Complexes of symplectic twistor operators

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- “Twistor operators in symplectic geometry” Adv. Applied Cliff. Analysis 32 (2022); or preprint `https://www2.karlin.mff.cuni.cz/~krysl/Twist.pdf`

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- *Related questions*: Complexes of infinite rank bundles, topological cohomology questions: Images with or without completion $\mathrm{Ker} D^i/\mathrm{Im} D^{i-1}$ or $\mathrm{Ker} D^i/\overline{\mathrm{Im} D^{i-1}}$? (Important in Analysis and Quantum Physics of constraint systems - BRST)

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- **Result:** Symplectic twistor sequences form complexes if their inducing connection is Weyl-flat. The same is true in classical spin geometry (but there one cannot choose a connection).

Symplectic Vector Spaces - foremost setting of notation

- (V, ω) real symplectic vector space of dimension $2n$
- $Sp(2n, \mathbb{R})$ symplectic group (non-compact!), $\pi_1(Sp(2n, \mathbb{R})) = \pi_1(U(n)) = \mathbb{Z}$
- There exists connected Lie group that covers $Sp(2n, \mathbb{R})$ twice
- unique as Lie group up to choice of neutral element and deck-transformation
- the metaplectic group, denoted by $\widetilde{G}$ or $Mp(2n, \mathbb{R})$
- Let us choose a complex structure on V such that $g(u, v) = \omega(Ju, v)$ is positive definite (*adapted cplx str.*), and a maximal ω-isotropic subspace $L \subseteq V$ (J is then g-orthogonal, ω-symplectic)

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$$sCliff(V, \omega) = T(V)/\langle v \otimes w - w \otimes v - \omega(v, w)1 |\, v, w \in V\rangle$$

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- There is no faithful unitary representation on a finite dimensional vector space (Weyl's unitary trick)
- $E = E_+ \oplus E_-$, even and odd square integrable functions on Lagrangian space L, it is a decomposition into irreducibles

Model for the Complex - Symplectic Spinor Valued Exterior Forms

Notation:

- The double cover $\lambda : \widetilde{G} \to Sp(2n, \mathbb{R}) \simeq \lambda^* : \widetilde{G} \to \mathrm{Aut}(V^*)$ is a representation

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- **Theorem** [Krysl, Lie Theory]: For each i, there are irreducible modules $E^{ij}_\pm$, $j = 0, \dots, k_i = n - |n - i|$, such that

$$\bigwedge^i V^* \otimes E_\pm = E^i_\pm = \oplus_{j=0}^{k_i} E^{ij}_\pm.$$

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- We set $E^{ij} = E^{ij}_{+} \oplus E^{ij}_{-}$.

Decomposition Diagram (in specific dimension)

Representations $E^{ij}_{\pm}$ described by highest weight (of the $\mathfrak{g}$-structure on C^∞-vectors) with respect to a Cartan subalgebra and a choice of positive roots, $2n = 6$.

$$\begin{array}{ccccccc}
E^0 & E^1 & E^2 & E^3 & E^4 & E^5 & E^6 \\
E^{00} & E^{10} & E^{20} & E^{30} & E^{40} & E^{50} & E^{60} \\
 & E^{11} & E^{21} & E^{31} & E^{41} & E^{51} & \\
 & & E^{22} & E^{32} & E^{42} & & \\
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\end{array}$$

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- Existence [Forger, Hess] - obstacle: second Stiefel-Whitney class non-zero; T^*N for N orientable, tori, $\mathbb{CP}^{2n+1}$

Symplectic Connections

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- **Theorem** [Tondeur ('60)]: The affine space of Fedosov connections is in an affine bijection with the affine space $(\Gamma(S^3M), 0)$.

