Symplectic Dirac operator and its generalization May 19–29, 2005, Toulouse, France

Svatopluk Krýsl

ABSTRACT At an infinitesimal level, we will give a classification of 1^{st} order invariant differential operators acting on fields defined over contact projective geometries and having values in higher symplectic spinors. These fields are symplectic analogues of ordinary spinor fields in Riemannian geometry (the orthogonal case). In particular, we shall present a symplectic analogues of Dirac, twistor and Rarita-Schwinger operators and other higher spinor operators in the realm of parabolic geometries of contact projective type.
Keywords: Kostant spinors, symplectic Clifford algebras, symplectic Dirac operator, invariant differential operators.

1 Introduction

In early 1970's, a symplectic analogue of spinors was introduced by B. Kostant [14] in the context of geometric quantization of Hamiltonian mechanics. About twenty years later, K. Habermann introduced a symplectic analogue of the Dirac operator known from Riemannian geometry or classical Clifford analysis. This operator is now commonly called symplectic Dirac operator. We will present her Dirac operator as a version of one special invariant first order differential operator defined in the realm of projective contact geometries. These geometries are certain Cartan geometries closely related to the symplectic ones. In the projective contact and symplectic cases, important operators which were studied are acting between fields with values in infinite dimensional representations, which are analogous to (higher) orthogonal spinors. See, e.g., Habermann [8], Klein [13] for the symplectic Dirac operator and Kadlčáková [11] for the symplectic twistor operator. In this article, we will show that these representations (although infinite dimensional) can be handled for the seeks of the theory

AMS Subject Classification: 17B10, 17B20, 53D10.

of first order invariant differential operators in a similar way as the finite dimensional ones. See, e.g., Slovák, Souček [19] for a treatment of theory of invariant differential operators for finite dimensional irreducible representations. We will introduce the so called higher symplectic spinor representations, which are analogues of higher orthogonal spinors, and present some of their properties.

In the second section, the basic definitions of symplectic Clifford and Heisenberg algebras, Heisenberg group and the Segal-Shale-Weil representation of the symplectic Lie group are written. In the third section, higher symplectic spinors are introduced using irreducible highest weight modules. Basics on projective contact geometries are treated in the fourth section. The fifth section is devoted to first order invariant differential operators acting on fields over projective contact geometry with values in the higher symplectic spinors and to the classification result at the infinitesimal level. At the end of the fifth section, three specific examples of such operators are briefly introduced, namely the analogues of Dirac, twistor and Rarita-Schwinger operators.

The author of this article was supported by the grant GAČR 201/06/P223 of the Grant Agency of Czech Republic for young researchers. The work is a part of the research project MSM 0021620839 financed by MŠMT ČR. Also supported by the SPP 1096 of the DFG.

2 Symplectic Clifford algebras and Segal-Shale-Weil representation

Let us begin with a definition of symplectic Clifford algebra.

Definition: Let $(\mathbb{V}, \omega_0)$ be a real symplectic vector space of dimension $2n$, $n \in \mathbb{N}$,[1] i.e., $\mathbb{V}$ is a $2n$ dimensional real vector space and $\omega_0 : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is $\mathbb{R}$-bilinear, antisymmetric and non-degenerate form. Symplectic Clifford algebra $sC\ell(\mathbb{V}, \omega_0)$ is the following quotient algebra $sC\ell(\mathbb{V}, \omega_0) := \mathcal{T}\mathbb{V}/I(\mathbb{V}, \omega_0)$, where $I(\mathbb{V}, \omega_0)$ is a non-homogeneous two-sided ideal in the tensor algebra $\mathcal{T}\mathbb{V}$ generated by $v \otimes w - w \otimes v - \omega_0(v, w)1$, $v, w \in \mathbb{V}$.

Further, we shall choose two Lagrangian subspaces $\mathbb{L}, \mathbb{L}' \subseteq \mathbb{V}$ of $\mathbb{V}$, such that $\mathbb{V} \simeq \mathbb{L} \oplus \mathbb{L}'$. Let us recall that a Lagrangian subspace $\mathbb{L}$ of $\mathbb{V}$ is an n dimensional subspace of $\mathbb{V}$ such that $\omega_{0|\mathbb{L}\times\mathbb{L}} = 0$.

In this article, we shall use a special symplectic basis adapted to the decomposition $\mathbb{V} = \mathbb{L} \oplus \mathbb{L}'$. Because the definition of symplectic basis is not unique in the literature, let us fix one which we will be using in this text. We call a basis $\{e_i\}_{i=1}^{2n}$ of $\mathbb{V}$ symplectic basis of $(\mathbb{V}, \omega_0)$, if $\omega_{ij} := \omega(e_i, e_j)$ satisfies $\omega_{ij} = 1$ if and only if $i \leq l$ and $j = i + n$; $\omega_{ij} = -1$ if and only if

[1] Throughout this article, the symbol $\mathbb{N}$ denotes the set of all non-negative integers, i.e., $\mathbb{N} = \{0, 1, 2, \dots\}$.

