Cvičení MAF, 3.11.2020, 9⁰⁰

Ještě limity (Cv. 4)

$$\boxed{\lim_{y\to 1} \frac{\ln y}{y-1} = 1}$$

(15)

$$\lim_{x\to 0+} \ln(x\cdot \ln a)\cdot \ln\left(\frac{\ln ax}{\ln \frac{x}{a}}\right) = \lim_{x\to 0+} \ln(x\cdot \ln a)\cdot \ln\left(\frac{\ln x + \ln a}{\ln x - \ln a}\right) =$$

$$\left[\lim_{x\to 0+} \frac{\ln x + \ln a}{\ln x - \ln a} = \lim_{x\to 0+} \frac{1 + \frac{\ln a}{\ln x}}{1 - \frac{\ln a}{\ln x}} = 1\right]$$

$\frac{\ln a}{\ln x} \to 0$

$$= \lim_{x\to 0+} \ln(x\cdot \ln a)\cdot\left(\frac{\ln x + \ln a}{\ln x - \ln a} - 1\right) = \ldots = \underline{\underline{2\ln a}}$$

$$= \lim_{x\to 0+} \ln(x\cdot \ln a)\cdot \frac{2\ln a}{\ln x - \ln a} = \lim_{x\to 0+} \underbrace{\frac{\ln x + \ln(\ln a)}{\ln x - \ln a}}_{\to 1 \text{ jako výše}} \cdot 2\ln a$$

(10) $$\lim_{x\to 0+} \frac{\arccos(1-x)}{\sqrt{x}} = \lim_{x\to 0+} \underbrace{\frac{\arcsin\left(\sqrt{x\cdot(2-x)}\right)}{\sqrt{x}\,\sqrt{2-x}}}_{\to 1} \underbrace{\sqrt{2-x}}_{\to\sqrt{2}} = \sqrt{2}$$

(vztah mezi arccos, arcsin)

$$\lim_{y\to 0} \frac{\sin y}{y} = 1$$

subst. $1-x = \cos y$
$y = \arccos(1-x)$
$x = 1-\cos y$

$x\to 0+ \iff y \to 0+$

$$= \lim_{y\to 0+} \frac{y}{\sqrt{1-\cos y}} = \lim_{y\to 0+} \sqrt{\frac{y^2}{1-\cos y}} = \underline{\underline{\sqrt{2}}}$$

(25) $$\lim_{x\to 1} \frac{\sin \pi x^\alpha}{\sin \pi x^\beta} = \lim_{x\to 1} \frac{\sin(\pi x^\alpha - \pi + \pi)}{\sin(\pi x^\beta - \pi + \pi)} = \lim_{x\to 1} \frac{\sin(\pi x^\alpha - \pi)\underbrace{\cos\pi}_{=-1} + \underbrace{\sin\pi}_{=0}\cdot\cos(\ldots)}{\sin(\pi x^\beta - \pi)\underbrace{\cos\pi}_{=-1} + \underbrace{\sin\pi}_{=0}\cdots}$$

$(\alpha,\beta\in\mathbb{R},\ \beta\neq 0)$

$$= \lim_{x\to 1} \frac{\sin(\pi(x^\alpha-1))}{\sin(\pi(x^\beta-1))} \overset{(*)}{=} \lim_{x\to 1} \frac{\cancel{\pi}(x^\alpha-1)}{\cancel{\pi}(x^\beta-1)} = \lim_{x\to 1} \frac{e^{\alpha\cdot\ln x}-1}{e^{\beta\cdot\ln x}-1} = \lim_{x\to 1} \frac{\alpha\ln x}{\beta\ln x} = \underline{\frac{\alpha}{\beta}}$$

$$\circledast = \lim_{x \to 1} \underbrace{\frac{\sin(\pi(x^{\alpha}-1))}{\pi(x^{\alpha}-1)}}_{\to 1} \cdot \underbrace{\frac{\pi(x^{\beta}-1)}{\sin(\pi(x^{\beta}-1))}}_{\to 1} \cdot \frac{\pi(x^{\alpha}-1)}{\pi(x^{\beta}-1)}$$

