Cvičení MAF 27.10.2020, 9^{00}

Základní limity:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$$

$$\lim_{x\to a} (f(x))^{g(x)} = e^{\lim\limits_{x\to a} g(x)\cdot \ln(f(x))}$$

používá se v případech typu „1^∞"

Věta o limitě složené fce:

Nechť $\lim\limits_{x\to a} f(x) = b$, $\lim\limits_{y\to b} g(y) = c$

a navíc platí:

buď (S) g je spojitá v bodě b

nebo (P) $\exists \delta > 0\ \forall x \in P(a,\delta): f(x) \neq b$.

Pak $\lim\limits_{x\to a} g(f(x)) = c$.

Pokud je g spojitá, spočteme $\lim\limits_{x\to a} f(x)$ a tu dosadíme do g.

Cv.4 – 1, 2, 3

(1) $\lim\limits_{x\to a} \dfrac{\mathrm{tg}\,x - \mathrm{tg}\,a}{x-a} = \lim\limits_{x\to a} \dfrac{\dfrac{\sin x}{\cos x} - \dfrac{\sin a}{\cos a}}{x-a} = \lim\limits_{x\to a} \dfrac{\sin x\cdot\cos a - \sin a\cdot\cos x}{\cos x\cdot\cos a\cdot(x-a)}$

$(a\in\mathbb{R})$

$\cancel{\sin(x-a)} = \lim\limits_{x\to a} \dfrac{\sin(x-a)}{\cos x\cdot\cos a\cdot(x-a)} = \underbrace{\lim\limits_{x\to a}\dfrac{\sin(x-a)}{x-a}}_{=1}\cdot\underbrace{\lim\limits_{x\to a}\dfrac{1}{\cos x\cdot\cos a}}_{=\frac{1}{\cos^2 a}}$

V. o lim. složené fce (P), $g(y) = \dfrac{\sin y}{y}$

$f(x) = x-a$

$= \dfrac{1}{\cos^2 a}$ pro $a \neq \dfrac{\pi}{2} + k\pi$ $(k\in\mathbb{Z})$

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(2) $\lim\limits_{x\to 0} \dfrac{\sqrt{1-\cos x^2}}{1-\cos x} = \lim\limits_{x\to 0} \dfrac{\sqrt{1-\cos^2 x^2}}{(1-\cos x)\sqrt{1+\cos x^2}} = \lim\limits_{x\to 0} \dfrac{|\sin x^2|}{(1-\cos x)\sqrt{1+\cos x^2}}\cdot\dfrac{x^2}{x^2} =$

$= \lim\limits_{x\to 0} \underbrace{\dfrac{|\sin x^2|}{x^2}}_{\to 1\ (*)}\cdot\underbrace{\dfrac{x^2}{1-\cos x}}_{\to 2}\cdot\underbrace{\dfrac{1}{\sqrt{1+\cos x^2}}}_{\to \frac{1}{\sqrt{2}}} = \sqrt{2}$

$(*)$ $\dfrac{|\sin x^2|}{x^2} = \left|\dfrac{\sin x^2}{x^2}\right| \to 1$, $\quad \dfrac{\sin x^2}{x^2} \to 1$

(2)

(2) jinak:

$$\lim_{x\to 0}\frac{\sqrt{1-\cos x^2}}{1-\cos x} = \lim_{x\to 0}\underbrace{\sqrt{\underbrace{\frac{1-\cos x^2}{x^4}}_{\to \frac{1}{2}}}}_{\text{slož. fce}\ \to\ \frac{1}{\sqrt{2}}}\cdot\underbrace{\frac{x^2}{1-\cos x}}_{\to 2} = \underline{\underline{\sqrt{2}}}$$

slož. fce

(3)

$$\lim_{x\to 0}\frac{\operatorname{tg} x-\sin x}{x^3} = \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3} = \lim_{x\to 0}\frac{\overbrace{\sin x}^{\to 1}\,\overbrace{(1-\cos x)}^{\to \frac{1}{2}}}{\underbrace{\cos x}_{\to 1}\cdot x\cdot x^2} = \underline{\underline{\frac{1}{2}}}$$

4, 5 – sami doma (pracnější)

(6) (7)

