9. HODINA

Integrování parciálních zlomků

— jen předběžné, ale stěžejní případy —

1. $\alpha \in \mathbb{R}$: $\int \frac{1}{x-\alpha} dx = \ln|x-\alpha| + C$ ‼ DŮLEŽITÉ

$x \neq \alpha$

2. $\alpha \notin \mathbb{R}$:

K $\frac{C}{x-\alpha}$ najít $\frac{\bar{C}}{x-\bar{\alpha}}$ a sečíst: $\frac{Cx + \bar{C}x - C\bar{\alpha} - \bar{C}\alpha}{x^2 - 2\mathrm{Re}(\alpha)x + |\alpha|^2}$

$c + \bar{c}$ je reálné

Tj. $\frac{Ax + B}{x^2 - 2\mathrm{Re}(\alpha)x + |\alpha|^2}$.

$C\bar{\alpha} + \bar{C}\alpha$ je také reálné

‼ Ponaučení: $\alpha \notin \mathbb{R} \wedge n = 1 \Rightarrow$ vidět jako kvadratické členy ve jmenovateli. Toto budeme integrovat na ln a arctg rozdělením:

Obecny pripad: $\frac{Ax}{x^2 - 2\mathrm{Re}(\alpha)x + |\alpha|^2} + \frac{B}{x^2 - 2\mathrm{Re}(\alpha)x + |\alpha|^2} =$

pr

$= \frac{A}{2} \frac{(x^2 - 2\mathrm{Re}(\alpha)x + |\alpha|^2)'}{x^2 - 2\mathrm{Re}(\alpha)x + |\alpha|^2} + \underbrace{\frac{\mathrm{Re}(\alpha)A \ldots}{x^2 - 2\mathrm{Re}(\alpha)x + |\alpha|^2} + \frac{B}{\ldots}}$

$\Downarrow \int$ $\Downarrow$

$\frac{A}{2} \ln|x^2 - 2\mathrm{Re}(\alpha)x + |\alpha|^2| + c$

TOTO NA arctg

Pozn.: v konkretnim pripade obvykle snazsi, takze se neucim nazpamet jako vzorec, ale ze spoctenych prikladu

$\int \frac{1}{x^2+1} = \mathrm{arctg}\, x + C$

Integral tzv. logaritmicke derivace: $\int \frac{f'(x)}{f(x)} dx = \int [\ln(f(x))]' dx =$

$\left[\int (\ln f(x))' dx = \int f'(x) \frac{1}{f(x)} dx \right]$

$= \ln|f(x)| + c$

Ve ② jde tedy o log. derivaci a pak ... o doplnění na čtverec

3. $\alpha \in \mathbb{C}$ avšak $\int (x-\alpha)^{-n} dx \quad n > 1$

SNAŽŠÍ $\underset{\mathbb{R}}{\cup}$

$\alpha \in \mathbb{R}$ zjevně: $\int (x-\alpha)^{-n} dx = \frac{-1}{-n+1} (x-\alpha)^{-n+1} + c$

$$\int (x-3)^{-5} dx = \frac{-1}{-4} (x-3)^{-4} =$$

$$= \frac{-1}{4} \frac{1}{(x-3)^4}$$

Pro $\alpha \in \mathbb{C}$ mohu udělat totéž:

$$\int \left[\frac{c}{(x-\alpha)^n} + \frac{\bar{c}}{(x-\bar{\alpha})^n}\right] dx = \int \frac{c\,dx}{(x-\alpha)^n} + \int \frac{\bar{c}\,dx}{(x-\bar{\alpha})^n} =$$

stále $n > 1$ $\qquad \int y^n = \frac{y^{n+1}}{n+1} + C$

$$= \frac{c(x-\alpha)^{-n+1}}{-n+1} + \frac{\bar{c}(x-\bar{\alpha})^{-n+1}}{-n+1} = \frac{1}{1-n} \left[\frac{\overbrace{c(x-\alpha)^{n-1}}^{\bar{A}} + \overbrace{\bar{c}(x-\bar{\alpha})^{n-1}}^{A}}{[(x-\alpha)(x-\bar{\alpha})]^{n+1}} \right]$$

$A + \bar{A} \in \mathbb{R}$!