Spojitost funkce

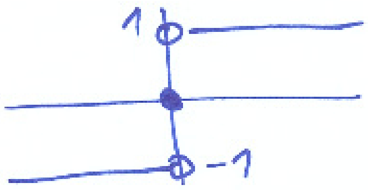
Def: f je spojitá v bodě $x_0 \in D_f$ $\equiv$ $\lim\limits_{x \to x_0} f(x) = f(x_0)$ $\equiv$

$$\equiv \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in U_{\delta}(x_0) : f(x) \in U_{\varepsilon}(f(x_0))$$

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Ozn: funkce signum (znaménko)

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



Platí:

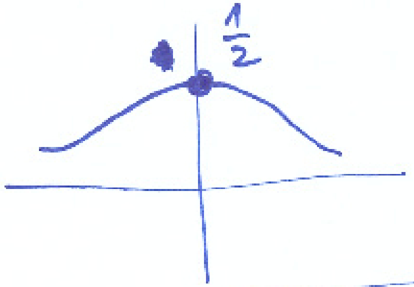
$$\left.\begin{aligned} |x| &= x \cdot \operatorname{sgn} x \\ x &= |x| \cdot \operatorname{sgn} x \end{aligned}\right\} \forall x \in \mathbb{R}$$

$$|x|' = \operatorname{sgn} x \quad (x \neq 0)$$

① $f(x)=\frac{1-\cos x}{x^2}$, $x\neq 0$

dodef. spojitě v nule:

$$f(0)=\frac{1}{2}=\lim_{x\to 0} f(x)$$



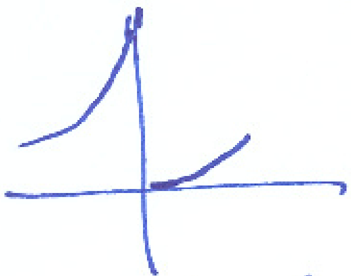
② Kde jsou nespojité:

a) $f(x)=e^{-\frac{1}{x}}$

$0\notin D_f$ ... je tam nespojitá

$$\lim_{x\to 0+} e^{-\frac{1}{x}}=0$$

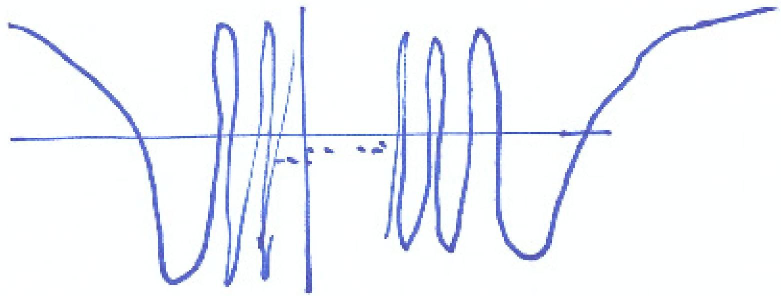
$$\lim_{x\to 0-} e^{-\frac{1}{x}}=+\infty$$



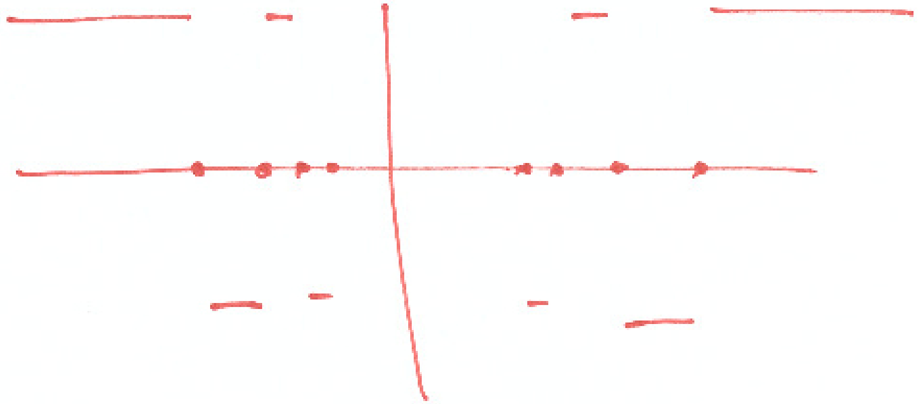
nejde spoj. dodef.

b) $f(x)=\operatorname{sgn}\cos\frac{1}{x}$ $D_f=\mathbb{R}\setminus\{0\}$

$\cos\frac{1}{x}$:



$\operatorname{sgn}\cos\frac{1}{x}$



body nespojitosti:

$$\frac{1}{x}=\frac{\pi}{2}+k\pi \quad (k\in\mathbb{Z})$$

$$x=\frac{1}{\frac{\pi}{2}+k\pi}$$

nelze spojitě dodef.

