Cvičení MAF, 10.11.2020, 9^{00}

Musí být
$\sin x > 0$ i
$\cos x > 0$
$\Rightarrow D_f = (0, \frac{\pi}{2}) + 2k\pi$

(26) $f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x} =$

$= e^{\cos x \cdot \ln(\sin x)} + e^{\sin x \cdot \ln(\cos x)}$

$f'(x) = e^{\cos x \cdot \ln(\sin x)} \cdot \left(-\sin x \cdot \ln(\sin x) + \cos x \cdot \frac{\cos x}{\sin x}\right) +$

$+ \ldots$

(15) ? $\underset{y=}{\operatorname{argsinh}} x = \ln(x + \sqrt{x^2+1})$, $x \in \mathbb{R}$

$x = \sinh y = \frac{e^y - e^{-y}}{2} = \frac{t - \frac{1}{t}}{2}$

subst. $t = e^y$

$2x = t - \frac{1}{t}$ $/ \cdot t$

$2xt = t^2 - 1$

$\rightarrow t^2 - 2xt - 1 = 0$

$D = 4x^2 + 4$, $\sqrt{D} = 2\sqrt{x^2+1}$

$t = \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1}$

pro „$-$" je $t < 0$ ... nejde

pro „$+$" je $t > 0$ ... $y = \ln(x + \sqrt{x^2+1})$

Př. z 2. testu D. Pelc.

Spočtěte derivaci (vč. jednostr.) fce $f(x) = \sqrt{1-\cos(2x)}$ všude, kde $\exists$.

$\sqrt{x}$

jednostr. der. $+\infty$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}} \xrightarrow[x\to 0+]{} +\infty$$

a) $$f'(x) = \frac{\sin(2x)\cdot \cancel{2}}{\cancel{2}\sqrt{1-\cos(2x)}}$$

$D_f = \mathbb{R}$ protože $\cos(2x) \in \langle -1, 1\rangle$

$1-\cos(2x) \in \langle 0, 2\rangle$

$D_{f'}$: $\cos 2x \neq 1$

$2x \neq 2k\pi$ ... $D_{f'} = \mathbb{R} \setminus \{k\pi; k\in\mathbb{Z}\}$

$x \neq k\pi$

b) v bodech $k\pi$:

f je π-period. $\Rightarrow$ stačí vyšetřit v 0

$$f'_{\pm}(0) = \lim_{x\to 0\pm} f'(x) = \lim_{x\to 0\pm} \frac{\sin(2x)}{\sqrt{1-\cos(2x)}} = \lim_{x\to 0\pm} \underbrace{\frac{\sin(2x)}{2x}}_{\to 1} \cdot \underbrace{\sqrt{\frac{(2x)^2}{1-\cos(2x)}}}_{\to \sqrt{2}} \cdot \underbrace{\frac{2x}{|2x|}}_{=\operatorname{sgn} x}$$

$$= \pm\sqrt{2}$$

Spočtěte derivaci (vč. jednostr.)
všude, kde existuje:

$$f(x) = \arcsin \frac{2x}{x^2+1}$$

$D_f = ?$ $\quad -1 \leq \frac{2x}{x^2+1} \leq 1$

$D_f = \mathbb{R}$ ✓

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{2x}{x^2+1}\right)^2}} \cdot \frac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2} =$$

$$= \dots = \frac{2}{x^2+1} \cdot \frac{1-x^2}{\sqrt{(1-x^2)^2}} =$$

$$= \underline{\underline{\frac{2}{x^2+1} \cdot \operatorname{sgn}(1-x^2)}} \quad \text{pro } x \neq \pm 1$$

v ± 1:

$$f'_+(1) = \lim_{x\to 1+} f'(x) = -1 = f'_-(-1)$$

$$f'_-(1) = \lim_{x\to 1-} f'(x) = 1 = f'_+(-1)$$

Derivace vyšších řádů

$f''(x) = (f'(x))'$

$f'''(x) = (f''(x))'$

$f^{(4)}(x) = (f'''(x))'$

atd. $f^{(n)}(x)$

$\rightarrow$ Sami Př. 32, 33

Parciální derivace

u fcí více proměnných

$$f(x_1, x_2, x_3) = f(x)$$

$$\frac{\partial f}{\partial x_1} = \partial_{x_1} f = \text{parc. derivace podle 1. prom.}$$

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right)$$

atd.