Symplectic curvature tensors

- Ricci tensor $\sigma(X,Y) = \mathrm{Tr}(Z \mapsto R^{\nabla}(Z,X)Y)$, $\sigma_{ij} = R^k{}_{ijk} = +R^k{}_{ikj}$, coordinates with respect to a local symplectic frame $(e_i)_{i=1}^{2n}$, R^{∇} classical curvature of affine connection ∇

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- Extended Ricci tensor: $\sigma_{ijkl} = \frac{1}{2n+2}(\omega_{il}\sigma_{jk} - \omega_{ik}\sigma_{jl} + \omega_{jl}\sigma_{ik} - \omega jk\sigma_{il} + 2\sigma_{ij}\omega_{kl})$, $\widehat{\sigma} = \sigma_{ijkl}\epsilon^i \otimes \epsilon^j \otimes \epsilon^k \otimes \epsilon^l$ where $(\epsilon^i)_i$ is the dual basis (not ω-dual) (See Izu [Vaisman])

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- **Definition:** A Fedosov connection is called symplectic *Weyl-flat* (or symplectic Ricci-type) if $W = 0$.

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- i-th curvature $R^i = \nabla^{i+1}\nabla^i$. Total curvature $R = \sum_{i=0}^{2n-2} R^i$

Family of Twistor operators

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- Decomposition of $\bigwedge^i V^* \otimes E$ into irreducible representations $\Longrightarrow$

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E^{i,j-1} & E^{i+1,j-1} \\
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Condition for forming a complex

- We would like to investigate the chain-complex condition

$$T_{\pm}^{i+k+1,j+k\pm 1} \circ T_{\pm}^{i+k,j+k} = 0, \text{ i.e.,}$$

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- $\implies T_{\pm}^{i+1,j\pm1} T_{\pm}^{i,j} = p^{i+2,j\pm2} \nabla\nabla^{ij} = p^{i+2,j\pm2} R^{\nabla}_{|\Gamma(\mathcal{E}^{ij})}$

Curvature structure of Weyl-Flat Connection - Representational approach

- $(\epsilon^i)_{i=1}^{2n}$ the dual frame to a local symplectic frame $(e_i)_{i=1}^{2n}$, $e_i \in \Gamma(U, TM)$, $U \subseteq M$ open
- $(e_i.f)(x^1, \ldots, x^n) = \imath x^i f(x)$, $e_{i+n}.f = \frac{\partial f}{\partial x^i}$, $1 \leq i \leq n$. Where $f \in \mathcal{S}(L)$ (Schwartz space of rapidly decreasing functions) $\subseteq L^2(L)$ See [Habermann, K., Habermann, L.]. So-called symplectic spinor multiplication/or Canonical Quantization (up to multiples)

Representation of $\mathfrak{osp}(1|2)$ on symplectic spinor forms

- $F^+(\alpha \otimes f) := \frac{\imath}{2}\sum_{i=1}^{n} \epsilon^i \wedge \alpha \otimes e_i \cdot f$, $\alpha \otimes f \in \bigwedge^i V^* \otimes \mathcal{S}(L)$

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- This define a representation of the five dimensional Lie superalgebra $\mathfrak{osp}(1|2) = \langle e^+, e^-, h, f^+, f^- \rangle / \simeq$

Representation of $\mathfrak{osp}(1|2)$ on symplectic spinor forms

- $F^+(\alpha \otimes f) := \frac{\imath}{2} \sum_{i=1}^{n} \epsilon^i \wedge \alpha \otimes e_i \cdot f$, $\alpha \otimes f \in \bigwedge^i V^* \otimes \mathcal{S}(L)$
- $F^-(\alpha \otimes f) := \frac{1}{2} \sum_{i=1}^{n} \omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot f$,
- $E^{\pm} := \pm 2\{F^{\pm}, F^{\pm}\}$, anti-commutator. (See [Krysl, Monats.] for a super-algebra setting.)
- $H = [E^+, E^-]$
- This define a representation of the five dimensional Lie superalgebra $\mathfrak{osp}(1|2) = \langle e^+, e^-, h, f^+, f^- \rangle / \simeq$
- $f^{\pm} \mapsto F^{\pm}$, $e^{\pm} \mapsto E^{\pm}$, $h \mapsto H$

Curvature of Weyl-flat connection - Representational Approach

Non-equivariant maps

- $\Sigma^{\sigma}(\alpha \otimes f) := \sum_{i,j=1}^{2n} \sigma^{i}{}_{j} \epsilon^{j} \wedge \alpha \otimes e_{i} \cdot f$