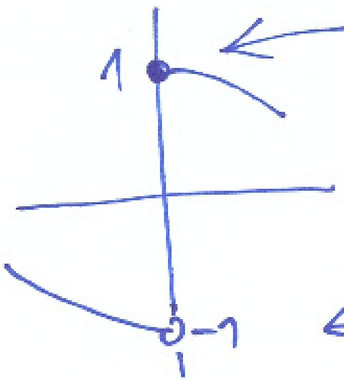
③ – sami

④ a) $f(x) = \begin{cases} \dfrac{\sin x}{|x|} & (x \neq 0) \\ 1 & (x = 0) \end{cases}$

je spojitá? v 0 není

$\lim\limits_{x \to 0+} f(x) = 1$

$\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0-} \dfrac{\sin x}{-x} = -1$



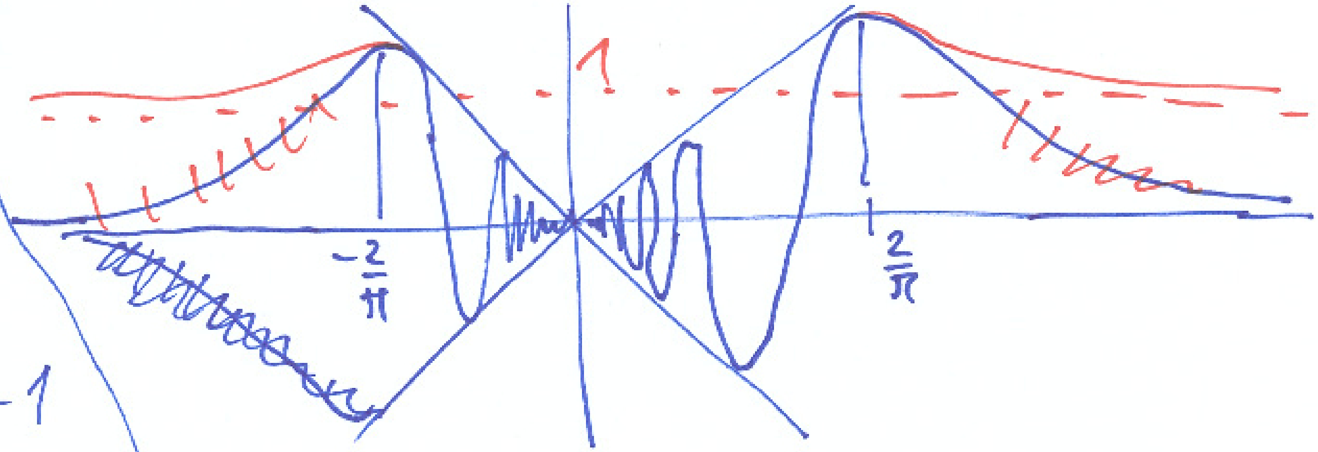
v 0 je spojitá zprava

ale ne zleva

$\lim\limits_{x \to +\infty} x \sin \frac{1}{x} = \lim\limits_{x \to +\infty} \dfrac{\sin \frac{1}{x}}{\frac{1}{x}} = \underline{\underline{1}}$

5]

b) $f(x) = \begin{cases} x \cdot \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$ $\quad D_f = \mathbb{R}$



$f(-x) = -x \cdot \sin\left(-\frac{1}{x}\right) = x \cdot \sin \frac{1}{x} = f(x)$

f je spojitá ← v $\mathbb{R} \setminus \{0\}$ ... kombinace spoj. fcí

v 0:

$\lim\limits_{x \to 0} f(x) = 0$ podle věty o 2 polic.