$i > l$ and $j = i - n$ and finally, $\omega_{ij} = 0$ in other cases. For the rest of this article, we shall fix a specific symplectic basis which will be referred to as $\{e_i\}_{i=1}^{2n}$ and which satisfy the conditions $\{e_i\}_{i=1}^{n} \subseteq \mathbb{L}$ and $\{e_i\}_{i=n+1}^{2n} \subseteq \mathbb{L}'$.

Having any symplectic basis, we can say that $sC\ell(\mathbb{V}, \omega_0)$ is generated by $\{e_i\}_{i=1}^{2n}$ and 1 obeying the relations

$$e_i e_j - e_j e_i = \omega_{ij} 1,$$

for $1 \leq i, j \leq 2n$. The dimension of $sC\ell(\mathbb{V}, \omega_0)$ is always infinite if the dimension of the vector space $\mathbb{V}$ is bigger then 0.

The associative algebra $sC\ell(\mathbb{V}, \omega_0)$ equipped by the bracket $[,]$ defined by $[x, y] := xy - yx$ becomes a Lie algebra. When stressing the Lie algebra structure of $sC\ell(\mathbb{V}, \omega_0)$, we shall denote it by $(sC\ell(\mathbb{V}, \omega_0); [,])$. The Lie subalgebra of $(sC\ell(\mathbb{V}, \omega_0); [,])$ consisting of elements of homogeneity degree less or equal to 1 is called Heisenberg algebra. Let us denote it by H_n. Thus $H_n \simeq \mathbb{R} \oplus (\mathbb{L} \oplus \mathbb{L}') \simeq \mathbb{R} \oplus (\mathbb{R}^n \oplus \mathbb{R}^n)$, if the vector space $\mathbb{V}$ is of dimension $2n$. There is also a group structure on H_n, namely

$$(t, v).(s, w) := (t + s + \frac{1}{2}\omega_0(v, w), v + w),$$

$(t, v), (s, w) \in H_n$. The vector space H_n equipped by this group structure is usually referred to as Heisenberg group. One can easily verify that

$$(s, v).(t.w).(-s, -v).(-t, -w) = [(s, v), (t, w)]$$

for $(s, v), (t, w) \in H_n$, i.e., that the commutator in the Heisenberg group coincides with the Lie algebra bracket in the Heisenberg algebra.

Consider the representation

$$(\pi(t, (x, y))f)(z) := e^{-i(t + \omega_0(y, z - \frac{1}{2}x))} f(z - x),$$

$z \in \mathbb{L}$, $(t, (x, y)) \in H_n$ and $f \in L^2(\mathbb{L})$ of the Heisenberg group H_n on the Hilbert space $L^2(\mathbb{L})$ of complex valued Lebesgue square integrable functions defined on the Lagrangian subspace $\mathbb{L}$. This unitary representation is usually called Schrödinger representation of H_n in the literature.

Theorem: (Stone-von Neumann) Up to a unitary equivalence, there is exactly one unitary irreducible representation of H_n on $L^2(\mathbb{L})$

$$\pi : H_n \to \mathrm{Aut}(L^2(\mathbb{L}))$$

satisfying the condition $\pi(t, 0) = e^{-it} \mathrm{id}_{|L^2(\mathbb{L})}$ for all $t \in \mathbb{R}$.

Proof. See Folland [5], chapter 1.5. $\square$

Using the Schrödinger representation, one can define the following family of unitary representation of the Heisenberg group H_n on $L^2(\mathbb{L})$

$$\pi^g(t, v) := \pi(t, gv),$$

$g \in Sp(\mathbb{V},\omega_0)$ and $(t,v) \in H_n$, where we have denoted the symplectic group of $(\mathbb{V},\omega_0)$ by $Sp(\mathbb{V},\omega_0)$. Obviously, $Sp(\mathbb{V},\omega_0) \simeq Sp(2n,\mathbb{R})$.

Due to the Stone-von Neumann theorem, we have:

$$\pi^g(t,v) = U(g)\pi(t,v)U(g)^{-1}$$

for each $g \in Sp(\mathbb{V},\omega_0)$, $(t,v) \in H_n$ and a unitary automorphism $U(g)$ of the Hilbert space $L^2(\mathbb{L})$. The unitary transformations $U(g)$, $g \in Sp(\mathbb{V},\omega_0)$, satisfy