$\boxed{\text{SUBSTITUCE}}$

Dvě věty: $\varphi : I \to J$, $\varphi' \exists$

1. $F(x) := \int f(x) dx \ \exists \Rightarrow \exists \int f(\varphi(x)) \varphi'(x) dx = F(\varphi(x))$

2. φ prosté a $\varphi' \neq 0$ a $\exists G(t) := \int f(\underbrace{\varphi(t)}_{(=x)}) \varphi'(t) dt$.
(nebo $\varphi' > 0$
popř. $\varphi' < 0$)

Pak $\exists \int f(x) dx = G \circ \varphi^{-1}(x)$.

Př.: 1. a) $\int x e^{-x^2} dx = -\frac{1}{2}\int \underline{(-2x)} e^{\underline{-x^2}} dx = -\frac{1}{2}\int (e^{-x^2})' dx =$

$= -\frac{1}{2} e^{-x^2} + C$

Umím $\int \underbrace{e^y}_{f(y)} dy$ a spočtu $\int f'(\varphi(x))\, g'(x)\, dx$ popř.

$\underline{\underline{1.VOS}}$

$\underline{-\frac{1}{2} \text{ krát}} \int f(\varphi(x))\, \varphi'(x)\, dx$

b) Jest tedy $f(y) = \underline{e^y}$, $\underline{\varphi(x) = -x^2}$

$\underline{-\frac{1}{2}\int f(\varphi(x))\varphi'(x)\, dx} = -\frac{1}{2}\int e^{-x^2}(-2x)\, dx = \int \underline{x e^{-x^2} dx}$

Věta říká, že jde o $-\frac{1}{2}(F\circ\varphi)(x) = -\frac{1}{2}\underline{e^{\varphi(x)}} = \underline{-\frac{1}{2} e^{-x^2}}$,

což nám vyšlo.

2. a) $\int \sin^3 x \cos x\, dx = \frac{1}{4}\int (\sin^4 x)' dx = \frac{1}{4}\sin^4 x + C$

NOVÁ FCE PŮVODNÍ

b) $f(y) = y^3$, $y = \varphi(x) = \sin x$ $\underline{\underline{1.VOS}}$

$F(y) = \frac{1}{4} y^4$ $\int f(\varphi(x))\varphi'(x)\, dx = \int \underline{\sin^3 x\ \cos x\, dx}$

!! 1. ∫-VOS: $F(\varphi(x)) = \underline{\frac{1}{4}\sin^4 x + C}$

c) ~~ne~~formálně $y = \sin x$ $\int \underline{y^3 dy} = \underline{\frac{1}{4}y^4 + C}$

$dy = \cos x\, dx$

STARÁ NOVÁ STARÁ JE FCE NOVÉ

3. $\int \sqrt{a^2 + x^2}\, dx = \left| \begin{array}{l} x = a \sinh t \\ dx = a\cosh t\, dt \\ \varphi: t \mapsto x \end{array} \right| = \int \underbrace{\sqrt{a^2 + a^2\sinh^2 t}}_{f(\varphi(t))}\ \underbrace{a \cosh t\, dt}_{\varphi'(t)}$

$a > 0$

$= \int a^2 \underline{\cosh^2 t\, dt}$ umím! Umím tedy $\int f(g(t))\, g'(t)\, dt$

a pomocí něj $\int f(x)\, dx$. Jde o $\underline{\underline{2.VOS}}$

$$\int \cosh^2 t\,dt = \left| \begin{array}{ll} f = \cosh t & f' = \sinh t \\ g' = \cosh t & g = \sinh t \end{array} \right| = \cosh t \sinh t - \int \sinh^2 t\,dt =$$

$$= \cosh t \sinh t - \int \underline{(\cosh^2 t - 1)}\,dt = \cosh t \sinh t - \int \cosh^2 t + \underline{t} \Rightarrow$$

$$\Rightarrow \int \cosh^2 t\,dt = \underline{\frac{1}{2}(t + \sinh t \cdot \cosh t)}$$

Celkově $\displaystyle\int \sqrt{a^2+x^2}\,dx = \frac{a^2}{2}\left(\operatorname{arcsinh}\frac{x}{a} + \frac{x}{a}\cosh\left(\operatorname{arcsinh}\frac{x}{a}\right)\right) + C$

[Výsledek má odpovídat $G(t) := \int f(\varphi(t))\varphi'(t)\,dt$

$G \circ \varphi^{-1}(x)$, tj. do G dosadím za t inverzní fci.

