Cvičení MAF, 3.11.2020, 12:20

$$(a^n - b^n) = (a-b)(a^{n-1} + \dots + b^{n-1})$$

Cv. 3 (16) $\lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+x} - \sqrt[n]{1+x}}{x} = \lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+x} - 1 + 1 - \sqrt[n]{1+x}}{x}$

$m, n \in \mathbb{N}$

Cv. 4 (25) $\lim\limits_{x\to 1} \dfrac{\sin \pi x^\alpha}{\sin \pi x^\beta} = \lim\limits_{x\to 1} \dfrac{\sin((\pi x^\alpha - \pi) + \pi)}{\sin((\pi x^\beta - \pi) + \pi)} = \lim\limits_{x\to 1} \dfrac{\sin(\pi x^\alpha - \pi)\overbrace{\cos\pi}^{=-1} + \overbrace{\sin\pi}^{=0} \cdot \cos(\dots)}{\sin(\pi x^\beta - \pi)\cdot \underbrace{\cos\pi}_{=-1} + \underbrace{\sin\pi}_{=0} \cdot \cos\dots}$

$(\alpha, \beta \in \mathbb{R}, \beta \neq 0)$

$$= \lim_{x\to 1} \frac{\sin(\pi(x^\alpha - 1))}{\sin(\pi(x^\beta - 1))} = \lim_{x\to 1} \underbrace{\frac{\sin(\pi(x^\alpha - 1))}{\pi(x^\alpha - 1)}}_{\to 1} \cdot \underbrace{\frac{\pi(x^\beta - 1)}{\sin(\pi(x^\beta - 1))}}_{\to 1} \cdot \frac{\pi(x^\alpha - 1)}{\pi(x^\beta - 1)} =$$

$$= \lim_{x\to 1} \frac{x^\alpha - 1}{x^\beta - 1} = \lim_{x\to 1} \frac{e^{\alpha \cdot \ln x} - 1}{e^{\beta \cdot \ln x} - 1} = \lim_{x\to 1} \underbrace{\frac{e^{\alpha \cdot \ln x} - 1}{\alpha \cdot \ln x}}_{\to 1} \cdot \underbrace{\frac{\beta \ln x}{e^{\beta \ln x} - 1}}_{\to 1} \cdot \frac{\alpha}{\beta} = \frac{\alpha}{\beta}$$

Protože pro $x \to 1$ je $\ln x \to 0$

$$\boxed{\lim_{y\to 0} \frac{e^y - 1}{y} = 1}$$

Spojitost funkce

Def: fce f je spojitá v bodě $x_0 \in D_f \equiv \lim\limits_{x\to x_0} f(x) = f(x_0) \equiv$

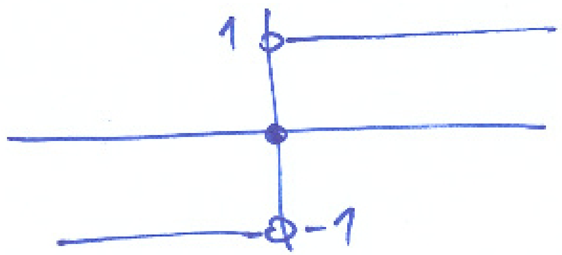
$\hookrightarrow$ f je def. na $U(x_0)$

$$\equiv \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in U_\delta(x_0): \ f(x) \in U_\varepsilon(f(x_0))$$

$$|x - x_0| < \delta \ : \ |f(x) - f(x_0)| < \varepsilon$$

Ozn: funkce signum (znaménko)

$$\operatorname{sgn} x = \begin{cases} 1 & (x>0) \\ 0 & (x=0) \\ -1 & (x<0) \end{cases}$$



Platí:

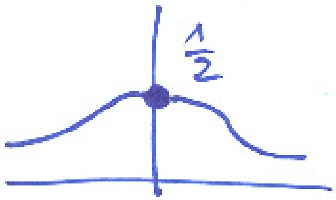
$$\left.\begin{aligned} |x| &= x \cdot \operatorname{sgn} x \\ x &= |x| \cdot \operatorname{sgn} x \end{aligned}\right\} \forall x \in \mathbb{R}$$