(6)

$$\lim_{x\to 1}\frac{\sin \pi x}{1-x} = \begin{bmatrix} \text{subst. } y = 1-x \\ x = 1-y \\ x\to 1 \Leftrightarrow y\to 0 \end{bmatrix} = \lim_{y\to 0}\frac{\sin(\overbrace{\pi(1-y)}^{\pi-\pi y})}{y} =$$

$$= \lim_{y\to 0}\frac{\overbrace{\sin\pi}^{=0}\cdot\cos\pi y-\overbrace{\cos\pi}^{=-1}\cdot\sin\pi y}{y} = \lim_{y\to 0}\overbrace{\frac{\sin\pi y}{\pi y}}^{\to 1}\cdot\pi = \underline{\pi}$$

(7) $$\lim_{x\to\frac{\pi}{4}} \operatorname{tg}(2x)\cdot \operatorname{tg}\left(\frac{\pi}{4}-x\right) = \lim_{x\to\frac{\pi}{4}} \frac{\underbrace{\sin(2x)}_{\to 1}}{\underbrace{\cos(2x)}_{\to 0}}\cdot\frac{\underbrace{\sin\left(\frac{\pi}{4}-x\right)}_{\to 0}}{\underbrace{\cos\left(\frac{\pi}{4}-x\right)}_{\to 1}} =$$

„$\infty\cdot 0$"

$$\overset{(*)}{=} \lim_{x\to\frac{\pi}{4}} \frac{\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x}{\cos^2 x - \sin^2 x} = \frac{\sqrt{2}}{2}\cdot\lim_{x\to\frac{\pi}{4}}\frac{\cos x - \sin x}{\cos^2 x - \sin^2 x} = \frac{\sqrt{2}}{2}\lim_{x\to\frac{\pi}{4}}\frac{1}{\underbrace{\cos x + \sin x}_{\sqrt{2}}}$$

$$= \underline{\underline{\frac{1}{2}}}$$

jinak (*) $$\lim_{x\to\frac{\pi}{4}}\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos(2x)} = \left[\begin{array}{l} \text{subst. } y=\frac{\pi}{4}-x \\ \qquad x=\frac{\pi}{4}-y \\ x\to\frac{\pi}{4} \Leftrightarrow y\to 0 \end{array}\right] = \lim_{y\to 0}\frac{\sin y}{\cos\left(\frac{\pi}{2}-2y\right)} =$$

$$= \lim_{y\to 0}\frac{\sin y}{\underbrace{\cos\frac{\pi}{2}}_{=0}\cos(2y) + \underbrace{\sin\frac{\pi}{2}}_{=1}\sin(2y)} = \lim_{y\to 0}\frac{\sin y}{\sin 2y} = \lim_{y\to 0}\underbrace{\frac{\sin y}{y}}_{\to 1}\cdot\underbrace{\frac{2y}{\sin 2y}}_{\to 1}\cdot\frac{1}{2} = \underline{\underline{\frac{1}{2}}}$$

(18) $$\lim_{x\to 0+}(1+x)^{\frac{1}{x}} = e^{\underbrace{\lim_{x\to 0+}\frac{1}{x}\ln(1+x)}_{\text{zákl. limita } = 1}} = e^1 = \underline{\underline{e}}$$

typ „1^∞"

(19) $$\lim_{x\to \frac{\pi}{2}}(\sin x)^{\operatorname{tg} x} = e^{\lim_{x\to\frac{\pi}{2}}\frac{\sin x}{\cos x}\cdot\ln(\sin x)} = e^0 = \underline{\underline{1}}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{\overset{\to 1}{\sin x}}{\underset{\to 0}{\cos x}}\overset{\to 0}{\ln(\sin x)} = \lim_{x\to\frac{\pi}{2}} \frac{\ln(\sin x)}{\sin x - 1}\cdot\frac{\sin x - 1}{\cos x} =$$

$$\left[\begin{array}{l} y = \sin x - 1 \\ \sin x = y+1 \end{array}\right] \quad x\to\frac{\pi}{2} \Leftrightarrow y\to 0$$

$$= \lim_{y\to 0}\frac{\ln(y+1)}{y} = 1$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\sin^2 x - 1}{\cos x\cdot(\sin x + 1)} = \lim_{x\to\frac{\pi}{2}}\frac{-\cos^2 x}{\cos x}\cdot\underset{\to\frac{1}{2}}{\frac{1}{\sin x + 1}} = \lim_{x\to\frac{\pi}{2}} -\frac{1}{2}\cos x = 0$$

Sami: ∀ další limity z Cv. 4
+ DÚ 2 do 3.11.