To jsme přesně učinili $\operatorname{arcsinh}\frac{x}{a} = t$]

H na přednášce. My jsme psali G.

(3')

$x = \sin t \to$ 1. VOS

Př. $\displaystyle\int \frac{1}{\sqrt{1-x^2}\,(\arcsin x)^2}\,dx \quad \left| \begin{array}{l} t = \arcsin x \to 1. \\ dt = \frac{1}{\sqrt{1-x^2}}\,dx \end{array} \right| \quad \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{\arcsin x}$.

$x \in (-1, 1)$

Př.: $\displaystyle\int \sin^7 x\,dx = \int \sin x\,(1-\cos^2 x)^3\,dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x\,dx \end{array} \right|$

$\displaystyle = -\int (1-t^2)^3\,dt = -\int (1-3t^2+3t^4-t^6)\,dt = -t + t^3 - \frac{3t^5}{5} + \frac{t^7}{7} =$

$\displaystyle = -\cos x + \cos^3 x - \frac{3\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$

Př.:

$\displaystyle\int 1 \arccos x\,dx = \left| \begin{array}{ll} f = \arccos x & f' = \frac{-1}{\sqrt{1-x^2}} \\ g' = 1 & g = x \end{array} \right| = x\arccos x + \int \frac{x\,dx}{\sqrt{1-x^2}}$

$\displaystyle = x\arccos x + \int \frac{1}{2}\,\underbrace{\frac{-2x}{\sqrt{1-x^2}}}_{(\sqrt{1-x^2})'}\,dx = x\arccos x - \int \left(\sqrt{1-x^2}\right)'\,dx$ (1. VOS-∫)

$= x\arccos x - \sqrt{1-x^2} + C.$

Pozn.: Obdobně $\int \operatorname{arcsinh} x\,dx$, $\int \operatorname{arctanh} x\,dx$.

Př.: $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x\,dx \end{array} \right| = -\int \frac{dt}{t} =$

$= -\ln|t| + C = -\ln|\cos x| + C$

0. Integrace parc. zlomků – podrobněji

Př.: Předp. příklad vede na $\int$ rac. lom fce. Nechť dojdeme k tomuto rozkladu

$$\frac{1}{x-3}+\frac{i}{(x-i)^2}-\frac{i}{(x+i)^2}+\frac{2+i}{x-1-2i}+\frac{2-i}{x-1+2i}$$

- $\int \frac{1}{x-3}\,dx = \ln|x-3| + c$

$\int$ zvlášť

- $\int\left[\frac{i}{(x-i)^2}-\frac{i}{(x+i)^2}\right]dx = \frac{i}{x-i}\,\frac{1}{-1} - \frac{i}{x+i}\,\frac{1}{-1} + c =$

$$= \frac{-i}{x-i}+\frac{i}{x+i} = \frac{-ix+1+ix+1}{x^2+1} = \frac{2}{x^2+1} \checkmark$$

Lze i nejdřív sečíst, pak $\int$, ale zdlouhavé a zde zbytečné!

- Pro $\alpha\in\mathbb{C}\setminus\mathbb{R} \wedge \underline{m=1}$ je jediná fce, co neumíme zintegrovat.

Nejdřív sečíst.

$$\frac{4x-8}{x^2-2x+4} = 2\,\frac{\boxed{2x-2}-2}{x^2-2x+4} =$$

$$= 2\,\frac{2x-2}{x^2-2x+4} - 4\,\frac{1}{x^2-2x+4} \Rightarrow$$

$$2\int\frac{2x-2}{x^2-2x+4}\,dx = 2\ln|\underbrace{x^2-2x+4}_{(>0)}| + C$$

$$\int\frac{1}{x^2-2x+4}\,dx = \int\frac{1}{(x-1)^2\underline{-1}+\underline{4}}\,dx = \int\frac{1}{(x-1)^2+3}\,dx =$$

