# Complexes of symplectic twistor operators

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- Related questions: Complexes of infinite rank bundles, topological cohomology questions: Images with or without completion  $\mathrm{Ker} D^i/\mathrm{Im} D^{i-1}$  or  $\mathrm{Ker} D^i/\mathrm{Im} D^{i-1}$ ? (Important in Analysis and Quantum Physics of constraint systems BRST)

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- Result: Symplectic twistor sequences form complexes if their inducing connection is Weyl-flat. The same is true in classical spin geometry (but there one cannot choose a connection).

# Symplectic Vector Spaces - foremost setting of notation

- $(V, \omega)$  real symplectic vector space of dimension 2n
- $Sp(2n,\mathbb{R})$  symplectic group (non-compact!),  $\pi_1(Sp(2n,\mathbb{R})) = \pi_1(U(n)) = \mathbb{Z}$
- There exists connected Lie group that covers  $Sp(2n,\mathbb{R})$  twice
- unique as Lie group up to choice of neutral element and deck-transformation
- the metaplectic group, denoted by  $\widetilde{G}$  or  $Mp(2n,\mathbb{R})$
- Let us choose a complex structure on V such that  $g(u,v)=\omega(Ju,v)$  is positive definite (adapted cplx str.), and a maximal  $\omega$ -isotropic subspace  $L\subseteq V$  (J is then g-orthogonal,  $\omega$ -symplectic)

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$$sCliff(V,\omega) = T(V)/\langle v \otimes w - w \otimes v - \omega(v,w)1|v,w \in V \rangle$$

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- There is no faithful unitary representation on a finite dimensional vector space (Weyl's unitary trick)
- $E = E_+ \oplus E_-$ , even and odd square integrable functions on Lagrangian space L, it is a decomposition into irreducibles



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- **Theorem** [Krysl, Lie Theory]: For each i, there are irreducible modules  $E_+^{ij}$ ,  $j = 0, \ldots, k_i = n |n i|$ , such that

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• We set  $E^{ij} = E^{ij}_+ \oplus E^{ij}_-$ .



# Decomposition Diagram (in specific dimension)

Representations  $E_{\pm}^{ij}$  described by highest weight (of the  $\mathfrak{g}$ -structure on  $C^{\infty}$ -vectors) with respect to a Cartan subalgebra and a choice of positive roots, 2n=6.

$$E^{0}$$
  $E^{1}$   $E^{2}$   $E^{3}$   $E^{4}$   $E^{5}$   $E^{6}$ 
 $E^{00}$   $E^{10}$   $E^{20}$   $E^{30}$   $E^{40}$   $E^{50}$   $E^{60}$ 
 $E^{11}$   $E^{21}$   $E^{31}$   $E^{41}$   $E^{51}$ 
 $E^{22}$   $E^{32}$   $E^{42}$ 
 $E^{33}$ 

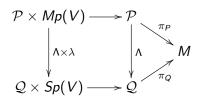
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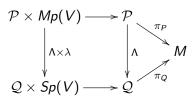
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• Existence [Forger, Hess] - obstacle: second Stiefel-Whitney class non-zero;  $T^*N$  for N orientable, tori,  $\mathbb{CP}^{2n+1}$ 

# Symplectic Connections

• **Definition:** Let  $(M, \omega)$  be a symplectic manifold. A covariant derivative on TM is called symplectic if it preserves the symplectic form  $(\nabla \omega = 0)$ . It is called *Fedosov* if it is symplectic and torsion-free.

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- **Theorem** [Tondeur ('60)]: The affine space of Fedosov connections is in an affine bijection with the affine space  $(\Gamma(S^3M), 0)$ .