Curvature of Weyl-flat connection - Representational Approach

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Curvature of Weyl-flat connection - Representational Approach

Non-equivariant maps

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- $\Theta^\sigma(\alpha \otimes f) = \sum_{i,j=1}^{2n} \alpha \otimes \sigma^{ij} e_i \cdot e_j \cdot f$
- **Lemma** (curvature, [Krysl, Monats]). If ∇ is a symplectic Weyl-flat connection, then

$$R = \frac{1}{n+2}(E^+\Theta^\sigma + 2F^+\Sigma^\sigma).$$

Curvature in Diagram Decomposition

- F^+:

$$\underline{E^{i,j}} \xrightarrow{F^+} E^{i+1,j}$$

- Σ^σ:

$$\begin{tikzcd}
E^{i,j-1} & E^{i+1,j-1} \\
\underline{E^{i,j}} \arrow[ur] \arrow[r] \arrow[dr] & E^{i+1,j} \\
E^{i,j+1} & E^{i+1,j+1}
\end{tikzcd}$$

- E^+:

$$E^{i,j} \xrightarrow{F^+} E^{i+2,j}$$

- Θ^σ:

$$\begin{tikzcd}
E^{i,j-1} \\
\underline{E^{i,j}} \arrow[u] \arrow[d] \\
E^{i,j+1}
\end{tikzcd}$$

Curvature and Connection

- Curvature $R = \frac{1}{n+2}(E^+\Theta^\sigma + 2F^+\Sigma^\sigma)$:

$$\begin{tikzcd}
E^{i,j-1} & E^{i+1,j-1} & E^{i+2,j-1} \\
\underline{E^{i,j}} \arrow[r] \arrow[urr] \arrow[drr] & E^{i+1,j} \arrow[r] & E^{i+2,j} \\
E^{i,j+1} & E^{i+1,j+1} & E^{i+2,j+1}
\end{tikzcd}$$

- Cov. derivative ∇^{ij} :

$$\begin{tikzcd}
\Gamma(\mathcal{E}^{i,j-1}) & \Gamma(\mathcal{E}^{i+1,j-1}) \\
\underline{\Gamma(\mathcal{E}^{i,j})} \arrow[r] \arrow[ur] \arrow[dr] & \Gamma(\mathcal{E}^{i+1,j}) \\
\Gamma(\mathcal{E}^{i,j+1}) & \Gamma(\mathcal{E}^{i+1,j+1})
\end{tikzcd}$$

Cov. derivative's target are right also if the connection has torsion and ω is pre-symplectic only ($d\omega \neq 0$).

Theorem and a proof

Theorem: Let (M, ω) be symplectic manifold admitting a metaplectic structure and let ∇ a Weyl-flat Fedosov connection on (M, ω). Then for all pairs of integers (i, j), the sequences $(\Gamma(\mathcal{E}^{i+k, j\pm k}), T_{\pm}^{i+k, j\pm k})_{k\in\mathbb{Z}}$ form complexes.

Proof. Basic steps, ideas

Composition of twistor operators

$p \circ \nabla \circ \nabla_{|E^{ij}}$

$p \circ R^{\nabla}_{|E^{ij}}$

From the structure of R^{∇} for Weyl-flat Fedosov connection, we see that $p \circ R^{\nabla}_{|E^{ij}} = 0$

Parallel

Parallel to: Let (M, J) be a complex manifold. Then for each couple of integers (i, j), the sequence $(\Gamma(E^{i,j+k}), \overline{\partial}_{i,j+k})_{k\in\mathbb{Z}}$ of holomorphic-antiholomorphic differential forms and anti-Dolbeault differentials is a complex. Recall: Complex manifold $=$ almost complex structure J is integrable (is complex), i.e., (Nirenberg–Newlander thm.), Nijenhuis' tensor $N(X, Y) = [X, Y] + J[JX, Y] + [X, JY] - [JX, JY]$ of (M, J) is zero for all $X, Y \in \Gamma(TM)$ (J complex $\Rightarrow$ $N = 0$ is easy $\Rightarrow$ the sequence is a complex).

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