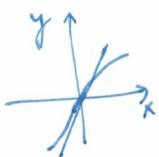


6. hodina

1

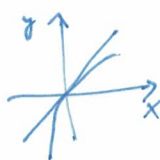
$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$



$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$



$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{y \rightarrow 0^+} \frac{\ln y}{y-1} = 1$$

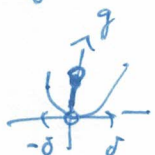


$$\lim_{x \rightarrow a} f(x)^{g(x)} = e^{\lim_{x \rightarrow a} g(x) \ln f(x)}$$

①  $\lim_{x \rightarrow 0} \frac{\sqrt{1-\cos x^2}}{1-\cos x} = \lim_{x \rightarrow 0} \sqrt{\frac{1-\cos x^2}{x^4}} \cdot \frac{x^2}{1-\cos x} \stackrel{VOL + VOLSF}{=} \sqrt{\frac{1}{2}} \cdot 2$ , *že jsme*

potřili  $\lim_{x \rightarrow 0} \frac{1-\cos x}{x^2} = \lim_{x \rightarrow 0} \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)x^2} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos x} = \frac{1}{2}$

až VOLSF. Podmínky  $f(y) = \frac{1-\cos y}{y^2}$   $g(x) = x^2$  :  
 1.  $\lim_{x \rightarrow 0} g(x) = 0$   
 2.  $\lim_{y \rightarrow 0} f(y) = \frac{1}{2}$  viz výše

+3.  $g(x) \neq 0$  na  $P_\delta(0)$  (dohovore  $\delta > 0$ ):  (Lze použít i 2.)

VOLSF, vím-li, že  $f$  je spojitá ...  $f$  ale takto def. není ani v  $y=0$  definovaná, ~~musí~~ bych dodefinovat ... snadně  $f(0) := \frac{1}{2}$ .

Dále VOLSF ještě na  $f(y) = \sqrt{y}$ ,  $y \geq 0$ ,  $g(x) = \frac{1-\cos x^2}{x^4}$  :

1.  $\lim_{x \rightarrow 0} g(x) = \frac{1}{2}$ , 2.  $\lim_{y \rightarrow \frac{1}{2}} f(y) = \sqrt{\frac{1}{2}}$ , 3.  $g(x) \neq \frac{1}{2}$ ? ... Raději 2.

VOLSF :  $\lim_{x \rightarrow 0} g(x) = \frac{1}{2}$ ,  $f$  spojitá  $\Rightarrow \lim_{x \rightarrow 0} (f \circ g)(x) = f\left(\frac{1}{2}\right)$

•  $\lim_{x \rightarrow 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$  do paměti či libitě  $\Rightarrow$  odvodit.

jak  $\frac{\sin x}{x}$  či  $\frac{1-\cos x}{x^2}$ .

②  $\lim_{x \rightarrow 0} \frac{\ln x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} \cdot \frac{\cos x}{\cos x} = \lim_{x \rightarrow 0} \frac{\sin x - \sin x \cos x}{x^3} =$

$$= \lim_{x \rightarrow 0} \frac{\sin x (1 - \cos x)}{x x^2} \stackrel{\text{VOAL}}{=} 1 \cdot \frac{1}{2} = \frac{1}{2} \text{ (opet d\u00edl. k\u00fasla).}$$

$$\textcircled{3} \lim_{x \rightarrow 0} \frac{\sin mx}{\sin nx} = \lim_{x \rightarrow 0} \frac{\sin mx}{mx} \cdot \frac{mx}{\sin mx} \cdot \frac{m}{n} = \frac{m}{n} \lim_{y_1 \rightarrow 0} \frac{\sin y_1}{y_1} = \frac{m}{n}$$

$\text{VOAL}$   
 $\text{VOLSF}$

$$\lim_{y_2 \rightarrow 0} \frac{y_2}{\sin y_2} = \frac{m}{m} \quad \forall m, n \in \mathbb{N}$$

$$\textcircled{4} \lim_{x \rightarrow \pi} \frac{\sin mx}{\sin nx} \left| \begin{array}{l} y = x - \pi \\ \text{VOLSF ok:} \\ y \text{ je injektivn\u00ed} \end{array} \right| = \lim_{y \rightarrow 0} \frac{\sin(m(y+\pi))}{\sin(n(y+\pi))} =$$

$$= \left| \begin{array}{l} \sin(my + m\pi) = \sin my \cos m\pi + \sin m\pi \cos my = \\ = -\sin my (-1)^m \end{array} \right| =$$