$$U(gh) = c(g,h)U(g)U(h)$$

for $g,h \in Sp(\mathbb{V},\omega_0)$ and some $c(g,h) \in \mathbb{C}$. Hence, one has a projective unitary representation of $Sp(\mathbb{V},\omega_0)$. It is known that one cannot choose the operators $U(g)$ in such a way that $c(g,h) = 1$ for all $g,h \in Sp(\mathbb{V},\omega_0)$, thus the representation cannot be considered as a true representation of the symplectic group. But it is well known and not so easy to prove that one can arrange the coefficients $c(g,h)$ in order $c(g,h) \in \{-1,1\}$, i.e., the projective representation $U : Sp(\mathbb{V},\omega_0) \ni g \mapsto U(g)$ can be lifted to a nontrivial double cover of the symplectic group. This double cover is usually called metaplectic group and will be denoted by $Mp(\mathbb{V},\omega_0)$ in this text. We shall denote the resulting unitary representation by Λ. Thus

$$\Lambda : Mp(\mathbb{V}_0,\omega_0) \to \mathcal{U}(L^2(\mathbb{L}))$$

(where $\mathcal{U}(L)$ is the group of unitary endomorphisms on a Hilbert space L) is a unitary representation of $Mp(\mathbb{V},\omega_0)$. For further details, see Kashiwara, Vergne [12] or Folland [5]. The representation Λ is usually called Segal-Shale-Weil representation.

3 Higher symplectic modules

Now, we would like to see the Segal-Shale-Weil representation Λ from the point of view of highest weight representations of the complex symplectic Lie algebra $\mathfrak{sp}(\mathbb{V},\omega_0)$ of the chosen symplectic vector space $(\mathbb{V},\omega_0)$. More precisely, we will introduce two irreducible representations of the Lie algebra $\mathfrak{sp}(\mathbb{V},\omega_0) \simeq \mathfrak{sp}(2n,\mathbb{C})$ such that their sum could be densely embedded into the (underlying vector space of the) Segal-Shale-Weil representation.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra of rank n and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Choosing a system Φ^+ of positive roots of the root system Φ, the system of fundamental weights $\{\varpi_i\}_{i=1}^n$ of $\mathfrak{g}$ is then uniquely defined. Thus we have a notion of dominant and integral weight. Recall that $\lambda \in \mathfrak{h}^*$ is called dominant and integral if λ could be written as $\lambda = \sum_{i=1}^n \lambda_i \varpi_i$ for $\lambda_i \in \mathbb{N}$, $i = 1,\dots,n$. Further, we shall define a vector $\delta := \sum_{i=1}^n \varpi_i$, the so called minimal regular weight. For $\lambda \in \mathfrak{h}^*$, we will denote the irreducible highest weight module with highest weight λ by

$L(\lambda)$. It is well known that $L(\lambda)$ is a finite dimensional irreducible module if and only if λ is dominant and integral. Let $\Pi(\lambda)$ be the set of all weights of $L(\lambda)$.

In general, for an arbitrary complex semisimple Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-module L is called a weight module if it is an algebraic direct sum $L = \bigoplus_\lambda L_\lambda$ of weight spaces

$$L_\lambda = \{v \in V; \forall H \in \mathfrak{h},\ H.v = \lambda(H)v\},$$

$\lambda \in \mathfrak{h}^*$. The $\mathfrak{g}$-module L is called module with bounded multiplicities if there exists a non-negative integer $k \in \mathbb{N}_0$, such that for each weight occurring in the decomposition $L = \bigoplus_\lambda L_\lambda$, we have $\dim L_\lambda \leq k$. The minimal such k is called the degree of L. The $\mathfrak{g}$-module L is called completely pointed if the degree of L is 1.

In the case of $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}, \omega_0) \simeq \mathfrak{sp}(2n, \mathbb{C})$, the Cartan algebra $\mathfrak{h}$ of $\mathfrak{g}$ is isomorphic to $\mathbb{C}^n$. For later use, we shall introduce a basis $\{\epsilon_i\}_{i=1}^n$ of $\mathfrak{h}^*$ given by the prescription $\varpi_j = \sum_{i=1}^j \epsilon_i,\ j = 1, \ldots, n$.

Theorem 1: If L is an infinite dimensional completely pointed $\mathfrak{sp}(\mathbb{V}, \omega_0)$-module then $L \simeq L(-\frac{1}{2}\varpi_n)$ or $L \simeq L(\varpi_{n-1} - \frac{3}{2}\varpi_n)$.

Proof. See Britten, Hooper, Lemire [1]. $\Box$

It is interesting to realize that the modules $L(-\frac{1}{2}\varpi_n)$ and $L(\varpi_{n-1} - \frac{3}{2}\varpi_n)$ serve a direct analogy to the spinor modules over complex orthogonal Lie algebras. Suppose that $(\mathbb{W}, B)$ is a $2n$ dimensional complex vector space endowed with a bilinear non-degenerate symmetric form B on $\mathbb{W}$, and denote the complex orthogonal Lie algebra associated to $(\mathbb{W}, B)$ by $\mathfrak{so}(\mathbb{W}, B)$, i.e., $\mathfrak{so}(\mathbb{W}, B) \simeq \mathfrak{so}(2n, \mathbb{C})$. Now, choose a maximal isotropic subspace $\mathbb{M}$ of $\mathbb{W}$. Then it is well known that for $\bigoplus_{i=0}^{2n} \bigwedge^i \mathbb{M}$ considered as a module over the orthogonal group $\mathfrak{so}(\mathbb{W}, B)$ possesses a decomposition