$$|x|' = \operatorname{sgn} x \quad (\forall x \neq 0)$$

Cvs. 1 $f(x) = \dfrac{1-\cos x}{x^2}, \ x \in \mathbb{R} \setminus \{0\}$

dodefinovat f v 0 spojitě

$$f(0) := \lim_{x\to 0} f(x) = \frac{1}{2}$$

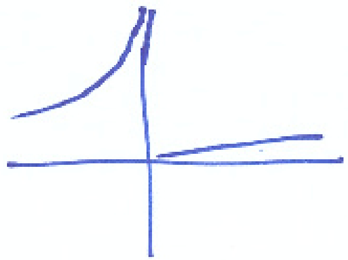


② a) $f(x) = e^{-\frac{1}{x}}$, $x \neq 0$

? lze dodef. $f(0) = ?$ spojitě

$$\lim_{x\to 0+} e^{-\frac{1}{x}} = 0$$

$$\lim_{x\to 0-} e^{-\frac{1}{x}} = +\infty$$



jednostr. limity jsou různé

$\Rightarrow$ nejde spojitě dodef.

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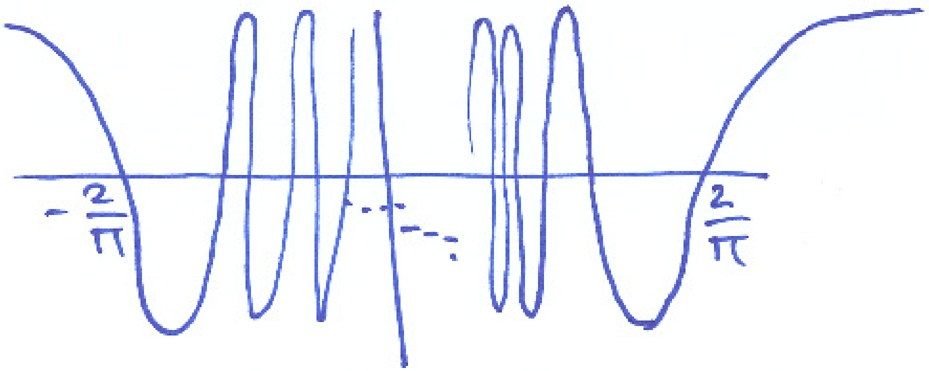
b) $f(x) = \operatorname{sgn} \cos \frac{1}{x}$

kde je nespojitá? lze spoj. dodef?

$D_f = \mathbb{R} \setminus \{0\}$ sudá fce

$\cos \frac{1}{x}$



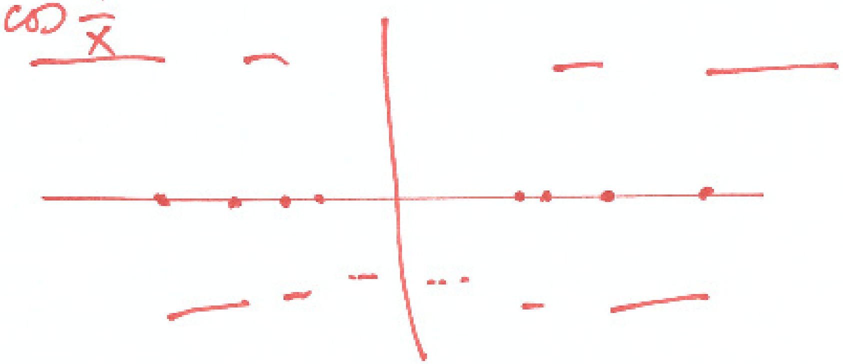
$\lim_{x \to 0} \cos \frac{1}{x}$ neexistuje

(ani jednostranné)

$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in U_\delta(0) : f(x) \notin U_\varepsilon(a)$

pro žádné reálné a

$\operatorname{sgn} \cos \frac{1}{x}$



body nespojitosti:

$$\frac{1}{x} = \frac{\pi}{2} + k\pi$$

$$x = \frac{1}{\frac{\pi}{2} + k\pi}$$

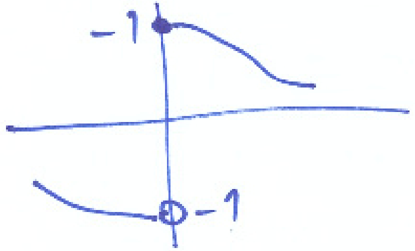
v nich nelze spojitě ~~dosad~~ dodefinovat