• Ricci tensor  $\sigma(X,Y) = \operatorname{Tr}(Z \mapsto R^{\nabla}(Z,X)Y)$ ,  $\sigma_{ij} = R^k{}_{ijk} = + R^k{}_{ikj}$ , coordinates with respect to a local symplectic frame  $(e_i)_{i=1}^{2n}$ ,  $R^{\nabla}$  classical curvature of affine connection  $\nabla$ 

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- Extended Ricci tensor:

$$\begin{split} &\sigma_{ijkl} = \frac{1}{2n+2} \big( \omega_{il} \sigma_{jk} - \omega_{ik} \sigma_{jl} + \omega_{jl} \sigma_{ik} - \omega_{jk} \sigma_{il} + 2 \sigma_{ij} \omega_{kl} \big), \\ &\widehat{\sigma} = \sigma_{ijkl} \epsilon^i \otimes \epsilon^j \otimes \epsilon^k \otimes \epsilon^l \text{ where } (\epsilon^i)_i \text{ is the dual basis (not } \omega\text{-dual) (See Izu [Vaisman])} \end{split}$$

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- $W = R^{\nabla} \hat{\sigma}$  symplectic Weyl curvature tensor
- **Definition:** A Fedosov connection is called symplectic Weyl-flat (or symplectic Ricci-type) if W = 0.

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- *i*-th curvature  $R^i = \nabla^{i+1} \nabla^i$ . Total curvature  $R = \sum_{i=0}^{2n-2} R^i$

# Family of Twistor operators

• Spaces indexed by integer couples "outside of triangle" are set zero for convenience, i.e.,  $E^{ij_i} = 0$  if  $i \notin \{0, \dots, 2n\}$  or  $j_i \notin \{0, \dots, k_i\}$ 

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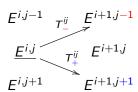
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- Decomposition of  $\bigwedge^i V^* \otimes E$  into irreducible representations



#### Condition for forming a complex

We would like to investigate the chain-complex condition

$$T_{\pm}^{i+k+1,j+k\pm 1}\circ T_{\pm}^{i+k,j+k}=0, \text{ i.e.},$$
 
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$$\bullet \implies T_{\pm}^{i+1,j\pm 1}T_{\pm}^{i,j} = p^{i+2,j\pm 2}\nabla\nabla^{ij} = p^{i+2,j\pm 2}R_{|\Gamma(\mathcal{E}^{ij})}^{\nabla}$$



## Curvature structure of Weyl-Flat Connection - Representational approach

- $(\epsilon^i)_{i=1}^{2n}$  the dual frame to a local symplectic frame  $(e_i)_{i=1}^{2n}$ ,  $e_i \in \Gamma(U, TM), \ U \subseteq M$  open
- $(e_i.f)(x^1,\ldots,x^n)=\imath x^i f(x),\ e_{i+n}.f=\frac{\partial f}{\partial x^i},\ 1\leq i\leq n.$  Where  $f\in\mathcal{S}(L)$  (Schwartz space of rapidly decreasing functions)  $\subseteq L^2(L)$  See [Habermann, K., Habermann, L.]. So-called symplectic spinor multiplication/or Canonical Quantization (up to multiples)

• 
$$F^+(\alpha \otimes f) := \frac{\imath}{2} \sum_{i=1}^n \epsilon^i \wedge \alpha \otimes e_i \cdot f, \ \alpha \otimes f \in \bigwedge^i V^* \otimes S(L)$$

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- This define a representation of the five dimensional Lie superalgebra  $\mathfrak{osp}(1|2)=\langle e^+,e^-,h,f^+,f^-\rangle/\simeq$

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- $E^{\pm}:=\pm 2\{F^{\pm},F^{\pm}\}$ , anti-commutator. (See [Krysl, Monats.] for a super-algebra setting.)
- $H = [E^+, E^-]$
- This define a representation of the five dimensional Lie superalgebra  $\mathfrak{osp}(1|2)=\langle e^+,e^-,h,f^+,f^-\rangle/\simeq$
- $f^{\pm} \mapsto F^{\pm}$ ,  $e^{\pm} \mapsto E^{\pm}$ ,  $h \mapsto H$

# Curvature of Weyl-flat connection - Representational Approach

#### Non-equivariant maps

• 
$$\Sigma^{\sigma}(\alpha \otimes f) := \sum_{i,j=1}^{2n} \sigma^{i}{}_{j} e^{j} \wedge \alpha \otimes e_{i} \cdot f$$