$$\bigoplus_{i=0}^{n} \bigwedge^{i} \mathbb{M} \simeq \mathbb{S}_+^{so} \oplus \mathbb{S}_-^{so}$$

into two inequivalent irreducible modules $\mathbb{S}_+^{so}$ and $\mathbb{S}_-^{so}$ which are called spinor modules. That is one of the easiest ways how the spinor modules could be defined.

Now, let us focus at the symplectic case. Let us denote the coordinates on the Lagrangian subspace $\mathbb{L}$ wr. to the basis $\{e_i\}_{i=1}^n$ by $(x^1, \ldots, x^n)$. There is a so called Chevalley realization [2] ψ of $\mathfrak{sp}(\mathbb{V}, \omega_0)$ in the algebra of differential operators acting on $\mathbb{C}[x^1, \ldots, x^n]$ given by the following

[2] In modern terms, ψ is actually an injective homomorphism of $\mathfrak{sp}(\mathbb{V}, \omega_0)$ into $\mathrm{End}_{\mathbb{C}}\mathbb{C}[x^1, \ldots, x^n]$.

prescription for the action of coroots $X_\alpha,\ \alpha \in \Phi,$ on $f \in \mathbb{C}[x^1,\dots,x^n]$

$$\begin{aligned}
&\psi(X_{\epsilon_i-\epsilon_{i+1}})f = x_{n-i}\partial_{n-i+1}f,\\
&\psi(X_{-(\epsilon_i-\epsilon_{i+1})})f = x_{n-i+1}\partial_{n-i}f, i=1,\dots,n-1\\
&\psi(X_{2\epsilon_n})f = -\frac{1}{2}\partial_1^2 f,\\
&\psi(X_{-2\epsilon_n})f = \frac{1}{2}x_1^2 f,
\end{aligned}$$

where the vectors $\epsilon_i,\ i=1,\dots,n,$ were introduced above.

With this realization, $\mathbb{C}[x^1,\dots,x^n]$ becomes a $\mathfrak{sp}(\mathbb{V},\omega_0)$-module as one can easily realize. Now, consider the module generated by 1; it can be checked that this is the module $L(-\frac{1}{2}\varpi_n) =: \mathbb{S}_+^{sp}$. The submodule generated by x^1 is the module $L(\varpi_{n-1}-\frac{3}{2}\varpi_n) =: \mathbb{S}_-^{sp}$. Thus we have obtained a decomposition of $\mathbb{C}[x^1,\dots,x^n]$ into "even" and "odd" polynomials:

$$\oplus_{i=0}^{\infty} \odot^i \mathbb{L} = \mathbb{S}_+^{sp} \oplus \mathbb{S}_-^{sp},$$

where $\odot^i\mathbb{L}$ (the ith symmetric power of $\mathbb{L}$) denotes the set of homogeneous polynomials in the symmetric algebra $\oplus_{i=0}^{\infty}\odot^i\mathbb{L} \simeq \mathbb{C}[x^1,\dots,x^n]$ of homogeneity degree $i,\ i \in \mathbb{N}.$ Thus the symplectic case is completely analogous to the orthogonal one: the word antisymmetric should be replaced by symmetric. We shall call $\mathbb{S}_+^{sp}$ and $\mathbb{S}_-^{sp}$ basic symplectic spinor modules.

Let us introduce the so called higher symplectic spinors. In the orthogonal case, each finite dimensional irreducible representation could be obtained as an irreducible direct summand in some tensor power of the defining representation or in a tensor product of this summand with one of the spinor representations. The latter are sometimes called higher spinors. We would like to translate this fact into the symplectic case. First, let us define the following set.

$$\mathbb{A} = \{\sum_{i=1}^{n}\lambda_i\varpi_i; \lambda_i \in \mathbb{N}, i=1,\dots,n-1, \lambda_n \in \mathbb{Z}+\frac{1}{2}, \lambda_{n-1}+2\lambda_n+3>0\}.$$

We would like to define higher symplectic spinor modules also as direct summands of a tensor product of a finite dimensional module over $\mathfrak{sp}(\mathbb{V},\omega_0)$ and one of the basic symplectic spinor modules. It is well known that a tensor product of an irreducible highest weight module and a finite dimensional module decomposes into a finite direct sum of irreducible submodules, see, e.g., Humphreys [9]. (In Kostant [15], one can find a more general version of this theorem.) In Britten, Hooper, Lemire [1] the following theorem characterizing the highest weight modules which can occur in the mentioned tensor products is proved.