$$= \lim_{y \rightarrow 0} \frac{\sin my (-1)^m}{\sin ny (-1)^n} \stackrel{\text{dle p\u0159edch. limity}}{=} (-1)^{n-m} \frac{m}{n}$$

$$\textcircled{5} \lim_{x \rightarrow 1} \frac{\sin \pi x}{1-x} = \left| \begin{array}{l} y = x-1 \\ x = y+1 \end{array} \right| = \lim_{y \rightarrow 0} \frac{\sin(\pi y + \pi)}{-y} = \lim_{y \rightarrow 0} \frac{-\sin \pi y}{-y} \pi =$$

$$= \left| \begin{array}{l} z = \pi y \\ \text{p\u0159edp.} \\ \text{VOLSF(1)} \end{array} \right| = \lim_{z \rightarrow 0} \frac{\sin z}{z} \pi = \pi$$

$$\textcircled{6} \lim_{x \rightarrow 0} \frac{\ln(1+x e^x)}{\ln(x + \sqrt{1+x^2})} = \lim_{x \rightarrow 0} \frac{\ln(1+x e^x)}{x e^x} \cdot \frac{x e^x}{\ln(x + \sqrt{1+x^2})} \cdot \frac{(x + \sqrt{1+x^2} - 1)}{(x + \sqrt{1+x^2} - 1)}$$

$$\stackrel{\text{VOLSF}}{=} \lim_{x \rightarrow 0} \left( \frac{\ln(1+x e^x)}{x e^x} \right) \cdot \lim_{x \rightarrow 0} \frac{x e^x}{\ln(x + \sqrt{1+x^2})} \cdot \lim_{x \rightarrow 0} \frac{x e^x}{x + \sqrt{1+x^2} - 1} = 1 \cdot 1.$$

$$\lim_{x \rightarrow 0} \frac{x e^x}{x + \sqrt{1+x^2} - 1} = \lim_{x \rightarrow 0} \frac{x e^x - x + x}{x} \cdot x = \lim_{x \rightarrow 0} \frac{x e^x - x + x}{x + \sqrt{1+x^2} - 1} =$$

kde / jak  $\frac{e^x - 1}{x}$       opět: pokud limitu napravo extrahuji!

$$= \lim_{x \rightarrow 0} \frac{[x \left(\frac{e^x - 1}{x}\right) + \frac{x}{x}] \cdot x}{x + \sqrt{1+x^2} - 1} = \lim_{x \rightarrow 0} \left[ \frac{x(e^x - 1) + x}{x} \right] \lim_{x \rightarrow 0} x$$

$$\frac{x}{x + \sqrt{1+x^2} - 1} \cdot \frac{x + \sqrt{1+x^2} + 1}{x + \sqrt{1+x^2} + 1} = \lim_{x \rightarrow 0} \frac{x^2 + x\sqrt{1+x^2} + x}{x^2 + 1 + x^2 + 2x\sqrt{1+x^2} - 1} =$$

$$= \lim_{x \rightarrow 0} \frac{x^2 + x\sqrt{1+x^2} + x}{x^2 + x^2 + 2x\sqrt{1+x^2}} = \lim_{x \rightarrow 0} \frac{x + \sqrt{1+x^2} + 1}{x + x + 2\sqrt{1+x^2}} = 1 \cdot \frac{2}{2} = 1$$

7)  $b \neq 0$   
 $\lim_{x \rightarrow 0} \frac{\ln(\cos ax)}{\ln(\cos bx)} = \lim_{x \rightarrow 0} \frac{\ln(\cos ax - 1 + 1)}{\ln(\cos bx - 1 + 1)} = \lim_{x \rightarrow 0} \frac{\ln(\cos ax - 1 + 1)}{\cos ax - 1}$

$$\lim_{x \rightarrow 0} \frac{\cos bx - 1}{\ln(\cos bx - 1 + 1)} \cdot \lim_{x \rightarrow 0} \frac{\cos ax - 1}{\cos bx - 1} = 1 \cdot 1 \cdot \lim_{x \rightarrow 0} \frac{\cos ax - 1}{(ax)^2}$$

$$\lim_{x \rightarrow 0} \frac{(bx)^2}{\cos bx - 1} \cdot \lim_{x \rightarrow 0} \frac{(ax)^2}{(bx)^2} = -\frac{1}{2} \cdot 2 \frac{a^2}{b^2} = \frac{a^2}{b^2} \quad (b \neq 0 \text{ etc})$$