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## Curvature of Weyl-flat connection - Representational Approach

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$$\Theta^{\sigma}(\alpha \otimes f) = \sum_{i,j=1}^{2n} \alpha \otimes \sigma^{ij} e_i \cdot e_j \cdot f$$

• Lemma (curvature, [Krysl, Monats]). If  $\nabla$  is a symplectic Weyl-flat connection, then

$$R = \frac{1}{n+2} (E^+ \Theta^{\sigma} + 2F^+ \Sigma^{\sigma}).$$

### Curvature in Diagram Decomposition

• F+:

 $\bullet$   $\Sigma^{\sigma}$ :

$$\underline{E^{i,j}} \xrightarrow{F^+} E^{i+1,j}$$

$$E^{i,j-1} \xrightarrow{E^{i+1,j-1}} E^{i+1,j-1}$$

$$E^{i,j} \xrightarrow{E^{i+1,j}} E^{i+1,j+1}$$

• E+:

$$E^{i,j} \xrightarrow{F^+} E^{i+2,j}$$

Θ<sup>σ</sup>:

$$E^{i,j-1} \\ \uparrow \\ \underline{E^{i,j}}_{\forall} \\ E^{i,j+1}$$



#### Curvature and Connection

• Curvature  $R = \frac{1}{n+2}(E^+\Theta^{\sigma} + 2F^+\Sigma^{\sigma})$ :

$$E^{i,j-1} \qquad E^{i+1,j-1} \qquad E^{i+2,j-1}$$

$$E^{i,j} \longrightarrow E^{i+1,j} \longrightarrow E^{i+2,j}$$

$$E^{i,j+1} \qquad E^{i+1,j+1} \longrightarrow E^{i+2,j+1}$$

• Cov. derivative  $\nabla^{ij}$ :

$$\Gamma(\mathcal{E}^{i,j-1}) \qquad \Gamma(\mathcal{E}^{i+1,j-1})$$

$$\Gamma(\mathcal{E}^{i,j}) \longrightarrow \Gamma(\mathcal{E}^{i+1,j})$$

$$\Gamma(\mathcal{E}^{i,j+1}) \qquad \Gamma(\mathcal{E}^{i+1,j+1})$$

Cov. derivative's target are right also if the connection has torsion and  $\omega$  is pre-symplectic only  $(d\omega \neq 0)$ .

#### Theorem and a proof

**Theorem:** Let  $(M,\omega)$  be symplectic manifold admitting a metaplectic structure and let  $\nabla$  a Weyl-flat Fedosov connection on  $(M,\omega)$ . Then for all pairs of integers (i,j), the sequences  $(\Gamma(\mathcal{E}^{i+k,j\pm k}),T_\pm^{i+k,j\pm k})_{k\in\mathbb{Z}}$  form complexes. *Proof.* Basic steps, ideas

Composition of twistor operators

$$p\circ\nabla\circ\nabla_{|E^{ij}}$$

$$p \circ R^{\nabla}_{|E^{ij}}$$

From the structure of  $R^{\nabla}$  for Weyl-flat Fedosov connection, we see that  $p\circ R_{|E^{ij}}^{\nabla}=0$ 



#### **Parallel**

Parallel to: Let (M,J) be a complex manifold. Then for each couple of integers (i,j), the sequence  $(\Gamma(E^{i,j+k}), \overline{\partial}_{i,j+k})_{k\in\mathbb{Z}}$  of holomorphic-antiholomorphic differential forms and anti-Dolbeault differentials is a complex. Recall: Complex manifold = almost complex structure J is integrable (is complex), i.e., (Nirenberg–Newlander thm.), Nijenhuis' tensor N(X,Y) = [X,Y] + J[JX,Y] + [X,JY] - [JX,JY] of (M,J) is zero for all  $X,Y \in \Gamma(TM)$  (J complex  $\Rightarrow N=0$  is easy  $\Rightarrow$  the sequence is a complex).

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