Theorem 2: Let L be an irreducible highest weight $\mathfrak{sp}(\mathbb{V},\omega_0)$-module, then the following are equivalent

1. L is a direct summand of $L(-\frac{1}{2}\varpi_n)\otimes L(\nu)$ for some integral dominant weight $\nu\in\mathfrak{h}^*$;
2. L is an infinite dimensional module with bounded multiplicities;
3. The highest weight λ of L is in $\mathbb{A}$.

Proof. See Britten, Hooper, Lemire [1] and Britten, Hooper [2].

Thus we may define a higher symplectic spinor module as a module of the form $L(\lambda)$ for $\lambda\in\mathbb{A}$.

4 Projective contact geometry

This section is devoted to geometries for which we will define and classify the invariant differential operators. We start describing the so called homogeneous model of these geometries. Let $(\tilde{\mathbb{V}},\tilde{\omega_0})$ be a $2n+2$ dimensional symplectic vector space. Consider the action of the symplectic group $\tilde{G}=Sp(\tilde{\mathbb{V}},\tilde{\omega_0})$ on the space $\tilde{\mathbb{V}}\simeq\mathbb{R}^{2n+2}$ by the defining representation. This action restricts to a transitive action of $\tilde{G}$ on $\tilde{\mathbb{V}}-\{0\}$ and factors to a transitive action on the projective space $\mathbb{P}\tilde{\mathbb{V}}\simeq\mathbb{RP}^{2n+1}$. Let us denote the stabilizer of a point in the $\mathbb{P}\tilde{\mathbb{V}}$ by $\tilde{P}$. It can be checked that $\tilde{P}$ is a parabolic subgroup of $\tilde{G}$.

In order to gain some information about the group $\tilde{P}$, let us pass to the Lie algebra level. First, let us start with the notion of a $|k|$-grading. A $|k|$-grading ($k\in\mathbb{N}$) of a simple real or complex Lie algebra $\mathfrak{g}$ is a vector space direct sum decomposition $\mathfrak{g}=\mathfrak{g}_{-k}\oplus\ldots\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\ldots\oplus\mathfrak{g}_k$ such that $[\mathfrak{g}_i,\mathfrak{g}_j]\subseteq\mathfrak{g}_{i+j}$ for $i,j\in\{-k,\ldots,k\}$.[3] It is well known that if $\mathfrak{g}$ is a $|k|$-graded simple Lie algebra, then $\mathfrak{p}:=\mathfrak{g}_0\oplus\ldots\oplus\mathfrak{g}_k$ is a parabolic subalgebra of $\mathfrak{g}$, see, e.g., Čap, Schichl [4]. Moreover, it is known that for any $|k|$-grading, there exists an element $Gr\in\mathfrak{g}$ satisfying $[Gr,X]=jX$ for all $X\in\mathfrak{g}_j$ and $j\in\{-k,\ldots,k\}$, see again Čap, Schichl [4], for instance. This element is usually called grading element and its existence follows from the fact that each derivation in a simple Lie algebra is inner. Further, it is known that $\mathfrak{g}_0$ is a reductive Lie algebra.

Now, let us focus our attention at the specific case of the symplectic Lie algebra $\tilde{\mathfrak{g}}:=\mathfrak{sp}(\tilde{\mathbb{V}},\tilde{\omega_0})$. Suppose the following grading of $\tilde{\mathfrak{g}}$ is given:

$$\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_{-2}\oplus\tilde{\mathfrak{g}}_{-1}\oplus\tilde{\mathfrak{g}}_0\oplus\tilde{\mathfrak{g}}_1\oplus\tilde{\mathfrak{g}}_2,$$

where the the summands $\tilde{\mathfrak{g}}_i$, $i\in\{-2,\ldots,2\}$, satisfy: $\tilde{\mathfrak{g}}_2\cong\mathbb{R}$, $\tilde{\mathfrak{g}}_1\cong\mathbb{R}^{2n}$, $\tilde{\mathfrak{g}}_0\cong\mathfrak{sp}(2n,\mathbb{R})\oplus\mathbb{R}Gr\cong\tilde{\mathfrak{g}}_0^{ss}\oplus\tilde{\mathfrak{z}}$, where $\tilde{\mathfrak{g}}_0^{ss}$ denotes the semisimple part

[3] For $|j|>k$, we set $\mathfrak{g}_j=\{0\}$.

