Cvičení MAF, 1.12.2020, 9^{00}

Písemka:

(1) $f(x) = \sqrt{1 - e^{-x^2}}$ $D_f = \mathbb{R}$

$$f' = \frac{x \cdot e^{-x^2}}{\sqrt{1 - e^{-x^2}}} \quad \text{pro } x \neq 0$$

$$f'_+(0) = \lim_{x \to 0+} f'(x) = \lim_{x \to 0+} \sqrt{\underbrace{\frac{-(-x^2)}{1 - e^{-x^2}}}_{(y = -x^2)\ \to 1}} \cdot \frac{e^{-x^2} \to 1}{1} \cdot \frac{x}{|x|} = 1$$

$$f'_-(0) = -1$$

(2) $$\int \frac{\ln^3 x + 1}{x(\ln^2 x - \ln x)}\, dx \quad \left| \begin{array}{l} y = \ln x \\ dy = \frac{dx}{x} \end{array} \right.$$

$$\int \frac{y^3 + 1}{y^2 - y}\, dy =$$

$$\int \underbrace{\frac{y^3 - y^2}{y^2 - y}}_{y} + \underbrace{\frac{y^2 - y}{y^2 - y}}_{1} + \frac{y + 1}{y^2 - y}\, dy =$$

$$\left[\frac{y+1}{y(y-1)} = \frac{-1}{y} + \frac{2}{y-1} \right]$$

$$= \frac{\overset{\ln x}{\cancel{y}}^2}{2} + \overset{\ln x}{\cancel{y}} - \ln|\overset{\ln x}{\cancel{y}}| + 2\ln|\overset{\ln x}{\cancel{y}} - 1| + C$$

D_f: $x > 0$, $\ln x \neq 0 \ldots x \neq 1$, $\ln x \neq 1 \ldots x \neq e$ $\Rightarrow (0,1) \cup (1,e) \cup (e, +\infty)$ ✓

$D_f = D_F$

Speciální substituce $\quad R = \text{rac. fce}$

(A) Goniometrické

(A1) $\int R(\cos x)\cdot \sin x \;\ldots\; y=\cos x$

$\int R(\sin x)\cdot \cos x \;\ldots\; y=\sin x$

(A2) $\int R(\cos^2 x, \sin^2 x) \;\ldots\; y=\operatorname{tg} x$

(A3) $\int R(\cos x, \sin x) \;\ldots\; y=\operatorname{tg}\frac{x}{2}$

„univerzální subst."

Př: $\int \frac{3\sin^2 x+\cos^2 x}{\sin^2 x+3\cos^2 x}\,dx$

sudé mocniny $\Rightarrow$ volíme (A2) $y=\operatorname{tg} x$

$$y=\operatorname{tg} x, \quad dy=\frac{dx}{\cos^2 x}, \quad dx=\frac{dy}{y^2+1}$$

$$y^2=\operatorname{tg}^2 x=\frac{\sin^2 x}{\cos^2 x}=\frac{1-\cos^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}-1$$

$$\Rightarrow \boxed{\cos^2 x=\frac{1}{y^2+1}}$$

$$\boxed{\sin^2 x=1-\cos^2 x=1-\frac{1}{y^2+1}=\frac{y^2}{y^2+1}}$$

$$\Rightarrow \int \frac{3\frac{y^2}{y^2+1}+\frac{1}{y^2+1}}{\frac{y^2}{y^2+1}+3\cdot\frac{1}{y^2+1}}\,\frac{dy}{y^2+1}=$$

$$=\int \frac{3y^2+1}{(y^2+3)(y^2+1)}\,dy=\int \frac{4}{y^2+3}-\frac{1}{y^2+1}\,dy$$

Pozn: $y=\operatorname{tg} x \;\ldots\; dy=\frac{dx}{\cos^2 x}$

$x=\operatorname{arctg} y \;\ldots\; dx=\frac{dy}{y^2+1}$

1. subst. $\qquad$ 2. subst.

%

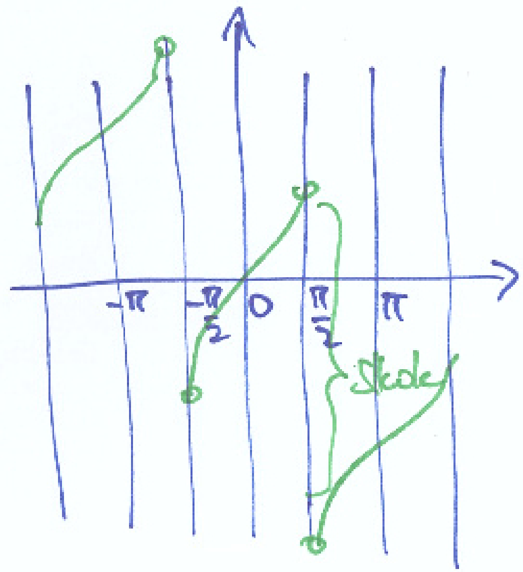
$$\to \int \frac{4}{y^2+3} - \frac{1}{y^2+1}\,dy = 4\int \frac{\sqrt{3}\cdot\frac{dy}{\sqrt{3}}}{3\left(\left(\frac{y}{\sqrt{3}}\right)^2+1\right)} - \operatorname{arctg} y = \underline{\frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{y}{\sqrt{3}}\right) - \operatorname{arctg} y}$$

$$\Rightarrow \underline{I(x) = \frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{\operatorname{tg} x}{\sqrt{3}}\right) - x}$$