$x \to 0+$: $\quad -x \le f(x) \le x$, $\quad -x \to 0$, $\quad f(x) \to 0$, $\quad x \to 0$

$x \to 0-$: $\quad x \le f(x) \le -x$

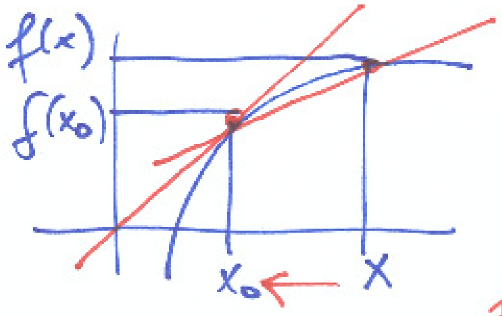
Derivace funkce

Def: $x_0 \in D_f$, derivace fce f v bodě x_0:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

+ derivace zprava / zleva:

$$f'_{\pm}(x_0) = \lim_{x \to x_0 \pm} \text{—"—}$$



$$\frac{f(x) - f(x_0)}{x - x_0} = \text{směrnice sečny}$$

$$f'(x) = \lim \text{—"—} = \text{směrnice tečny}$$

Př: $f(x) = a \Rightarrow f'(x) = 0$ v $\forall x \in \mathbb{R}$

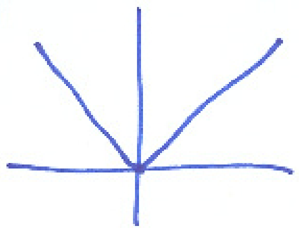
$f(x) = x \Rightarrow f'(x) = 1$

$$f(x) = |x| = \begin{cases} x & (x \geq 0) \\ -x & (x \leq 0) \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 1 & (x > 0) \\ -1 & (x < 0) \end{cases}$$

$f'(0)$ není def.

v 0 nemá tečnu

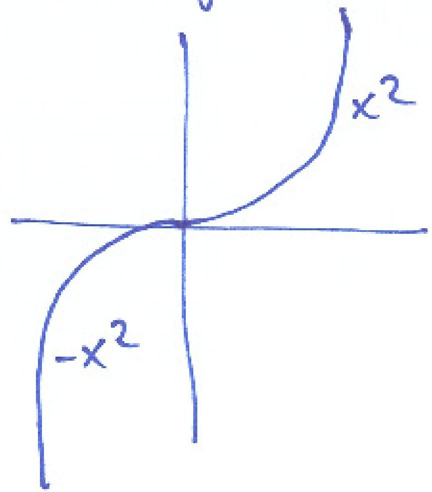


(7) $f(x) = x \cdot |x|$ ... ? $\exists f'(0)$

ano, $f'(0) = 0$:

$$f'(0) = \lim_{x \to 0} \frac{x \cdot |x| - 0 \cdot |0|}{x - 0} = \underline{\underline{0}}$$

lze psát $x \cdot |x| = \operatorname{sgn} x \cdot x^2$



(8) $f(x) = \begin{cases} |x|^\alpha \cdot \sin\frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$

Pro která $\alpha \in \mathbb{R}$ má f derivaci v 0? Pro která $\alpha \in \mathbb{R}$ je f' spojitá v 0?

$$f'(0) = \lim_{x\to 0} \frac{|x|^\alpha \cdot \sin\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0} \frac{|x|^\alpha \cdot \sin\frac{1}{x}}{|x| \cdot \operatorname{sgn} x} =$$

$$= \lim_{x\to 0} |x|^{\alpha-1} \cdot \operatorname{sgn} x \cdot \sin\frac{1}{x} \in \langle -1, 1 \rangle$$

Také platí: $f(x) \to 0$, $g(x)$ omez. $\Rightarrow f(x) \cdot g(x) \to 0$

pokud $\alpha > 1 \Leftrightarrow \lim_{x\to 0} |x|^{\alpha-1} = 0$

V o 2 polic.: ~~lim~~ $-|x|^{\alpha-1} \leq \underbrace{|x|^{\alpha-1} \cdot \operatorname{sgn} x \cdot \sin\frac{1}{x}}_{\downarrow} \leq |x|^{\alpha-1}$

pro $\alpha > 1$: $0 \Rightarrow \underline{0} \Leftarrow 0$

./.