of the reductive algebra $\tilde{\mathfrak{g}}_0$ and $\tilde{\mathfrak{z}}$ the center of $\tilde{\mathfrak{g}}_0$. This grading could be also displayed as follows

$$\mathfrak{sp}(\tilde{\mathbb{V}}, \tilde{\omega_0}) = \mathfrak{g} \ni A = \left(\begin{array}{c|c|c} \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1 & \tilde{\mathfrak{g}}_2 \\ \hline \tilde{\mathfrak{g}}_{-1} & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1 \\ \hline \tilde{\mathfrak{g}}_{-2} & \tilde{\mathfrak{g}}_{-1} & \tilde{\mathfrak{g}}_0 \end{array} \right)$$

with respect to a basis for which $\tilde{\omega_0}((z^1, \ldots, z^{2l+2}), (w^1, \ldots, w^{2l+2})) = w^1 z^{2l+2} + \ldots + w^{l+1} z^{l+2} - w^{l+2} z^{l+1} - \ldots - w^{2l+2} z^1$. Using this matrix realization, one can easily check that $[\tilde{\mathfrak{g}}_i, \tilde{\mathfrak{g}}_j] \subseteq \tilde{\mathfrak{g}}_{i+j}$ for appropriate i, j. One can also realize that the restricted Lie bracket $[,] : \tilde{\mathfrak{g}}_{-1} \times \tilde{\mathfrak{g}}_{-1} \to \tilde{\mathfrak{g}}_{-2}$ is nondegenerate. A $|2|$-graded simple Lie algebra $\mathfrak{f}$ for which $\dim \mathfrak{f}_{-2} = 1$ and $[,] : \mathfrak{f}_{-1} \times \mathfrak{f}_{-1} \to \mathfrak{f}_{-2}$ is non-degenerate is called contact. It could be checked that the Lie algebra of the isotropy group $\tilde{P}$ is isomorphic to the Lie algebra $\tilde{\mathfrak{p}} := \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$ with $\tilde{\mathfrak{g}}_i$, $i = 0, 1, 2$, introduced above using the matrix description.

From the geometrical point of view, the groups $\tilde{G}$ and $\tilde{P}$ do not determine only the homogeneous space $\tilde{G}/\tilde{P}$ but also some further structures. We have already seen one of them at the level of Lie algebras, namely the so called contact structure. In differential geometry, a contact structure on a manifold M^{2n+1} is defined to be a corank 1 subbundle of the tangent bundle which is non integrable (in the Frobenius sense) in each point of the underlying manifold. Besides the contact structure on $\tilde{G}/\tilde{P} \simeq \mathbb{RP}^{2n+1}$, one can also canonically introduce a projective class of partial affine connection on the contact structure, see Fox [6].

One of the canonical ways how to define a deformation of a homogeneous space is to use the Cartan idea of espace généralisé. We shall do it for the specific $\tilde{G}$ and $\tilde{P}$ chosen in the first paragraph of this section. Formally, projective contact structure is defined to be a pair $(\mathcal{G}, \omega)$, where $\mathcal{G}$ is a principle fiber bundle over a manifold M^{2n+1} with a structure group $\tilde{P}$, and ω is the so called Cartan connection of type $(\tilde{G}, \tilde{P})$, i.e., $\omega : T\mathcal{G} \to \mathfrak{g}$ is a differential 1-form satisfying certain properties. (Here, $T\mathcal{G}$ denotes the tangent bundle of the total space $\mathcal{G}$.) We shall not give a precise definition of Cartan connection which differs from the definition of the principal bundle connection, and refer the interested reader to, e.g., Sharpe [18] or Čap, Schichl [4].

In our article, we shall consider a general Cartan connection of type $(\tilde{G}, \tilde{P})$ of a contact projective geometry.

5 First order invariant differential operators

From now on, $\tilde{G}$ and $\tilde{P}$ will have the meaning introduced in the preceding section. Let $(\mathcal{G}, \omega)$ be a projective contact geometry on a manifold M^{2n+1} and suppose two irreducible highest weight representations $\rho : \tilde{P} \to \mathrm{Aut}(E)$ and $\sigma : \tilde{P} \to \mathrm{Aut}(F)$ of $\tilde{P}$ on vector spaces E, F are given. Form the associated vector bundles $EM := \mathcal{G} \times_\rho E$ and $FM := \mathcal{G} \times_\sigma F$. Let us denote the vector space of sections of $FM \to M$ and $EM \to M$ by $\Gamma(M, EM)$ and $\Gamma(M, FM)$, respectively. The process of classification of first order invariant differential operators

$$D : \Gamma(M, EM) \to \Gamma(M, FM)$$

leads to an investigation of the set

$$\mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(\mathbb{C}^{2n} \otimes \mathbb{E}, \mathbb{F}),$$

where the space $\mathbb{C}^{2n}$ is considered to be the defining representation of $(\mathfrak{g}_0^{ss})^{\mathbb{C}} \cong \mathfrak{sp}(2n, \mathbb{C})$ and $\mathbb{E}$ and $\mathbb{F}$ are the underlying Harish-Chandra modules of E and F, respectively.

In the case of finite dimensional modules E and F, a connection between the space of intertwining $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-homomorphisms and the space of 1^{st} order invariant differential operators is described in Slovák, Souček [19]. For the case of infinite dimensions, see Krýsl [16] where more details about the considered modules E and F are explained.