8)  $\lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} e^{\frac{\ln(1+x)}{x}} = e^{\lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x}} = e^1 = e$

9)  $\lim_{x \rightarrow 0} \frac{1 + \lg x}{1 + \sin x}$       Nejprve spoteme:  $\lim_{x \rightarrow 0} \frac{1}{\sin^3 x} \ln\left(\frac{1 + \lg x}{1 + \sin x}\right)$

$f(x)$

$$\lim_{x \rightarrow 0} \left[ \ln \frac{1 + \lg x}{1 + \sin x} \cdot \frac{1}{\sin^3 x} \right] = \lim_{x \rightarrow 0} \frac{\ln \left( \frac{1 + \lg x}{1 + \sin x} \right) \left( \frac{1 + \lg x}{1 + \sin x} - 1 \right)}{\frac{1 + \lg x}{1 + \sin x} - 1} \cdot \frac{1}{\sin^3 x} \quad (4)$$

$$\stackrel{!}{=} \lim_{x \rightarrow 0} \left( \frac{1 + \lg x}{1 + \sin x} - 1 \right) \frac{1}{\sin^3 x} = \lim_{x \rightarrow 0} \left( \frac{1 + \lg x - 1 - \sin x}{1 + \sin x} \cdot \frac{1}{\sin^3 x} \right) =$$

$$= \lim_{x \rightarrow 0} \frac{\lg x - \sin x}{1 + \sin x} \cdot \frac{1}{\sin^3 x} \stackrel{\text{bylo...}}{=} \lim_{x \rightarrow 0} \left( \frac{\lg x - \sin x}{x^3} \cdot \frac{x^3}{\sin^3 x} \cdot \frac{1}{1 + \sin x} \right)$$

$$= \frac{1}{2} \lim_{x \rightarrow 0} \left( \frac{x}{\sin x} \right)^3 \cdot \frac{1}{1} = \frac{1}{2} \Rightarrow \lim_{x \rightarrow 0} f(x) = e^{1/2}$$

$$(10) \lim_{x \rightarrow 0} \frac{1 - \cos x \cos 2x \cos 3x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{1 - \cos x \cos 2x \cos 3x}{x^2} \cdot \frac{x^2}{1 - \cos x} \stackrel{!}{=}$$

$$= 2 \lim_{x \rightarrow 0} \frac{1 - \cos x \cos 2x \cos 3x}{x^2} = 2 \lim_{x \rightarrow 0} \frac{1 - \cos x + \cos x (1 - \cos 2x \cos 3x)}{x^2} =$$

$$\stackrel{!}{=} \left[ \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} + \cos 0 \lim_{x \rightarrow 0} \frac{1 - \cos 2x \cos 3x}{x^2} \right] = 2 \left[ \frac{1}{2} + \lim_{x \rightarrow 0} \frac{1 - \cos 2x + \cos 2x (1 - \cos 3x)}{x^2} \right]$$

$$\left[ \frac{1 - \cos 3x}{x^2} \right] \stackrel{\text{VLSF } y=2x}{=} \left[ \frac{1}{2} + \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{4x^2} + \cos 0 \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{9x^2} \cdot 9 \right] \stackrel{!}{=}$$

$$= 2 \left( \frac{1}{2} + \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 9 \right) = 2 \frac{1 + 4 + 9}{2} = 14$$

Pozn.: Podmíněnost rovnosti:



$$\lim_{x \rightarrow 0} \frac{x}{x} \stackrel{\text{Lemna}}{=} \lim_{x \rightarrow 0} \left( \frac{x}{x} + \frac{1}{x} - \frac{1}{x} \right) \stackrel{\text{VOAL}}{=} \lim_{x \rightarrow 0} \left( \frac{x}{x} + \frac{1}{x} \right) - \lim_{x \rightarrow 0} \frac{1}{x}$$

$\lim_{x \rightarrow 0} \frac{1}{x}$  ale neexistuje!

$$\lim_{x \rightarrow 0} \frac{1}{x} \text{ neexistuje } \left( \lim_{x \rightarrow 0} \left( \frac{x}{x} + \frac{1}{x} \right) \text{ také} \right)$$

Odtud neplatí, že  $\lim_{x \rightarrow 0} \frac{x}{x}$  neexistuje. VOAL říká: Pokud

limity napravo  $\exists$ , existuje lim. ualevo.

