Cvičení MAF, 1.12.2020, 12:20

Písemka: (1) $f(x) = \arccos \frac{1}{1+x^2}$ $\quad D_f = \mathbb{R}$

pro $x \neq 0$

$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}} \cdot \frac{-2x}{(1+x^2)^2} = \dots = \frac{2x}{\sqrt{2x^2+x^4}\,(1+x^2)} = \frac{2\,\mathrm{sgn}\,x}{\underbrace{\sqrt{x^2+2}}_{\to\sqrt{2}}\,\underbrace{(1+x^2)}_{\to 1}}$$

$$f'_+(0) = \lim_{x\to 0+} f'(x) = \sqrt{2}, \quad f'_-(0) = -\sqrt{2}$$

(2) $$\int \frac{e^{3x}+4e^x}{e^{2x}-4}\,dx = \int \frac{e^{2x}+4}{e^{2x}-4}\,e^x dx \quad \left[\begin{array}{l} y = e^x \\ dy = e^x dx \end{array}\right. : \int \frac{y^2+4}{y^2-4}\,dy =$$

$$= \int \left(\frac{y^2-4}{y^2-4} + \frac{8}{y^2-4}\right) dy \qquad \left[\frac{8}{y^2-4} = \frac{2}{y-2} - \frac{2}{y+2} = \frac{8}{(y-2)\cdot(y+2)}\right]$$

$$= \int 1 + \frac{2}{y-2} - \frac{2}{y+2}\,dy = y + 2\ln|y-2| - 2\ln|y+2| = \underline{y + 2\ln\left|\frac{y-2}{y+2}\right|}$$

D_f: $e^x \neq 2$, ($e^x = -2$ … nikdy): $D_f = \mathbb{R} \setminus \{\ln 2\}$

$\| $

D_F

$$\Rightarrow F(x) = e^x + 2\ln\left|\frac{e^x-2}{e^x+2}\right| + c$$

1

$$\int \frac{1}{y+2}\,dy = \ln|y+2|$$

$$\int \frac{e^x}{e^x+2}\,dx = \ln|e^x+2|$$

$$\int \frac{y}{y+2}\,dy$$

$$\int \frac{e^x}{e^x+2}\,e^x dx$$

Př: $\displaystyle\int \frac{3\sin^2 x + \cos^2 x}{\sin^2 x + 3\cos^2 x}\,dx$

sudé mocniny $\Rightarrow$ (A2), $y = \operatorname{tg} x$

Speciální substituce

R = rac. fce

(A) Goniometrické

(A1) $\int R(\cos x)\cdot \sin x\,dx \;\dots\; y = \cos x$

$\int R(\sin x)\cdot \cos x\,dx \;\dots\; y = \sin x$

(A2) $\int R(\cos^2 x, \sin^2 x)\,dx \;\dots\; y = \operatorname{tg} x$

(A3) $\int R(\cos x, \sin x)\,dx \;\dots\; y = \operatorname{tg}\frac{x}{2}$

„univerzální subst."

$y = \operatorname{tg} x$

$$y^2 = \operatorname{tg}^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$$

$$\Rightarrow \boxed{\cos^2 x = \frac{1}{y^2+1}}$$

$$\boxed{\sin^2 x = 1-\cos^2 x = 1 - \frac{1}{y^2+1} = \frac{y^2}{y^2+1}}$$

$y = \operatorname{tg} x \;\dots\; dy = \dfrac{dx}{\cos^2 x}$ ← 1. subst.

$x = \operatorname{arctg} y \;\dots\; dx = \dfrac{dy}{y^2+1}$ ← 2. subst.

$$\Rightarrow \int \frac{3\frac{y^2}{y^2+1} + \frac{1}{y^2+1}}{\frac{y^2}{y^2+1} + 3\cdot\frac{1}{y^2+1}}\cdot\frac{dy}{y^2+1} = \int \frac{3y^2+1}{(y^2+3)(y^2+1)}\,dy = \int\left(\frac{4}{y^2+3} + \frac{-1}{y^2+1}\right)dy$$

./.

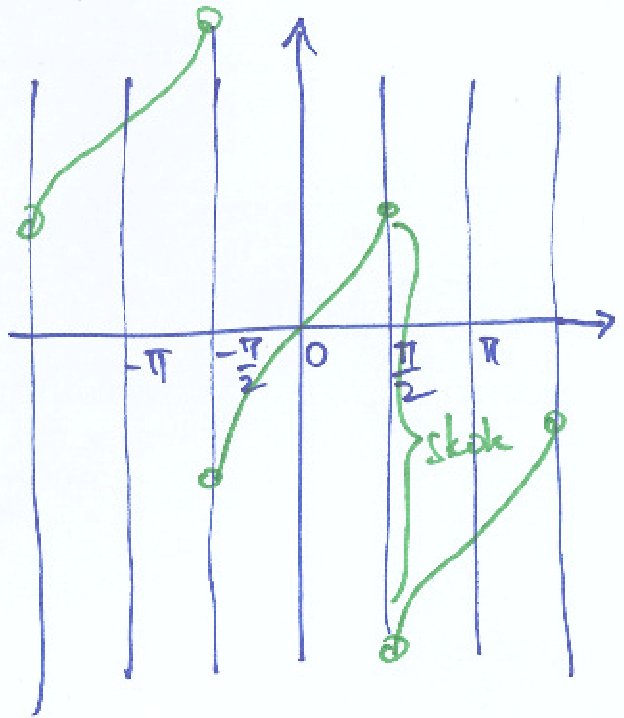
$$\int \frac{4}{y^2+3} - \frac{1}{y^2+1}\,dy = 4\int \frac{\sqrt{3}\,\frac{dy}{\sqrt{3}}}{3\left(\left(\frac{y}{\sqrt{3}}\right)^2+1\right)}\,dy - \operatorname{arctg} y = \underline{\frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{y}{\sqrt{3}}\right) - \operatorname{arctg} y + c}$$

$$\Rightarrow \underline{I(x) = \frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{\operatorname{tg} x}{\sqrt{3}}\right) - x + \cancel{c}}$$

