Cvičení MAF, 10. 11. 2020, 12:20

Leibnizovo pravidlo:

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

(15) dokažte: $\underset{y=}{\operatorname{argsinh}}\, x = \ln\left(x + \sqrt{x^2+1}\right)$, $x \in \mathbb{R}$

$$x = \sinh y = \frac{e^y - e^{-y}}{2} = \frac{t - \frac{1}{t}}{2} \quad /\cdot 2$$

označíme $t = e^y$

$$2x = t - \frac{1}{t} \quad /\cdot t$$

$$t^2 - 2xt - 1 = 0$$

$$D = 4x^2 + 4, \quad \sqrt{D} = 2\sqrt{x^2+1}$$

$$t = \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1}$$

$x - \sqrt{x^2+1} < 0 \Rightarrow$ nehodí se

$x + \sqrt{x^2+1} > 0 \Rightarrow y = \ln t = \ln\left(x + \sqrt{x^2+1}\right)$

Př. z 2. testu D. Yok.

Spočtěte derivaci (vč. jednostr.) fce $f(x)=\sqrt{1-\cos(2x)}$ všude, kde $\exists$.

$D_f=\mathbb{R}$: $\cos 2x \in \langle -1,1\rangle$

$\Rightarrow 1-\cos(2x)\in\langle 0,2\rangle$

$D_{f'}$: $\cos(2x)\neq 1$

$2x \neq 2k\pi$

$x = k\pi,\ k\in\mathbb{Z}$

$D_{f'}=\mathbb{R}\setminus\{k\pi,\ k\in\mathbb{Z}\}$

a) $$f'(x)=\frac{\sin(2x)\cdot \cancel{2}}{\cancel{2}\sqrt{1-\cos(2x)}}$$

b) v bodech $k\pi$:

f je π-periodická $\Rightarrow$ stačí vyšetřit v 0

$$f'_{\pm}(0)=\lim_{x\to 0\pm} f'(x)=\lim_{x\to 0\pm}\frac{\sin(2x)}{\sqrt{1-\cos(2x)}}\cdot\frac{\sqrt{1+\cos(2x)}}{\sqrt{1+\cos(2x)}}=\lim_{x\to 0\pm}\frac{\sin(2x)\cdot\sqrt{1+\cos(2x)}}{\sqrt{\sin^2(2x)}}=$$

$$=\lim_{x\to 0\pm}\frac{\sin(2x)}{|\sin(2x)|}\cdot\sqrt{2}=\lim_{x\to 0\pm}\operatorname{sgn}(\sin(2x))\cdot\sqrt{2}=\underline{\underline{\pm\sqrt{2}}}$$

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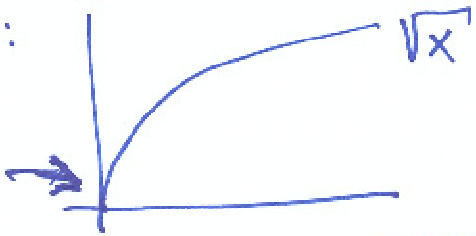
Jinak: $\lim\limits_{x\to 0\pm} \underbrace{\frac{\sin(2x)}{2x}}_{\to 1} \cdot \underbrace{\sqrt{\frac{(2x)^2}{1-\cos(2x)}}}_{\to\sqrt{2}} \cdot \underbrace{\frac{2x}{|2x|}}_{=\operatorname{sgn} x} = \underline{\underline{\pm\sqrt{2}}}$