Měli bychom psát  $\lim_{x \rightarrow 0} \left( \frac{x}{x} + \frac{1}{x} - \frac{1}{x} \right) \stackrel{\text{VOAL}}{=} \lim_{x \rightarrow 0} \left( \frac{x}{x} + \frac{1}{x} \right) - \lim_{x \rightarrow 0} \frac{1}{x}$

Pozn.: Dále: neplatnost část. dosazení:

$$\lim_{x \rightarrow 0^+} \left( 1 + x \right)^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} 1^{\frac{1}{x}} \stackrel{\text{Lemna o rovnosti na prstencovém okolí}}{=} \lim_{x \rightarrow 0^+} 1 = 1$$

$\uparrow$   $\text{ekvativ} = 1$

Chyba je zde: na to nemáme větvu

Zkusme  $\lim_{x \rightarrow 0^+} e^{\frac{1}{x} \ln(1+x)} = e^{\lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x}} = e^1$

• VOALSF najde pro  $f(y) = (1+y)^{\frac{1}{y}}$   $g(x) = 1+x$

•  $(1+x)^{\frac{1}{x}} = e^{\frac{\ln(1+x)}{x}}$  :  $f(y) = e^y$   $g(x) = \frac{\ln(1+x)}{x}$  To to

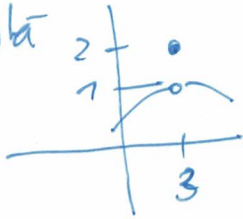
skládání ale už jde

# Nespojitost - spojitosť

(6)

(A) Spojitosť v  $a \Rightarrow f$  v  $a$  určité def  $\wedge \lim_{x \rightarrow a} f(x) = f(a)$ , wj. wj. wj. wj.

(B) Ma' lim  $\nRightarrow$  spojita



$$\lim_{x \rightarrow 3} ex \dots 1$$

$$f(3) \dots 2$$

$$\lim_{x \rightarrow 3} f(x) \neq f(3)$$

nespojita' (nepretrvna'aja, not continuous, unstetig)

Nonexistence of limit (0.)  $\lim_{x \rightarrow a} f(x) \neq \lim_{x \rightarrow a} f(x)$  nebojedu. uexistuju

1. z def:  $\forall \epsilon > 0 \exists \delta > 0 \forall x \in P_\delta(a) \quad f(x) \notin U_\epsilon(A)$

(limita není A:  $\exists \epsilon > 0 \forall \delta > 0 \exists x \in P_\delta(a) \quad f(x) \notin U_\epsilon(A)$ )

$$[U_\epsilon(A) := (A - \epsilon, A + \epsilon) = \{x \mid |x - A| < \epsilon\}]$$

Pr.: Dir(x) =  $\begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$  není spojita v žádném bodě (=  $\forall \neq$

bodě je nespojita) a nikde nemá limitu.

$x_0 \in \mathbb{R}$  a pro spor lim Dir(x) = L.

a)  $L < 0$   $|f(x) - L| < \epsilon$ ?  $|f(x) - L| \geq |L|$

Zvol  $\epsilon = |L| \dots \delta > 0$  lib.  $x \in P_\delta(x_0)$  lib.

$$|f(x) - L| < |L| < |L|, \text{ h. w. } |1 - L| < |L|!$$

$$|f(x) - L| < |L| \neq |L|, \quad x \in \mathbb{R} \setminus \mathbb{Q}$$

b)  $L > 1$   $|f(x) - L| \geq |L - 1|$ , obdobně zvol  $\epsilon = |L - 1|$ .

c)  $L \in [0, 1]$ ;  $\epsilon := \frac{1}{2} \max\{L, 1 - L\}$   $\delta > 0$   $x \in P_\delta(x_0)$

1.  $\epsilon = \frac{1}{2}L$   $|f(x) - L| = L \neq \frac{1}{2}L$ ,  $x \text{ irrac}$

2.  $\epsilon = \frac{1-L}{2}$   $|f(x) - L| = 1 - L \neq \frac{1-L}{2}$ ,  $x \text{ rac}$

Používáme  $\forall \delta > 0$   $(a - \delta, a + \delta)$  obsahuj' racionální i' iracionální u'pr

Ilustrace (snažíš fakt) : Neumí líni  $\lim_{x \rightarrow 0} \text{Dir}(x) = 1$

Dk. :  $\epsilon = \frac{1}{2}$  :  $|\text{Dir}(x) - 1| < \frac{1}{2}$  na  $(-\delta, \delta)$  ?