3'''

$$\left| \begin{matrix} y = x-1 \\ dy = dx \end{matrix} \right| \quad \int \frac{1}{y^2+3}\,dy = \left| \begin{matrix} \varphi(z) = \sqrt{3}z \\ y = \sqrt{3}\,z \\ dy = \sqrt{3}\,dz \end{matrix} \right| = \int \frac{\sqrt{3}}{3z^2+3}\,dz =$$

$$= \frac{1}{\cancel{3}\sqrt{3}} \int \frac{1}{z^2+1}\,dz = \frac{1}{\cancel{3}\sqrt{3}} \operatorname{arctg}\left(\sqrt{3}\,y\right) + C = \frac{1}{\cancel{3}\sqrt{3}} \operatorname{arctg}\left[\sqrt{3}\,(x-1)\right] + C.$$

Př.: Cvičení se jiná jen v $\dfrac{px+q}{ax^2+bx+c}$.

Př.: Spočtěte: $\displaystyle\int \frac{x+1}{x^2+3x+\cancel{7}\,4}\,dx = \frac{1}{2}\int \frac{2x+3}{x^2+3x+4}\,dx - \frac{1}{2}\int \frac{dx}{x^2+3x+4}$

$$= \frac{1}{2}\ln\left|x^2+3x+\cancel{7}\,4\right| - \frac{1}{2}\int \frac{dx}{x^2+3x+4}$$

Na whiteboardu misto 4 priklad se 7.

• $D = 9-16 = -5 < 0$, tj. $\displaystyle\int \frac{1}{\underbrace{x^2}_{a} + 2\cdot\underbrace{\tfrac{3}{2}}_{b}\underbrace{x}_{a} + \underbrace{\tfrac{9}{4}}_{b^2} + 4 - \tfrac{9}{4}}\,dx =$

$$= \int \frac{dx}{\left(x+\frac{3}{2}\right)^2 + \frac{5}{4}} = \int \frac{dx}{\frac{5}{4}\left[\frac{4}{5}\left(x+\frac{3}{2}\right)^2 + 1\right]} =$$

$$= \frac{4}{5}\int \frac{dx}{\left[\frac{2}{\sqrt{5}}\left(x+\frac{3}{2}\right)\right]^2+1}\,dx \quad \left| \, y = \frac{2}{\sqrt{5}}\left(x+\frac{3}{2}\right) \, \right| =$$

$$= \frac{4}{5}\int \frac{\frac{\sqrt{5}}{2}\,dy}{y^2+1}\,dy = \frac{2}{\sqrt{5}} \operatorname{arctg} y + C =$$

$$= \frac{2}{\sqrt{5}} \operatorname{arctg} \frac{2}{\sqrt{5}}\left(x+\frac{3}{2}\right) + C$$

Celkem $\displaystyle\frac{1}{2}\ln\left|x^2+3x+7\right| - \frac{1}{\sqrt{5}} \operatorname{arctg} \frac{2}{\sqrt{5}}\left(x+\frac{3}{2}\right) + C$

(Jde předevsim o 1. vos)

1. $R(e^{ax})$ $\quad y = e^{ax}$, více exponenc., pak volím nejmenší a...

Př.: $\displaystyle \int \frac{1}{e^{2x}+e^{x}-2}\,dx = \left| \begin{array}{l} y = e^{x} \\ dy = e^{x}dx \\ dx = e^{-x}dy = \frac{dy}{y} \end{array} \right| = \int \frac{1}{y}\frac{dy}{(y^2+y-2)} =$

$$\frac{1}{y}\,\frac{1}{y^2+y-2} = \frac{1}{y(y-1)(y+2)} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{y+2}$$

$$\Rightarrow A = -\frac{1}{2},\quad B = \frac{1}{3},\quad C = \frac{1}{6}$$

$$-\frac{1}{2}\int \frac{1}{y}\,dy + \frac{1}{3}\int \frac{1}{y-1}\,dy + \frac{1}{6}\int \frac{1}{y+2}\,dy = -\frac{1}{2}\ln|y| + \frac{1}{3}\ln(e^x-1) + \frac{1}{6}\ln(e^x+2)$$

$$+ C = -\frac{1}{2}x + \frac{1}{6}\ln (e^x-1)^2(e^x+2) + C.$$

2. $\dfrac{R(\ln x)}{x}$ $\quad y = \ln x$

Př.: $\displaystyle \int \frac{\ln^2 x + \ln x + 1}{x}\,dx \quad \left| \begin{array}{l} y = \ln x \\ dy = \frac{1}{x}dx \\ x>0 \end{array} \right| = \int (y^2+y+1)\,dy =$

$$= \frac{y^3}{3} + \frac{y^2}{2} + y + C = \frac{\ln^3 x}{3} + \frac{\ln^2 x}{2} + \ln x + C$$