Víme:



der. $= +\infty$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

Spočtěte derivaci, všude kde existuje (vč. jednostr.):

$$f(x) = \arcsin\frac{2x}{x^2+1}$$

D_f: pro arcsin je def. obor $\langle -1, 1\rangle$

chceme $-1 \le \dfrac{2x}{x^2+1} \le 1$

$$\frac{2x}{x^2+1} \le 1$$

$$2x \le x^2+1$$

$$0 \le x^2-2x+1 = (x-1)^2 \checkmark$$

podobně pro $\ldots \ge -1$ ✓

pro $\forall x \in \mathbb{R}$

$D_f = \mathbb{R}$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{2x}{x^2+1}\right)^2}} \cdot \frac{2\cdot(x^2+1)-2x\cdot 2x}{(x^2+1)^2}$$

$$= \ldots = \frac{2}{x^2+1} \cdot \frac{1-x^2}{\sqrt{(1-x^2)^2}} =$$

$$= \underline{\underline{\frac{2}{x^2+1} \cdot \operatorname{sgn}(1-x^2)}} \quad \text{pro } x \ne \pm 1$$

3]

$$f'_+(1) = \lim_{x\to 1+} \frac{2}{x^2+1} \cdot \underbrace{\operatorname{sgn}(1-x^2)}_{=-1} = -1 = f'_-(-1)$$

$$f'_-(1) = \lim_{x\to 1-} \text{———} \,''\, \text{———} = 1 = f'_+(-1)$$

Derivace vyšších řádů

$$f''(x) = (f'(x))'$$

$$f'''(x) = (f''(x))'$$

$$f^{(4)}(x) = (f'''(x))'$$

atd. $f^{(n)}(x)$

→ Sami 32, 33

PF.

Parciální derivace u funkcí více proměnných

$$f(x) = f(x_1, x_2, x_3)$$

$$\frac{\partial f}{\partial x_1} = \partial_{x_1} f = \text{parc. derivace podle } x_1$$

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right)$$

30 Ověřte, že $u(x) = \frac{1}{|x|} = \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}$ splňuje

Laplaceovu rovnici $\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} = 0.$

$$\frac{\partial u}{\partial x_1} = \frac{\partial}{\partial x_1}\left(x_1^2+x_2^2+x_3^2\right)^{-\frac{1}{2}} = -\frac{1}{2}\cdot\left(x_1^2+x_2^2+x_3^2\right)^{-\frac{3}{2}}\cdot 2x_1 =$$

$$= -x_1\cdot\left(x_1^2+x_2^2+x_3^2\right)^{-\frac{3}{2}}$$

$$\frac{\partial^2 u}{\partial x_1^2} = \cancel{\frac{\partial}{\partial x_1}} -\left[1\cdot\left(x_1^2+x_2^2+x_3^2\right)^{-\frac{3}{2}} + x_1\cdot\left(-\frac{3}{2}\right)\cdot\left(x_1^2+x_2^2+x_3^2\right)^{-\frac{5}{2}}\cdot 2x_1\right] =$$

$$= \frac{-\left(x_1^2+x_2^2+x_3^2\right)+3x_1^2}{\left(x_1^2+x_2^2+x_3^2\right)^{5/2}}$$ a podobně pro x_2, x_3

$$\Delta u = \frac{-3\left(x_1^2+x_2^2+x_3^2\right)+3\left(x_1^2+x_2^2+x_3^2\right)}{\left(x_1^2+x_2^2+x_3^2\right)^{5/2}} = \underline{\underline{0}}$$

Sami: 31 Př.

Leibnizův vzorec pro vyšší derivace:

$$(f\cdot g)' = f'\cdot g + f\cdot g'$$

$$(f\cdot g)'' = f''\cdot g + 2f'\cdot g' + f\cdot g''$$

$$(f\cdot g)''' = f'''\cdot g + 3f''\cdot g' + 3f'\cdot g'' + f\cdot g'''$$

Základní primitivní fce:

$$\int 1\,dx = x$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}, \quad n\in\mathbb{N}, x\in\mathbb{R}$$

$$\int x^a\,dx = \frac{x^{a+1}}{a+1}, \quad a\in\mathbb{R}, a\neq -1, \; x\in\mathbb{R}_+ \text{ (nebo větší } D_f)$$

$$\int \frac{1}{x}\,dx = \int\frac{dx}{x} = \ln|x| \quad (x\neq 0)$$

$$\int e^x\,dx = e^x$$

$$\int \sin x\,dx = -\cos x, \quad \int \cos x = \sin x$$

$$\int \sinh x\,dx = \cosh x, \quad \int \cosh x = \sinh x$$

$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x, \quad \int\frac{dx}{\sin^2 x} = -\operatorname{cotg} x$$

$$\int \frac{dx}{x^2+1} = \operatorname{arctg} x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$