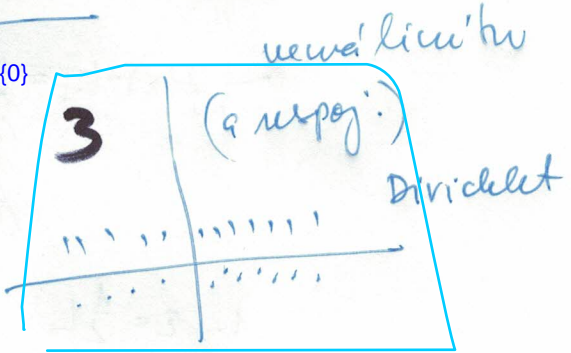
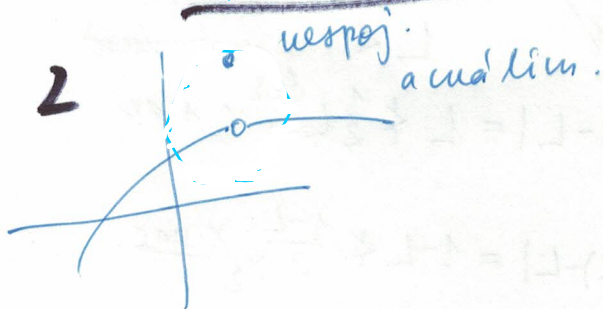
$x$  irac  $\in (-\delta, \delta)$  :  $|1 - 1| < \frac{1}{2}$  OK

Zvol ale  $x$  irac  $\in (-\delta, \delta)$  :  $|0 - 1| \not< \frac{1}{2}$  Neumí OK.

Stací tedy vařit lib. irac.

[Covíme dokázat?  $\exists \epsilon > 0 \forall \delta \exists x, (x \in (-\delta, \delta) \text{ irac})$   
 $|\text{Dir}(x) - 1| \not< \frac{1}{2}$ ]

Vždy můžeme dělat:



nespoj. a uca' líni

uca' líni'ku

# Spojitosť

1.  $f(x) = \frac{1 - \cos x}{x^2}, x \in \mathbb{R} \setminus \{0\}$

$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}, f(0) := \frac{1}{2}$

$f$  je spojita:  $\lim_{x \rightarrow 0} f(x) = \frac{1}{2} = f(0)$ .

2. a)  $f(x) = e^{-\frac{1}{x}}, D_f = \mathbb{R} \setminus \{0\}$ .  $f$  je spojita v svom def oboru:  $\frac{1}{x}$  spoj.

a exp. spoj.  $\nabla$  spojita!

Lze ji spoj. dodefi novat v nule? ( $\lim_{x \rightarrow 0^+} e^{-\frac{1}{x}} = e^{-\lim_{x \rightarrow 0^+} \frac{1}{x}} = 0$ )

$\lim_{x \rightarrow 0^-} e^{-\frac{1}{x}} = +\infty$  ← to jeste ale uverime!  
ale je videt, ze ne, nebot by se limity zleva a zprava v 0, nerovnal

b)  $f(x) = \text{sgn} \cos \frac{1}{x}, \frac{1}{x}$  spojita v  $\mathbb{R} \setminus \{0\}$ ,  $\cos$  take.

$\text{sgn}$  ale ne. Tabule uverime. Zkusime dat. uspojitosť.

$\cos \frac{1}{x} = 0 \iff \frac{1}{x} = \frac{\pi}{2} + k\pi \iff x = \frac{1}{\frac{\pi}{2} + k\pi} = \frac{2}{\pi + 2k\pi} = \frac{2}{\pi} \frac{1}{2k+1}$



Pro dk sta i k=0

$x = \frac{2}{\pi}, \cos \frac{\pi}{2} = 0, \cos > 0$  na  $(+\frac{\pi}{2} - \delta, \frac{\pi}{2})$   $\text{sgn} \cos \geq 1$   
 $\cos < 0$  na  $(\frac{\pi}{2}, \frac{\pi}{2} + \delta)$   $\text{sgn} \cos = -1$

Neni spoj. v  $\frac{2}{\pi}$ . Obdobu v  $\frac{2}{\pi} \frac{1}{2k+1}$

3.  $f(x) = \text{sgn} x, g(x) = x(1-x^2)$

$f(g(x)) = ?$

neni spojita.

	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$g$	+	-	+	-
$\text{sgn} g$	1	-1	1	-1

$g(f(x)) = ?$

$\text{sgn} (-\infty, 0) = 0, (0, \infty) = 1$   
 $g(f(x)) = 0 \implies$  je spojita.