In order to describe the set $\mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(\mathbb{C}^{2n} \otimes \mathbb{E}, \mathbb{F})$, we need a decomposition of the tensor product into $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-modules.

Theorem 3: Let $\lambda \in \mathbb{A}$ then

$$\mathbb{C}^{2n} \otimes L(\lambda) = \bigoplus_{\kappa \in \mathbb{A}_\lambda} L(\kappa),$$

where $\mathbb{A}_\lambda := \mathbb{A} \cap \{\kappa = \lambda + \nu; \nu \in \Pi(\varpi_1)\}$ and $\Pi(\varpi_1)$ is the set of weights of the defining representation $\mathbb{C}^{2n} \simeq L(\varpi_1)$.

Proof. See Krýsl [16]. □

Let us remark that the proof of this decomposition is based on the so called Kac-Wakimoto formal character formula, see Kac, Wakimoto [10].

Using a Dixmier generalization of Schur lemma, we derive the following theorem in which all 1^{st} order invariant differential operators are classified at least at the infinitesimal level. Using the so called minimal globalization functor (an adjoint functor to the Harish-Chandra forgetful functor), one is able to describe the set $\mathrm{Hom}_{G_0}(E \otimes \mathbb{C}^{2n}, F)$,[4] and thus to classify the invariant differential operators of first order also at the global level.

[4]Here, we have denoted the Levi factor of the parabolic group $\tilde{P}$ by G_0. In our case,

Theorem 4: Let $\lambda, \mu \in \mathbb{A}$ (see section 3) and let L resp. L' be the irreducible highest weight $\tilde{\mathfrak{p}}^{\mathbb{C}}$-modules on which the nilpotent part of $\tilde{\mathfrak{p}}^{\mathbb{C}}$ acts trivially, the grading element Gr acts by $w \in \mathbb{C}$ and $w' \in \mathbb{C}$, respectively, and L and L' are highest weight $\mathfrak{sp}(2n,\mathbb{C})$-modules with highest weight λ and μ, respectively. Then

$$\mathrm{Hom}_{\tilde{\mathfrak{p}}^{\mathbb{C}}}(\mathbb{C}^{2n} \otimes L, L') = \begin{cases} \mathbb{C} & \mu \in \mathbb{A}_\lambda \quad w = c^{\mu}_{\lambda\varpi_1} = w' - 1 \\ \{0\} & \text{in other cases,} \end{cases}$$

where

$$c^{\mu}_{\lambda\varpi_1} := \frac{1}{2}((\lambda + 2\delta, \lambda) + (\varpi_1 + 2\delta, \varpi_1) - (\mu + 2\delta, \mu)),$$

element δ is the half-sum of positive roots in $\mathfrak{sp}(\mathbb{V}, \omega_0)$ (see section 3), and $(,)$ is the rescaled Killing form, the normalization is chosen in such a way that $(Gr, Gr) = 1$ for the unique grading element Gr.

Proof. See Krýsl [16]. $\square$

For a brevity, let us introduce symbol $L(\lambda, w)$ for the module L defined in formulation of the the preceding theorem, i.e., the nilpotent part of $\tilde{\mathfrak{p}}^{\mathbb{C}}$ acts trivially on it, the grading element acts by $w \in \mathbb{C}$ and $L(\lambda, w)$ is the irreducible highest weight representation with highest weight λ when restricted to $\mathfrak{sp}(\mathbb{V}, \omega_0) \simeq \mathfrak{sp}(2n, \mathbb{C})$. We shall denote the minimal globalization of this module by the same letter.

The last part of this section is devoted to some examples. One can use theorem 3, to obtain the following decompositions $\mathbb{C}^{2n} \otimes \mathbb{S}^{sp}_{+} \cong \mathbb{S}^{sp}_{-} \oplus \mathbb{T}^{sp}$, where $\mathbb{T}^{sp} = L(\varpi_1 - \frac{1}{2}\varpi_n)$ is the so called symplectic twistor space. Further $\mathbb{C}^{2n} \otimes \mathbb{T}^{sp} \cong L(2\varpi_1 - \frac{1}{2}\varpi_n) \oplus L(-\frac{1}{2}\varpi_n) \oplus L(\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n) \oplus L(\varpi_2 - \frac{1}{2}\varpi_n)$.

Due to the previous theorem, there is up to a multiple, only one invariant differential operator between the section of bundles associated to $S_+ := L(-\frac{1}{2}\varpi_n, \frac{1+2n}{2})$ and $S_- := L(\varpi_{n-1} - \frac{3}{2}\varpi_n, \frac{3+2n}{2})$. Let us say more precisely how this operator is defined. Let $(\mathcal{G}, \omega)$ be a contact projective geometry over a manifold M^{2n+1}. Form the associated bundles S_+M and S_-M. Now the symplectic Dirac operator maps the sections of S_+M into the sections of S_-M and is given as the composition of the absolute invariant derivative ∇^ω canonically associated to the Cartan connection ω followed by a $\tilde{P}$-invariant homomorphism D from $\mathbb{C}^{2n} \otimes \mathbb{S}^{sp}_{+}$ to $\mathbb{S}^{sp}_{-}$, which is unique up to a constant multiple. For the definition of associated absolute invariant derivative, see Slovák, Souček [19].