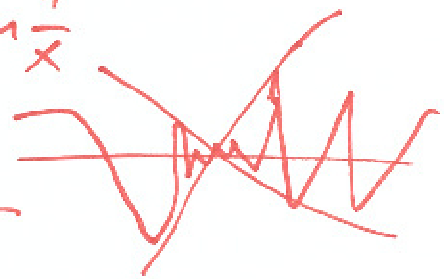
Hodnoty derivace pro $x \neq 0$: $\boxed{|x|' = \operatorname{sgn} x \quad (x \neq 0)}$

$$f'(x) = \alpha \cdot |x|^{\alpha-1} \cdot \operatorname{sgn} x \cdot \sin\frac{1}{x} + |x|^{\alpha} \cdot \cos\frac{1}{x} \cdot \frac{-1}{x^2} =$$

$(x \neq 0)$

$$= |x|^{\alpha-2} \underbrace{\left(\alpha \cdot |x| \cdot \operatorname{sgn} x \cdot \sin\frac{1}{x} - \cos\frac{1}{x}\right)}_{\text{omezená}} \Longleftarrow$$

Víme $x \cdot \sin\frac{1}{x}$ je omezená



$\alpha > 2 \iff 0$

Tedy $\lim\limits_{x \to 0} f'(x) = 0 \iff \alpha > 2$

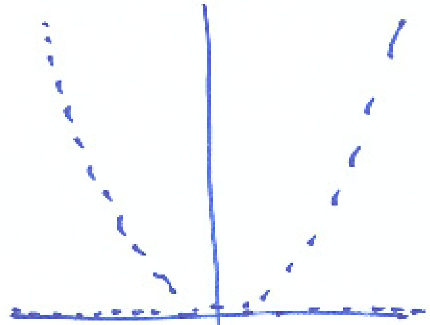
a tedy f' je spojitá v $0 \iff \underline{\underline{\alpha > 2}}$.

$f'(0) = 0$

limita = fční hodnota

---

(9) $f(x) = \begin{cases} x^2 & x \text{ racion.} \\ 0 & x \text{ iracion.} \end{cases}$



$x \neq 0 \Rightarrow$ nespojitá, nemá derivaci

má derivaci pouze v 0.

2 policajti:

$0 \leq f(x) \leq x^2$, kde $0 \to 0$, $f(x) \to 0$, $x^2 \to 0$

$x = 0$: $f'(0) = \lim\limits_{x \to 0} \frac{f(x) - 0}{x - 0}$

$0 \leq f'(x) \leq |x|$, kde $0 \to 0$, $f'(x) \to 0$, $|x| \to 0$

Derivace základních fcí

$a' = 0$

$(x^n)' = n \cdot x^{n-1} \quad (n \in \mathbb{N},\ x \in \mathbb{R})$

$(x^a)' = a \cdot x^{a-1} \quad (a \in \mathbb{R},\ x \in \mathbb{R}_+)$

$(\sqrt[n]{x})' = (x^{\frac{1}{n}})' = \frac{1}{n} \cdot x^{(\frac{1}{n}-1)}$

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

$(e^x)' = e^x$

$(a^x)' = a^x \cdot \ln a$

$(\ln x)' = \frac{1}{x}$

$(\log_a x)' = \frac{1}{x \cdot \ln a}$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\mathrm{tg}\, x)' = \frac{1}{\cos^2 x}$

$(\mathrm{cotg}\, x)' = \frac{-1}{\sin^2 x}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$

$(\mathrm{arctg}\, x)' = \frac{1}{x^2+1}$

$(\mathrm{arccotg}\, x)' = \frac{-1}{x^2+1}$

$$\left[\ \sinh x = \frac{e^x - e^{-x}}{2},\quad \cosh x = \frac{e^x + e^{-x}}{2}\ \right]$$

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

Pravidla: $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

Leibniz: $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$

chain rule $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

13.–29.
sami