$D_f = \mathbb{R}$

ale!! $D_I = \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi;\ k \in \mathbb{Z}\right\}$

f spojitá v int. J $\Rightarrow$ má v J prim. fci $\Rightarrow$ **lepíme**



Potřebujeme jednostr. limity:

$$\lim_{x\to -\frac{\pi}{2}+} I(x) = \frac{4}{\sqrt{3}} \cdot \frac{-\pi}{2} + \frac{\pi}{2} = \frac{\pi}{2}\left(1 - \frac{4}{\sqrt{3}}\right)$$

$$\lim_{x\to \frac{\pi}{2}-} I(x) = \frac{4}{\sqrt{3}} \cdot \frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}\left(-1 + \frac{4}{\sqrt{3}}\right)$$

$$\lim_{x\to \frac{\pi}{2}+} I(x) = \frac{4}{\sqrt{3}} \cdot \frac{-\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}\left(-1 - \frac{4}{\sqrt{3}}\right)$$

$\ominus$ $\frac{\pi}{2} \cdot \frac{8}{\sqrt{3}} = \underline{\underline{\frac{4\pi}{\sqrt{3}}}}$

skok

Pozn: tento skok je stejný ve $\forall$ bodech tvaru $\frac{\pi}{2} + k\pi$, ač limity se liší

$$\lim_{x\to -\frac{\pi}{2}+} \operatorname{tg} = -\infty$$

$\Rightarrow$ Definujeme primit. fci $F(x)$ takto:

$$F(x) = I(x) + k \cdot \frac{4\pi}{\sqrt{3}} \quad \text{pro } x \in \left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right), \; k \in \mathbb{Z}$$

$$F\left(\frac{\pi}{2}+k\pi\right) = \frac{\pi}{2}\left(-1+\frac{4}{\sqrt{3}}\right) + k\frac{4\pi}{\sqrt{3}} = \left(\frac{4-\sqrt{3}+8k}{\sqrt{3}}\right)\frac{\pi}{2}$$

je spojitá v $\mathbb{R}$

Př: $\displaystyle\int \frac{dx}{\sin x - \cos x} =$

$$= \int \frac{2\cos^2\frac{x}{2}}{\sin x - \cos x} \cdot \frac{dx}{2\cos^2\frac{x}{2}} =$$

$= \dots$ atd.

$$y = \operatorname{tg}\frac{x}{2} \qquad dy = \frac{dx}{2\cos^2\frac{x}{2}}$$

$$y^2 = \operatorname{tg}^2\frac{x}{2} = \frac{1}{\cos^2\frac{x}{2}} - 1$$

$$\cos^2\frac{x}{2} = \frac{1}{1+y^2}, \quad \sin^2\frac{x}{2} = 1-\cos^2\frac{x}{2} = \frac{y^2}{1+y^2}$$

$$\boxed{\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-y^2}{1+y^2}}$$

$$\boxed{\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2} = \frac{2\sqrt{y^2}}{1+y^2} = \frac{2y}{1+y^2}}$$

$\left[\; x \in (0,\pi) \quad y = \operatorname{tg}\frac{x}{2} > 0 \right.$
$\left. \sin x > 0 \;\right]$

Sami doma:

Cv. 7 : 7, 8, 9, 10

(B) s odmocninami

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{s}}\right) dx$$

subst. $y = \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{s}}$

zahrnuje i $\sqrt{x}$, $\sqrt[3]{x}$, ...

Sami doma: Př 3 (či 7)

+ DÚ4 (2,4).

(C) Eulerovy substituce:

$$\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$$

(a) ax^2+bx+c má 2 reálné kořeny x_1, x_2

$$\Rightarrow y = \left(\frac{x-x_1}{x-x_2}\right)^{\frac{1}{2}}$$

(b) $\sqrt{ax^2+bx+c} = \sqrt{a}\cdot x + t$ $(a>0)$

$\sqrt{ax^2+bx+c} = \sqrt{c} + t\cdot x$ $(c>0)$

(5) $\int \frac{x+\sqrt{1+x+x^2}}{1+x+\sqrt{1+x+x^2}} dx \Rightarrow \sqrt{x^2+x+1} = x+t \ /(\ )^2$

napřed vyjádříme x pomocí t:

$$\cancel{x^2}+x+1 = \cancel{x^2}+2tx+t^2$$

$$x = \frac{t^2-1}{1-2t} = \frac{1-t^2}{2t-1}$$

$$\frac{dx}{dt} = \frac{-2t^2+2t-2}{(2t-1)^2}$$

$$\sqrt{1+x+x^2} = x+t = \frac{t^2-t+1}{2t-1} \Rightarrow$$

$$\int \frac{\frac{1-t^2}{2t-1}+\frac{t^2-t+1}{2t-1}}{\frac{2t-1}{2t-1}+\frac{1-t^2}{2t-1}+\frac{t^2-t+1}{2t-1}} \cdot \frac{-2t^2+2t-2}{(2t-1)^2} dt = \dots$$

5)

Sami doma: Př. 3,4,6

(11,12 – gonio, hyperb.)