Now, let us briefly mention some other distinguished operators the analogies of which are used in the orthogonal case. The twistor operator corresponds to the projection operator $T : \mathbb{C}^{2n} \otimes \mathbb{S}^{sp}_{+} \to \mathbb{T}^{sp}$, and finally the

---

$G_0 \simeq Sp(2n, \mathbb{R})$.

Rarita-Schwinger operator corresponds to the projection $R : \mathbb{C}^{2n} \otimes \mathbb{T}^{sp} \to L(\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n)$.

In the future, we should investigate a coordinate description of the mentioned operators, their spectra and relations to the considered geometric structure. Let us also remark, that the symplectic Dirac operators are also used in physics, see Reuter [17] and Green, Hull [7].

The author of this text was supported by the grant GAČR 201/06/P223 of the Grant Agency of Czech Republic and by the grant GAUK 447/2004. Also supported by SPP 1096 of the DFG. The work is a part of the research project MSM 0021620839 financed by MŠMT ČR.

References

[1] Britten, D.J.; Hooper, J.; Lemire F.W., Simple C_n-modules with multiplicities 1 and application, Canadian Journal of Physics 72, 326-335 (1994).

[2] Britten, D.J; Lemire, F. W., On modules of bounded multiplicities for the symplectic algebra, Transactions of the AMS Vol. 351 No. 8, 3413-3431 (1999).

[3] Dixmier, J., Enveloping Algebras, North-Holland Publ. Comp., Akademie-Verlag Berlin, (1976).

[4] Čap, A., Schichl, H., Parabolic Geometries and Canonical Cartan connections, Hokkaido Math. J. 29 No.3, 2000, 453-505; electronically available at www.esi.ac.at.

[5] Folland, G., Harmonic Analysis in Phase Space, Princeton University Press, (1989).

[6] Fox, D., Contact Projective Structures, Ph.D. thesis (2004); electronically available at math.DG:0402332.

[7] Green,M.B., Hull, C.M., Covariant quantum mechanics of the superstring, Phys. Lett. B, 225, (1989), pp. 57 -65.

[8] Habermann, K., Symplectic Dirac Operators on Kähler Manifolds. Math. Nachr., 211, (2000).

[9] J.E. Humphreys, Finite and infinite dimensional modules for semisimple Lie algebras, Lie theories and their applications, (Proc. Ann. Sem. Can. Math. congr., Queen's University, Kingston, Ont. 1977); Queen's Papers in Pure and Appl. Math. 48, Queeen's University, Kingston, Ont., 1978, pp. 1- 64.

[10] V.G. Kac, M. Wakimoto, Modular invariant representations of infinite dimensional Lie algebras and superalgebras, Proc. Natl. Acad. Sci. USA, Vol. 85, No. 14, Nat. Acad. Sci., Washington, DC., pp. 4956 - 4960, 1988.

[11] Kadlčáková, L., Dirac operator in parabolic contact symplectic geometry, Ph.D. thesis, Charles University Prague, (2001).

[12] Kashiwara,M, Vergne, M., On the Segal-Shale-Weil representation and harmonic polynomials, Inventiones Mathematicae 44, 1978, 1-47.

[13] Klein, A., Eine Fouriertransformation für symplektische Spinoren und Anwendungen in der Quantisierung, Diploma Thesis, Technische Universität Berlin, (2000).

[14] Kostant, B., Symplectic Spinors, Symposia Mathematica, vol. XIV, (1974).

[15] Kostant, B., On the tensor Product of a Finite and an Infinite Dimensional Representations, Journal of Functional Analysis 20, 1975, 257-285.

[16] Krýsl, S., Invariant differential operators for projective contact geometries, Ph.D. thesis, Charles University Prague, (2004).

[17] Reuter, M., Symplectic Dirac-Kähler Fields, J.Math.Phys., 40, (1999), pp. 5593-5640, electronically available at hep-th/9910085.

[18] Sharpe, R.W., Differential geometry: Cartan's generalization of Klein's Erlangen program, Graduate Texts in Mathematics, Vol. 166., Springer.

[19] Slovák, J., Souček, V., Invariant operators of the first order on manifolds with a given parabolic structure, Séminaires et Congr è s 4, SMF, (2000),pp. 251-276.

Svatopluk Krýsl

Mathematical institute

Charles University, Prague

Sokolovská 83, 186 78 Prague 8 - Karlin

E-mail: krysl@karlin.mff.cuni.cz

Submitted: November 15, 2005; Revised: TBA.