Primitivní fce = Neurčité integrály

= „opačná derivace k derivování"

$$F(x) = \int f(x)\,dx \iff f(x) = F'(x)$$

Pozn: primit. fce je na intervalu určena jednoznačně až na aditivní konstantu, často se píše

$$F(x) = \int f(x)\,dx + C \quad (C\in\mathbb{R})$$

Aditivita: $\int (f \pm g) = \int f \pm \int g$

$a \in \mathbb{R} \Rightarrow \int af = a \int f$

Per partes:

$$\int f' g = f \cdot g - \int f \cdot g'$$

Cvič. 6: 1, 2, 3

(1) $$\int \left(\frac{1-x}{x}\right)^2 dx = \int \left(\frac{1}{x} - 1\right)^2 dx =$$

$$= \int \left(\frac{1}{x^2} - \frac{2}{x} + 1\right) dx = \underline{\underline{-\frac{1}{x} - 2\ln|x| + x + c}}$$

(2) $$\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx = \int 2 \cdot \left(\tfrac{1}{5}\right)^x - \tfrac{1}{5}\left(\tfrac{1}{2}\right)^x dx$$

$$= \int \left(2 \cdot e^{x \ln(\frac{1}{5})} - \frac{1}{5} \cdot e^{x \cdot \ln(\frac{1}{2})}\right) dx$$

$$= \left[\int e^{ax} dx = \frac{e^{ax}}{a} \right] =$$

$$= 2 \cdot \frac{e^{x \ln(\frac{1}{5})}}{\ln(\frac{1}{5})} - \frac{1}{5} \cdot \frac{e^{x \ln(\frac{1}{2})}}{\ln(\frac{1}{2})}$$

$= \dots$

(3) $\int tg^2 x \, dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - 1 \right) dx = \underline{\underline{tg\, x - x}} + c$

Per partes:

$= \int (tg\, x)\cdot(tg\, x)\, dx = \left[\begin{array}{ll} f' = tg\, x & f = ? \\ g = tg\, x & g' = \frac{1}{\cos^2 x} \end{array} \right]$

$\int tg\, x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x\, dx}{\cos x}$

$y = \cos x$

$dy = -\sin x\, dx$

$\sim -\int \frac{dy}{y} = -\ln|y|$

$\Rightarrow \int tg\, x\, dx = -\ln|\cos x| = f(x)$

$\int f g'$ je <u>ošklivý</u>

jinak: $\int \sin^2 x \cdot \frac{1}{\cos^2 x} dx = \left[\begin{array}{ll} f = \sin^2 x & f' = 2\sin x \cdot \cos x \\ g' = \frac{1}{\cos^2 x} & g = tg\, x \end{array} \right] = \sin^2 x \cdot tg\, x - \int 2\sin^2 x\, dx$

příště

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$$\int x e^x dx = \left[\begin{matrix} f' = e^x & f = e^x \\ g = x & g' = 1 \end{matrix} \right] = x \cdot e^x - \int 1 \cdot e^x dx = x e^x - e^x =$$

$$= \underline{\underline{e^x (x-1) + c}}$$

$$\int \ln x \, dx = \left[\begin{matrix} f' = 1 & f = x \\ g = \ln x & g' = \frac{1}{x} \end{matrix} \right] = x \cdot \ln x - \int x \cdot \frac{1}{x} dx = x \cdot \ln x - x =$$

$$= \underline{\underline{x(\ln x - 1) + c}}$$

<u>Sami</u>: Cv. 6 : 8, 16, 17, 18, 22