(4) a) $f(x) = \begin{cases} \dfrac{\sin x}{|x|} & (x \neq 0) \\ 1 & (x = 0) \end{cases}$

je spojitá? v 0 není:

$$\lim_{x \to 0+} \frac{\sin x}{|x|} = 1$$

$$\lim_{x \to 0-} \frac{\sin x}{|x|} = \lim_{x \to 0-} \frac{\sin x}{-x} = -1$$



v ost. bodech je spojitá

b) $f(x) = \begin{cases} x \cdot \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$

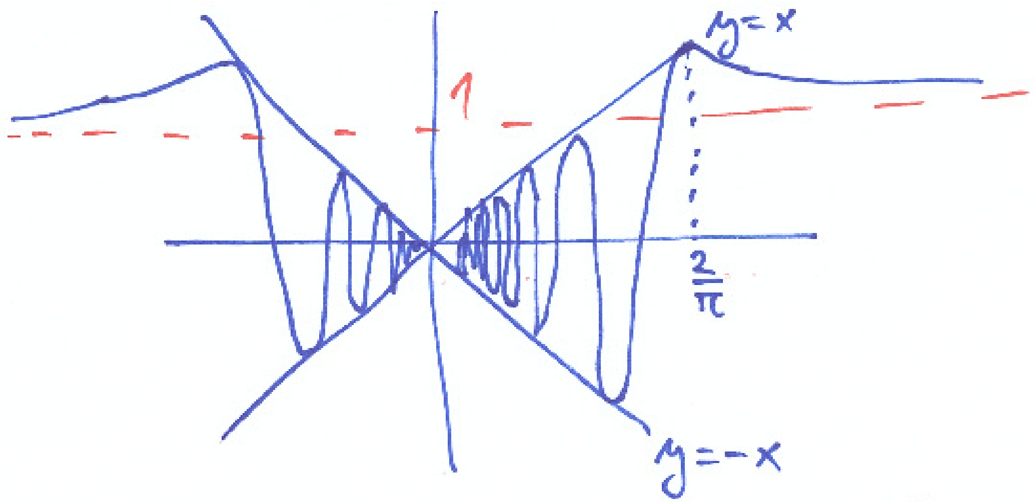
je spojitá? pro $\forall x \neq 0$ je spoj.

a co v $x = 0$?

? $\lim_{x \to 0} x \cdot \sin \frac{1}{x} = \lim_{x \to 0} \dfrac{\sin \frac{1}{x}}{\frac{1}{x}} \neq 1$!

$\frac{1}{x} \to$ nejde k 0

není známá limita



$f(-x) = (-x) \cdot \sin \frac{1}{-x} = (-x) \cdot \left(-\sin \frac{1}{x}\right) = f(x)$

$\Rightarrow f$ sudá

$$\lim_{x \to +\infty} x \cdot \sin \frac{1}{x} = \lim_{x \to +\infty} \frac{\sin \frac{1}{x}}{\underbrace{\frac{1}{x}}_{\to 0}} = 1$$

$\lim f(x) = 0$ podle věty o 2 polic.

pro $x>0$:

$$\begin{array}{ccccc} & -x & \le f(x) \le & x \\ & \downarrow & \downarrow & \downarrow \\ x\to 0+ & 0 & \Rightarrow \; 0 \; \Leftarrow & 0 \end{array}$$

pro $x<0$:

$$\begin{array}{ccccc} & x & \le f(x) \le & -x \\ & \downarrow & \downarrow & \downarrow \\ x\to 0- & 0 & \Rightarrow \; 0 \; \Leftarrow & 0 \end{array}$$

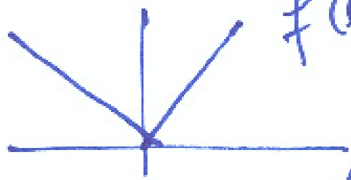
Př: $f(x) = a \Rightarrow f'(x) = 0$

$f(x) = x \Rightarrow f'(x) = 1$

$$f(x) = |x| = \begin{cases} x & (x \ge 0) \\ -x & (x \le 0) \end{cases}$$

$$\Rightarrow f'(x) = \operatorname{sgn} x = \begin{cases} 1 & (x>0) \\ -1 & (x<0) \end{cases}$$