3. Goniometrické substituce

1. $y = \operatorname{tg}\frac{x}{2}$ („univerzální") NEBO $t = \operatorname{tg}\frac{x}{2}$, ALE t a tg „blízko", tak radši y
2. $y = \operatorname{tg} x$ (sudé při současné záměně $\sin \to -\sin$, $\cos \to -\cos$)

Př.

$$\int \frac{\sin^2 x}{1+\sin^2 x}\,dx \left| \begin{array}{l} R(-\cos x, -\sin x) = R(\cos x, \sin x) \\ y = \operatorname{tg} x \;, \; x = \operatorname{arctg} y \\ y^2 = \dfrac{\sin^2 x}{1-\sin^2 x} \Rightarrow \sin^2 x = \dfrac{y^2}{1+y^2} \\ dx = \dfrac{1}{1+y^2}\,dy \end{array} \right| =$$

$$= \int \frac{y^2}{1+y^2} \cdot \frac{1}{1+\dfrac{y^2}{1+y^2}} \cdot \frac{1}{1+y^2}\,dy = \int \frac{y^2}{(1+y^2)(1+2y^2)}\,dy$$

$$= \int \left(\frac{1}{1+y^2} - \frac{1}{1+2y^2} \right) dy = \operatorname{arctan} y - \int \frac{dy}{2y^2+1} \left| \begin{array}{l} z = \sqrt{2}\,y \\ dz = \sqrt{2}\,dy \end{array} \right.$$

$$= \operatorname{arctg} y - \int \frac{\frac{dz}{\sqrt{2}}}{z^2+1}\,dz = \operatorname{arctg} \operatorname{tg} x -$$

$$- \frac{1}{\sqrt{2}} \operatorname{arctg} \sqrt{2}\,y + C = x - \frac{1}{\sqrt{2}} \operatorname{arctg}\left(\sqrt{2} \operatorname{tg} x\right) + C$$

TABLET

Př.

$$\int \frac{1}{2\sin x - \cos x + 5}\,dx \left| \begin{array}{ll} & x \in (-\pi, \pi) \\ y = \operatorname{tg} \dfrac{x}{2} & \cos x = \dfrac{1-y^2}{1+y^2} \\ dx = \dfrac{2}{1+y^2} & \sin x = \dfrac{2y}{1+y^2} \end{array} \right| =$$

$$= \int \frac{1}{\dfrac{4y}{1+y^2} + \dfrac{y^2-1}{y^2+1} + 5} \cdot \frac{2}{1+y^2}\,dy = \int \frac{2}{6y^2+4y+4}\,dy =$$

$$= \int \frac{1}{3y^2+2y+2}\,dy = \frac{1}{3} \int \frac{1}{y^2+\frac{2}{3}y+\frac{2}{3}}\,dy =$$

$$= \frac{1}{3} \int \frac{dy}{y^2 + \underbrace{2 \cdot \tfrac{1}{3}\, y}_{2ab} + \frac{1}{9} + \frac{2}{3} - \frac{1}{9}} = \frac{1}{3} \int \frac{dy}{\left(y+\frac{1}{3}\right)^2 + \frac{5}{9}} =$$

$$= \frac{1}{3}\frac{9}{5}\int \frac{dy}{\frac{9}{5}\left(y+\frac{1}{3}\right)^2 + \underline{\underline{1}}} = \frac{3}{5}\int \frac{dy}{\underbrace{\left(\frac{3y+1}{\sqrt{5}}\right)^2}_{\text{lineární}}+1}$$

$z = \frac{3y+1}{\sqrt{5}}$

$dz = \frac{3}{\sqrt{5}}\,dy$

1. subst.