30 Ověřte, že $u(x) = \frac{1}{|x|} = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$

splňuje Laplaceovu rovnici

$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} = 0$$

$$\frac{\partial u}{\partial x_1} = \frac{\partial}{\partial x_1}\left(x_1^2 + x_2^2 + x_3^2\right)^{-\frac{1}{2}} =$$

$$= -\frac{1}{2} \cdot \left(x_1^2 + x_2^2 + x_3^2\right)^{-\frac{3}{2}} \cdot 2x_1 =$$

$$= -x_1 \cdot \left(x_1^2 + x_2^2 + x_3^2\right)^{-\frac{3}{2}}$$

$$\frac{\partial^2 u}{\partial x_1^2} = -\left[1 \cdot \left(x_1^2 + x_2^2 + x_3^2\right)^{-\frac{3}{2}} + \right.$$

$$\left. + x_1 \cdot \left(-\frac{3}{2}\right) \cdot \left(x_1^2 + x_2^2 + x_3^2\right)^{-\frac{5}{2}} \cdot 2x_1\right] =$$

$$= \frac{-(x_1^2 + x_2^2 + x_3^2) + 3x_1^2}{(x_1^2 + x_2^2 + x_3^2)^{5/2}}$$ a podobně pro x_2, x_3

$$\Rightarrow \Delta u = \frac{-3(x_1^2 + x_2^2 + x_3^2) + 3(x_1^2 + x_2^2 + x_3^2)}{(x_1^2 + x_2^2 + x_3^2)^{5/2}} =$$

$$= 0$$

Sami doma: 31

Primitivní fce = Neurčité integrály

= „opačná operace k derivování"

$$F(x) = \int f(x)\,dx \iff f(x) = F'(x)$$

<u>Pozn</u>: primit. fce je na intervalu určena jednoznačně až na aditivní konstantu, často se píše $F(x) = \int f(x)\,dx + c \quad (c \in \mathbb{R})$

$$\int \frac{dx}{x^2+1} = \operatorname{arctg} x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$

Základní primit. fce:

$$\int 1\,dx = x$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}, \quad n \in \mathbb{N}, \; x \in \mathbb{R}$$

$$\int x^a\,dx = \frac{x^{a+1}}{a+1}, \quad a \in \mathbb{R} \setminus \{-1\}, \; x \in \mathbb{R}_+$$

(nebo větší D_f)

$$\int \frac{dx}{x} = \int \frac{1}{x}\,dx = \ln|x| \quad (x \neq 0)$$

$$\int e^x\,dx = e^x$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \sinh x\,dx = \cosh x$$

$$\int \cosh x\,dx = \sinh x$$

$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x$$

$$\int \frac{dx}{\sin^2 x} = \cancel{\operatorname{arctg}} -\operatorname{cotg} x$$

Aditivita: $\int f \pm g = \int f \pm \int g$

$a \in \mathbb{R} \Rightarrow \int a \cdot f = a \cdot \int f$

Per partes:

$\int f' \cdot g = f \cdot g - \int f \cdot g'$

① $\int \left(\frac{1-x}{x}\right)^2 dx = \int \left(\frac{1}{x} - 1\right)^2 dx =$

$= \int \left(\frac{1}{x^2} - \frac{2}{x} + 1\right) dx =$

$= \frac{-1}{x} - 2\ln|x| + x + c$

Cr.6 ① ② ③

② $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx = \int 2 \cdot \left(\frac{1}{5}\right)^x - \frac{1}{5}\left(\frac{1}{2}\right)^x dx =$

$= \int \left(2 \cdot e^{x \cdot \ln\left(\frac{1}{5}\right)} - \frac{1}{5} e^{x \cdot \ln\left(\frac{1}{2}\right)}\right) dx =$ $\boxed{\int e^{ax} dx = \frac{e^{ax}}{a}}$

$= 2 \cdot \frac{e^{x \cdot \ln\left(\frac{1}{5}\right)}}{\ln\left(\frac{1}{5}\right)} - \frac{1}{5} \frac{e^{x \ln\left(\frac{1}{2}\right)}}{\ln\left(\frac{1}{2}\right)} = \ldots$

(3) $\int tg^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx = \int\left(\frac{1}{\cos^2 x} - 1\right)dx =$

$= \underline{\underline{tg\,x - x + c}}$

$\int f'g = f\cdot g - \int f\cdot g'$

$\int x e^x dx = \begin{bmatrix} f'(x) = e^x & f(x) = e^x \\ g(x) = x & g'(x) = 1 \end{bmatrix} = x\cdot e^x - \int 1\cdot e^x dx =$

$= x\cdot e^x - e^x = \underline{\underline{e^x(x-1) + c}}$

$\int \ln x\,dx = \begin{bmatrix} f'(x) = 1 & f(x) = x \\ g(x) = \ln x & g'(x) = \frac{1}{x} \end{bmatrix} = x\cdot \ln x - \int \frac{x}{x}dx =$

$= x\cdot \ln x - x = \underline{\underline{x(\ln x - 1) + c}}$

Sami: Cv. 6 : 8, 16, 17, 18, 22