}+\dots+x+1)-1]} \underset{\text{dosaď } x=1}{=} \frac{98}{48} = \underline{\underline{\frac{49}{24}}}$$

③

5 jinak: $\lim\limits_{x\to 1} \dfrac{x^{100}-1-2x+2}{x^{50}-1-2x+2} = \lim\limits_{x\to 1} \dfrac{(x-1)\left[x^{99}+\dots+x+1-2\right]}{(x-1)\left[x^{49}+\dots+x+1-2\right]} = \dots$

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10 $\lim\limits_{x\to 0} \dfrac{\frac{2}{x^2}+1}{\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}} = \lim\limits_{x\to 0} \dfrac{\cancel{\frac{1}{x^2}}\left(2+x^2\right)}{\cancel{\frac{1}{x^2}}\sqrt{3-6x^2+5x^4}} \underset{\substack{\text{dosaď} \\ x=0}}{=} \dfrac{2}{\sqrt{3}}$

alt.: $\dfrac{\frac{2}{x^2}+1}{\sqrt{}} \cdot \dfrac{x^2}{x^2} =$ atd.

$\boxed{(a-b)(a+b)=a^2-b^2}$

11 $\lim\limits_{x\to 0+} \dfrac{\sqrt{\frac{1}{x^2}+1}-\sqrt{\frac{1}{x^2}-1}}{x} \cdot \dfrac{\sqrt{\frac{1}{x^2}+1}+\sqrt{\frac{1}{x^2}-1}}{\sqrt{\frac{1}{x^2}+1}+\sqrt{\frac{1}{x^2}-1}} = \lim\limits_{x\to 0+} \dfrac{\cancel{\frac{1}{x^2}}+1-\left(\cancel{\frac{1}{x^2}}-1\right)}{x\left(\sqrt{\frac{1}{x^2}+1}+\sqrt{\frac{1}{x^2}-1}\right)} =$

$= \lim\limits_{x\to 0+} \dfrac{2}{\sqrt{1+x^2}+\sqrt{1-x^2}} \underset{\substack{\text{dosaď} \\ x=0}}{=} \dfrac{2}{2} = 1$

$x\to 0+ \;\dots\; x>0 \;\dots\; \sqrt{x^2}=x$ (obecně $\sqrt{x^2}=|x|$)

Sami: totéž pro $x\to 0-$

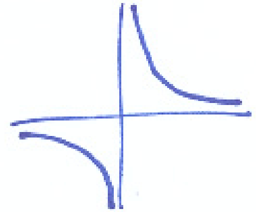
Př: 13a, 13b, 14

$$\frac{1}{4} \qquad \frac{1}{2} \qquad 0$$

13b

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} \overset{?}{=} \lim_{x\to 0}\left(\frac{\sqrt{x+1}}{x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{\sqrt{x+1}}{x}-\lim_{x\to 0}\frac{1}{x}$$

NE!



$$=\lim_{x\to 0}\frac{(\sqrt{x+1}-1)\cdot(\sqrt{x+1}+1)}{x\ (\sqrt{x+1}+1)}=$$

$$=\lim\frac{\cancel{x}+\cancel{1}-\cancel{1}}{\cancel{x}(\sqrt{x+1}-1)}=\frac{1}{2}$$

13a

$$\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}=$$

$$=\lim_{x\to 16}\frac{\cancel{\sqrt[4]{x}-2}}{\cancel{(\sqrt[4]{x}-2)}(\sqrt[4]{x}+2)}=\frac{1}{2+2}=\frac{1}{4}$$

14

$$\lim_{x\to 0}\frac{\sqrt{1-2x-x^2}-(1-x)}{x}=\lim_{x\to 0}\frac{\sqrt{1-2x-x^2}-(1-x)}{x}\cdot\frac{\sqrt{1-2x-x^2}+(1-x)}{〃}=$$

$$=\lim_{x\to 0}\frac{1-2x-x^2-(1-x)^2}{x\,\underbrace{(\sqrt{1-2x-x^2}+(1-x))}_{\text{dosad } x=0}}=\lim_{x\to 0}\frac{\cancel{1}-\cancel{2x}-x^2-(\cancel{1}-\cancel{2x}+x^2)}{x\cdot 2}=\lim_{x\to 0}\frac{-2x^2}{x}=$$

$$=\lim_{x\to 0}-2x=0$$

(5)

(15) $$\lim_{x\to 0} \frac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\sqrt[3]{x^4}} \cdot \frac{\left(\sqrt[3]{27+x}^2+\sqrt[3]{(27+x)(27-x)}+\sqrt[3]{27-x}^2\right)}{\left(\qquad \text{''} \qquad\right)} =$$

$$\boxed{a^3-b^3=(a-b)(a^2+ab+b^2)}$$

$$=\lim_{x\to 0} \frac{\cancel{27}+x-(\cancel{27}-x)}{x\cdot(1+2\sqrt[3]{x})} \frac{1}{\left(\sqrt[3]{\ }+\sqrt[3]{\ }+\sqrt[3]{\ }\right)} =$$

$$=\lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}(\underbrace{1+2\sqrt[3]{x}}_{1})} \frac{1}{3^2+3^2+3^2} = \frac{2}{27}$$

Řešení:

(6) $\frac{mn(n-m)}{2}$

(7) $\frac{n(n+1)}{2}$

(8) $\frac{n(n+1)}{2}$

(9) $\frac{m-n}{2}$

(12) 1