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Základní limity:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$$

$$\lim_{x\to a} \left(f(x)\right)^{g(x)} = e^{\lim\limits_{x\to a}(g(x)\cdot \ln(f(x)))}$$

↳ používá se v případech typu „1^∞"

Věta o limitě složené fce:

Nechť $\lim\limits_{x\to a} f(x) = b$, $\lim\limits_{y\to b} g(y) = c$

a navíc platí:

buď (S) g je spojitá v bodě b

nebo (P) $\exists \delta>0 \ \forall x\in P(a,\delta): f(x)\neq b$.

Pak $\lim\limits_{x\to a} g(f(x)) = c$.

Pokud g je spojitá, spočteme $\lim\limits_{x\to a} f(x)$ a tu dosadíme do g.

Cv.4 : 1, 2, 3

(1) $$\lim_{x\to a}\frac{\operatorname{tg}x-\operatorname{tg}a}{x-a}=\lim_{x\to a}\frac{\frac{\sin x}{\cos x}-\frac{\sin a}{\cos a}}{x-a}=\lim_{x\to a}\frac{\sin x\cdot\cos a-\sin a\cdot\cos x}{\cos x\cdot\cos a\cdot(x-a)}=$$

$$=\lim_{x\to a}\frac{\sin(x-a)}{\underbrace{\cos x}_{\to\cos a}\cdot\underbrace{\cos a}_{\cos a}\,(x-a)}=\frac{1}{\cos^2 a}\cdot\lim_{x\to a}\frac{\sin(x-a)}{x-a}=\frac{1}{\cos^2 a}\cdot 1=\underline{\underline{\frac{1}{\cos^2 a}}}$$

Věta o lim. slož. fce: $g(y)=\frac{\sin y}{y}$, $f(x)=x-a$ (P)

Platí pro $a\neq\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$

2, 3

(2) $$\lim_{x\to 0}\frac{\sqrt{1-\cos x^2}}{1-\cos x}=\lim_{x\to 0}\frac{\sqrt{1-\cos x^2}\cdot\underbrace{(1+\cos x)}_{\to 2}}{1-\cos^2 x}=2\lim_{x\to 0}\frac{\sqrt{1-\cos x^2}}{\sin^2 x}=$$

$$=2\cdot\lim_{x\to 0}\frac{\sqrt{1-\cos^2 x^2}}{\sin^2 x\cdot\underbrace{\sqrt{1+\cos x^2}}_{\to\sqrt{2}}}=\sqrt{2}\lim_{x\to 0}\frac{\sqrt{\sin^2 x^2}}{\sin^2 x}=\sqrt{2}\lim_{x\to 0}\underbrace{\frac{|\sin x^2|}{|x^2|}}_{\to|1|=1}\cdot\underbrace{\frac{x^2}{\sin^2 x}}_{\to 1}$$

$$=\underline{\sqrt{2}}$$

3.)

(2) jinak:

$$\lim_{x\to 0} \frac{\sqrt{1-\cos x^2}}{1-\cos x} = \lim_{x\to 0} \underbrace{\sqrt{\underbrace{\frac{1-\cos x^2}{x^4}}_{\to \frac{1}{2}}}}_{\frac{1}{\sqrt{2}}} \cdot \underbrace{\frac{x^2}{1-\cos x}}_{\to 2} = \sqrt{2}$$

(3)

$$\lim_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x\to 0} \frac{\sin x\,(1-\cos x)}{\cos x \cdot x^3} =$$

$$= \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \underbrace{\frac{1-\cos x}{x^2}}_{\to \frac{1}{2}} \cdot \underbrace{\frac{1}{\cos x}}_{\to 1} = \frac{1}{2}$$

6, 7

(6)

$$\lim_{x\to 1} \frac{\sin \pi x}{1-x} = \left[\begin{array}{r} \text{subst. } y = 1-x \\ x = 1-y \\ x\to 1 \Leftrightarrow y \to 0 \end{array}\right] = \lim_{y\to 0} \frac{\sin(\pi - \pi y)}{y} =$$

$$= \lim_{y\to 0} \frac{\overset{=0}{\sin \pi} \cdot \cos \pi y - \overset{=-1}{\cos \pi} \cdot \sin \pi y}{y} = \lim_{y\to 0} \underbrace{\frac{\sin \pi y}{\pi y}}_{\to 1} \cdot \pi = \pi$$

18, 19

(18) $$\lim_{x\to 0+} (1+x)^{\frac{1}{x}} = e^{\underbrace{\lim_{x\to 0+}\left(\frac{1}{x}\cdot \ln(1+x)\right)}_{=1}} = e^1 = \underline{\underline{e}}$$

známá limita

typ „1^{∞}"

(19) $$\lim_{x\to \frac{\pi}{2}} (\sin x)^{\operatorname{tg} x} = e^{\lim_{x\to\frac{\pi}{2}} \operatorname{tg} x \cdot \ln(\sin x)} = e^0 = \underline{\underline{1}}$$

$$\lim_{x\to\frac{\pi}{2}} \underbrace{\operatorname{tg} x}_{\downarrow\,+\infty\ \downarrow\,-\infty} \cdot \underbrace{\ln(\sin x)}_{\to 0} = \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos x} \cdot \frac{\ln(\sin x)}{\sin x - 1} \cdot (\sin x - 1) = \lim_{x\to\frac{\pi}{2}} \frac{\sin^2 x - 1}{\cos x \cdot (\sin x + 1)} =$$

$\sin x \to 1$, $\frac{\ln(\sin x)}{\sin x - 1} \to 1$, $\sin x + 1 \to 2$

slož. fce: $g(y) = \frac{\ln(y+1)}{y}$

$y = f(x) = \sin x - 1$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\cos^2 x}{(\cos x)\cdot 2} = \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{2} = 0$$