$f'(0)$ není def. pro $|x|$



v 0 nemá tečnu

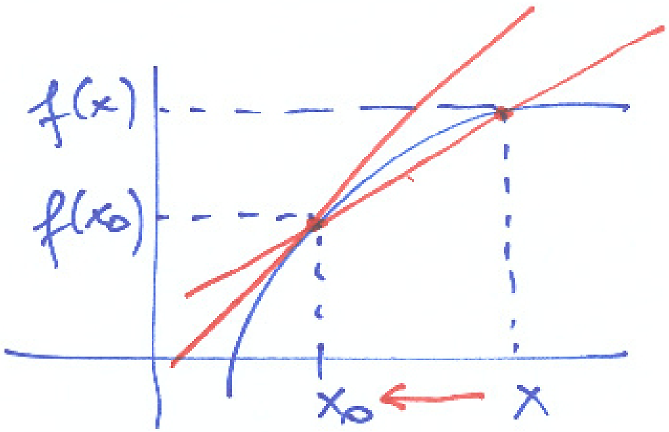
Sami doma: 3, 5, 6

Derivace funkce

Def: derivace fce f v bodě x_0:

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

5] Derivace zprava/zleva: $f'_\pm(x_0) = \lim_{x\to x_0\pm} \dots$

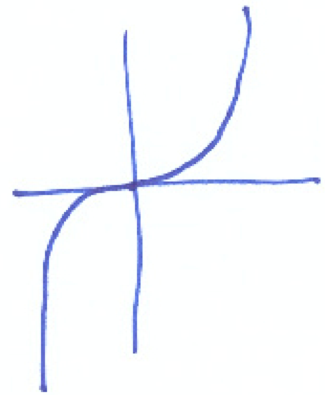


$$\frac{f(x)-f(x_0)}{x-x_0} = \text{směrnice sečny}$$

$$\lim_{x\to x_0} \text{—"—} = \text{směrnice tečny}$$

⑦ ? ex. derivace fce $f(x) = x\cdot|x| = \operatorname{sgn} x \cdot x^2$ v bodě 0 ?

ano, je $f'(0) = 0$



vidíme

$f'_+(0) = f'_-(0) = 0$

neboť $\lim\limits_{x\to 0} \dfrac{x\cdot|x| - 0}{x - 0} = \underline{\underline{0}}$

$\alpha > 1$:

$$f'(0) = \lim_{x\to 0} \frac{|x|^\alpha \cdot \sin\frac{1}{x} - 0}{x - 0} =$$

$$= \lim_{x\to 0} \frac{|x|^\alpha \cdot \sin\frac{1}{x}}{|x|\cdot \operatorname{sgn} x} = \lim_{x\to 0} |x|^{\alpha-1} \cdot \underbrace{\operatorname{sgn} x \cdot \sin\frac{1}{x}}_{\in\langle -1, 1\rangle}$$

$$-|x|^{\alpha-1} \le |x|^{\alpha-1}\cdot \operatorname{sgn} x \cdot \sin\frac{1}{x} \le |x|^{\alpha-1}$$

$x \to 0$: $\quad 0 \qquad 0 \qquad 0 \quad \Rightarrow \quad \underline{\underline{0}}$

pro $\alpha > 1$

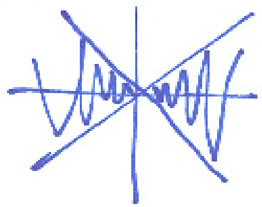
⑧ $f(x) = \begin{cases} |x|^\alpha \cdot \sin\frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$

Pro jaká $\alpha \in \mathbb{R}$ existuje $f'(0)$?

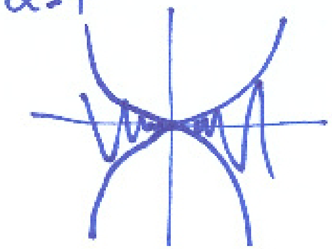
——— " ——— je f' spojitá v 0 ?