$$= \frac{3}{5}\int \frac{\sqrt{5}}{3}\frac{dz}{z^2+1} = \frac{1}{\sqrt{5}}\,\mathrm{arctg}\,\frac{3y+1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\,\mathrm{arctg}\left(\frac{3\,\mathrm{tg}\,\frac{x}{2}+1}{\sqrt{5}}\right)+C$$

4 Integrály*) $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{1/s}\right)dx$, $\underline{t = \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{s}}}$

Př.: $\int \frac{\sqrt{2x+3}+x}{\sqrt{2x+3}-x}\,dx$

$t = \sqrt{2x+3}$

$t^2 = 2x+3$

$x = \frac{1}{2}(t^2-3)$

$dx = \frac{1}{2}(2t\,dt) = t\,dt$

$2x+3>0$

$$= \int \frac{t+\frac{1}{2}(t^2-3)}{t-\frac{1}{2}(t^2-3)}\,t\,dt = \int \frac{t^2+2t-3}{-t^2+2t+3}\,t\,dt =$$

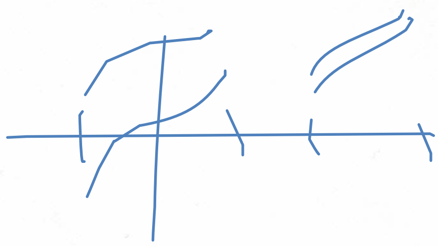
dělte a rozložte

$$= \int\left(\underline{-t-4}-\frac{9}{t-3}+\frac{1}{t+1}\right)dt =$$

$$= -\frac{1}{2}t^2-4t-9\ln|t-3|+\ln|t+1|+C =$$

$$= -\frac{1}{2}(2x+3)^2-4(2x+3)-9\ln\left|\sqrt{2x+3}-3\right|+\ln\left|\sqrt{2x+3}+1\right|+C$$

✓



*) Mocninné subst.

Př.: $\int \sqrt{\frac{\cancel{x}-1}{x+2}}\, x\, dx$ | $t = \sqrt{\frac{x-1}{x+2}}$

$t^2 x + 2t^2 = x - 1$

$t^2 x - x = -1 - 2t^2$

$x(t^2-1) = -1-2t^2$

$x = \frac{-1-2t^2}{t^2-1} = \frac{1+2t^2}{1-t^2}$

$dx = \frac{4t-4t^3+2t+4t^3}{(1-t^2)^2}\, dt = \frac{6t}{(1-t^2)^2}\, dt$

$\int t\, \underbrace{\frac{1+2t^2}{1-t^2} \frac{6t}{(1-t^2)^2}}\, dt$. Dál parciální zlomky.

Všechny kořeny reálné.

Př. $\int \sqrt{1-x^2}\, dx$ je také typu výše: $\int \sqrt{(1-x)(1+x)}\, dx = \int \sqrt{\underbrace{\frac{1-x}{1+x}}_{=t}}\,(1+x)\,dx$

Avšak lze počítat i substitucí $x = \sin t$

$\int \sqrt{1-\sin^2 t}\, \cos t\, dt = \int \cos^2 t\, dt$ = per-partes → proveďte

Omezme se na $(0,\pi)$

Anebo

$\cos 2x = \cos^2 x - \sin^2 x =$
$= \cos^2 x - 1 + \cos^2 x =$
$= 2\cos^2 x - 1$

$\int \frac{\cos 2x + 1}{2}\, dx = \frac{1}{2}\frac{\sin 2x}{2} + \frac{1}{2}x + C$

$\int \cos 2x\, dx \;\Big|\; y = 2x \;\Big| = \int \cos y\, \frac{dy}{2} =$

$= \frac{\sin y}{2} + C = \frac{\sin 2x}{2}$

(5) Eulerovy substituce $\int R(x,y)\, dx$, $y = \sqrt{ax^2+bx+c}$

a) Dva reálné: $x_1 = x_2 \Rightarrow a|x - x_1|$

$x_1 \neq x_2 \Rightarrow \sqrt{a(x-x_1)(x-x_2)} = \sqrt{a\frac{x-x_1}{x-x_2}}\,(x-x_2)$