Def. obor: $D_f = \mathbb{R}$,

ale!! $D_I = \mathbb{R} - \left\{\frac{\pi}{2} + k\pi;\ k \in \mathbb{Z}\right\}$

f spojitá $\Rightarrow$ má prim. fci v celém D_f $\Rightarrow$ <u>lepíme</u>



Potřebujeme jednostr. limity:

$$\lim_{x\to -\frac{\pi}{2}+} I(x) = \frac{4}{\sqrt{3}}\cdot\frac{-\pi}{2} + \frac{\pi}{2} = \frac{\pi}{2}\left(1 - \frac{4}{\sqrt{3}}\right)$$

$$\lim_{x\to \frac{\pi}{2}-} I(x) = \frac{4}{\sqrt{3}}\cdot\frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}\left(-1 + \frac{4}{\sqrt{3}}\right)$$

$$\lim_{x\to \frac{\pi}{2}+} I(x) = \frac{4}{\sqrt{3}}\cdot\frac{-\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}\left(-1 - \frac{4}{\sqrt{3}}\right)$$

$(-)$: $\frac{\pi}{2}\cdot\frac{8}{\sqrt{3}} = \frac{4\pi}{\sqrt{3}}$ = <u>skok</u>

Pozn: tento skok je <u>stejný</u> ve $\forall$ bodech tvaru $\frac{\pi}{2} + k\pi$, ač hodnoty limit se liší

$$\lim_{x\to -\frac{\pi}{2}+} \operatorname{tg} x = -\infty$$

$\Rightarrow$ Definujeme prim. fci $F(x)$ takto:

$$\left.\begin{aligned} F(x) &= I(x) + k\cdot\frac{4\pi}{\sqrt{3}} \quad \text{pro } x\in\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right), \ (k\in\mathbb{Z}) \\ F\left(\frac{\pi}{2}+k\pi\right) &= \left(\frac{4-\sqrt{3}+8k}{\sqrt{3}}\right)\cdot\frac{\pi}{2} \end{aligned}\right\} \text{ je spojitá v } \mathbb{R}$$

Sami doma: Cv7. : 7, 8, 9, 10

Ještě: Př: $\displaystyle\int \frac{dx}{\sin x - \cos x} =$

$$= \int \frac{2\cdot\cos^2\frac{x}{2}}{\sin x - \cos x}\cdot\frac{dx}{2\cos^2\frac{x}{2}}$$

atd.

$$y = \operatorname{tg}\frac{x}{2} \qquad dy = \frac{dx}{2\cos^2\frac{x}{2}}$$

$$y^2 = \operatorname{tg}^2\frac{x}{2} = \frac{1}{\cos^2\frac{x}{2}} - 1$$

$$\cos^2\frac{x}{2} = \frac{1}{1+y^2}, \quad \sin^2\frac{x}{2} = \frac{y^2}{1+y^2}$$

$$\boxed{\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-y^2}{1+y^2}}$$

$$\boxed{\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}} = 2\,\frac{\sqrt{y^2}}{1+y^2} =$$

$$= \boxed{\frac{2y}{1+y^2}}$$

$$\left[\begin{aligned} &x>0 \Rightarrow y = \operatorname{tg}\frac{x}{2} > 0 \\ &x\in(0, \pi) \end{aligned}\right]$$

(B) s odmocninami

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{s}}\right) dx$$

subst. $y = \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{s}}$

Zahrnuje i $x^{\frac{1}{s}}$, zejména $\sqrt{x}, \sqrt[3]{x}, \dots$

(C) Eulerovy subst.

$$\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$$

(a) ax^2+bx+c má 2 reálné kořeny x_1, x_2

$\Rightarrow$ subst. $y = \left(\frac{x-x_1}{x-x_2}\right)^{\frac{1}{2}}$

(b) $\sqrt{ax^2+bx+c} = \sqrt{a}\cdot x + t$ $(a>0)$

$\sqrt{ax^2+bx+c} = \sqrt{c} + tx$ $(c>0)$

Sami doma: Př 3

\+ DÚ4 (2,4)

(5) $\displaystyle\int \frac{x+\sqrt{1+x+x^2}}{1+x+\sqrt{1+x+x^2}}\, dx \Rightarrow \sqrt{x^2+x+1} = x+t \quad /()^2$

napřed vyjádřit x pomocí t: $x^2+x+1 = x^2+2tx+t^2$

$\Rightarrow \quad x = \frac{t^2-1}{1-2t} = \frac{1-t^2}{2t-1}$

$$\frac{dx}{dt} = \frac{-2t^2+2t-2}{(2t-1)^2}$$

$$\sqrt{1+x+x^2} = x+t = \frac{t^2-t+1}{2t-1}$$

$$\Rightarrow \int \frac{\frac{1-t^2}{2t-1} + \frac{t^2-t+1}{2t-1}}{\frac{2t-1}{2t-1} + \frac{1-t^2}{2t-1} + \frac{t^2-t+1}{2t-1}} \cdot \frac{-2t^2+2t-2}{(2t-1)^2}\, dt =$$

atd.