$f'(0)$ ex. $(a = 0) \iff \alpha > 1$

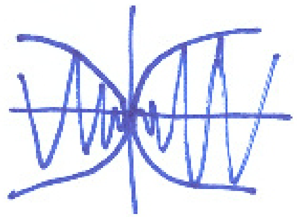
$\alpha = 1$



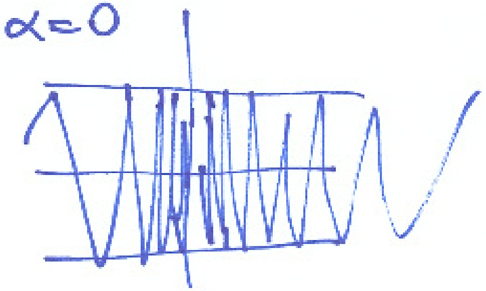
$\alpha > 1$



$\alpha \in (0,1)$



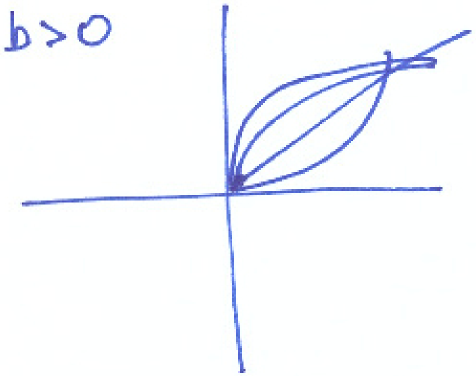
$\alpha = 0$



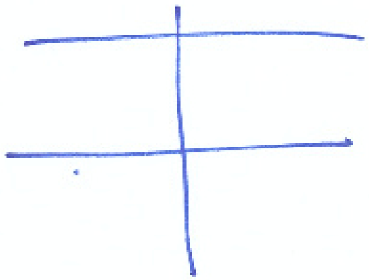
$\alpha < 0$

x^b:

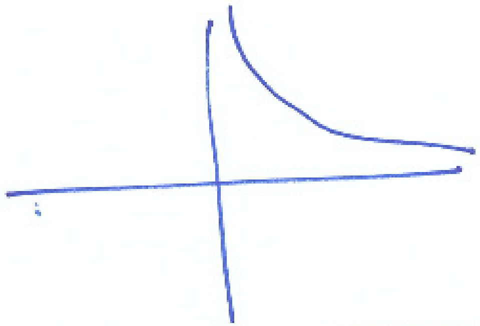
$b>0$



$b=0 \ldots x^0=1$



$b<0$



$\lim x^b = 0 \Leftrightarrow b>0$

Kdy je $f'(x)$ spojitá v 0?

už víme $f'(0)=0$

ale co $f'(x)=?$ $(x\neq 0)$

$$f'(x) = \left(|x|^\alpha \cdot \sin\frac{1}{x}\right)' = \qquad (x\neq 0)$$

$$= \alpha \cdot |x|^{\alpha-1} \cdot \operatorname{sgn} x \cdot \sin\frac{1}{x} + |x|^\alpha \cdot \cos\frac{1}{x} \cdot \frac{-1}{x^2} =$$

$$= \underbrace{|x|^{\alpha-2}}_{\substack{\downarrow \\ 0 \\ \text{pro } \alpha>2}} \underbrace{\left(\alpha \cdot \underbrace{|x| \cdot \operatorname{sgn} x \cdot \sin\frac{1}{x}}_{\substack{\text{omezená} \\ \text{(víme)}}} - \underbrace{\cos\frac{1}{x}}_{\text{omez.}}\right)}_{\text{omezená}}$$

Věta: $f(x)\to 0$, $g(x)$ omez. $\Rightarrow f(x)\cdot g(x) \to 0$

V. o 2 pol.: $-|x|^{\alpha-2} \leq f'(x) \leq |x|^{\alpha-2}$

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ 0 & \Rightarrow & 0 & \Leftarrow & 0 \end{array} \Leftrightarrow \alpha > 2$$

Derivace základních fcí

$$a' = 0$$

$$(x^n)' = n \cdot x^{n-1} \quad (n \in \mathbb{N},\ x \in \mathbb{R})$$

$$(x^a)' = a \cdot x^{a-1} \quad (a \in \mathbb{R},\ x \in \mathbb{R}_+)$$

$$(\sqrt[n]{x})' = (x^{\frac{1}{n}})' = \frac{1}{n} \cdot x^{(\frac{1}{n}-1)}$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$$(e^x)' = e^x$$

$$(a^x)' = a^x \cdot \ln a$$

$$(\ln x)' = \frac{1}{x}$$

$$(\log_a x)' = \frac{1}{x \cdot \ln a}$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$$

$$(\operatorname{cotg} x)' = \frac{-1}{\sin^2 x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$$

$$(\operatorname{arctg} x)' = \frac{1}{1+x^2}$$

$$(\operatorname{arccotg} x)' = \frac{-1}{1+x^2}$$

$$\left[\ \sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}\ \right]$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

Pravidla: $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

Leibniz: $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$

chain rule: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Procvičení doma: 13.–29.