Vytknutím a ⇝ typ minulý

b) $a>0$ $\sqrt{ax^2+bx+c} = \pm\sqrt{a}\, x + t$

c) $c>0$ $\sqrt{ax^2+bx+c} = \sqrt{c} \pm x t$

Př.

$$\int \frac{x^2}{2\sqrt{1-x^2}} = \int \frac{x^2}{2\sqrt{(1-x)(1+x)}}\,dx =$$

$$= \int \frac{x^2}{2}\sqrt{\frac{1+x}{1-x}}\,(1+x) \quad \text{Máme minulý typ.}$$

Př.:

$$\int \frac{dx}{x+\sqrt{x^2+x+1}} \qquad \sqrt{x^2+x+1} = x+t$$

$$\int \frac{1}{2\frac{t^2-1}{1-2t}+t} \qquad x = \frac{t^2-1}{1-2t}$$

$$\frac{dx}{dt} = \frac{2t(1-2t)+2(t^2-1)}{(1-2t)^2} = \frac{2t-4t^2+2t^2-2}{(1-2t)^2} = \frac{-2t^2+2t-2}{(1-2t)^2} = -2\frac{t^2-t+1}{(1-2t)^2}$$

$$\int \frac{-2\frac{t^2-t+1}{(1-2t)^2}\,dt}{2\frac{t^2-1}{1-2t}+t}$$

Dale integruji rac. lomenou fci dle probranych metod.

$$\int \left(1 + \frac{3t}{2t^2-5t+2}\right) dt$$

Realne koreny, takze rozlozim na soucin dvou 1/linearni a integruji oba na ln. Pokud maji cleny v sobe nasobek, nezapomenu zohlednit pri integrovani.

int dx/(3x + 1) ---> 1/3 ln|3x+1| +C

$$-2\int\left(t+1+\frac{3}{t-2}\right)dt = -2\left[\frac{t^2}{2}+t+3\ln|t-2|\right]+C$$

$$= -t^2-2t-6\ln|t-2|+C=$$

$$= -\left(\sqrt{x^2+x+1}-x\right)^2-2\left(\sqrt{x^2+x+1}-x\right)-6\ln\left|\sqrt{x^2+x+1}-x-2\right|+C$$

Př.

$$\int\frac{2x+3}{(x-2)(x+5)}\,dx \quad \Bigg| \quad \frac{2x+3}{(x-2)(x+5)}=\frac{A}{x-2}+\frac{B}{x+5}$$

$$A=\frac{7}{7},\ B=\frac{-7}{-7}=1$$

$$\ln|x-2|+\ln|x+5|$$

Doporučuji k písemce: Dopočíst příklady nespočtené, pak Kopáček, event. rel. složité příklady od M. Pokorného – „můj" web

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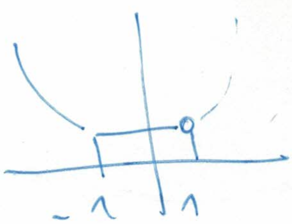
Lepení: $a<b<c$, $f:(a,c)\to\mathbb{R}$ spojitá | F_1 na (a,b), F_2 na (b,c) prim. k f

Pak
$$F(x)=\begin{cases} F_1(x) & x\in(a,b)\\ \lim\limits_{x\to b^-} F_1(x) & x=b\\ F_2(x)-\lim\limits_{x\to b^+}F_2(x)+\lim\limits_{x\to b^-}F_1(x), & x\in(b,c)\end{cases}$$
je prim. k f na (a,c).

Př.: Spočtěte $\int \max\{1,x^2\}\,dx$ na $(-\infty,1)$

$x\in(-\infty,-1)$ $f(x)=x^2$ $\int f(x)=\frac{x^3}{3}+C_1=F_1(x)$

$x\in(-1,1)$ $f(x)=1$ $\int f(x)=x+C_2=F_2(x)$



$\lim\limits_{x\to-1^-}F_1(x)=-\frac{1}{3}+C_1$ $\quad \| \quad C_2=\frac{2}{3}+C_1$

$\lim\limits_{x\to-1^+}F_2(x)=-1+C_2$

$$\int\max\{1,x^2\}\,dx=\begin{cases}\frac{x^3}{3}+C_1 & x<-1\\ x+C_2=x+\frac{2}{3}+C_1 & x\in\langle-1,1)\end{cases}$$