# Images of elliptic operators on Hilbert bundles on compact manifolds are orthogonally complemented

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**Abstract**

We prove that the image of each elliptic operator on a smooth separable Hilbert fibre bundle on a compact manifold is closed with respect to the topology generated by a natural inner product. We consider the completed injective tensor product of the elliptic operator with the identity operator on the typical fibre of the bundle and compare its image with the image of the original operator. Further we show that the operators image is orthogonally complemented.

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## 1 Introduction

Thanks to the Hodge theory it is known that kernels and cokernels of elliptic operators on *finite rank* hermitian vector bundles on a compact manifold $M$ are finite dimensional vector spaces and that the images of these operators are closed with respect to the topology generated by a natural inner product on the space of smooth sections of the vector bundle. The closed image property of the images is tightly related to the known fact that continuous extensions of elliptic operators to Sobolev spaces are Fredholm. See, e.g., Palais [42] and Wells [62]. In the paper, we prove that images of elliptic operators on smooth sections of

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separable infinite rank Hilbert bundles on compact manifolds are closed and that they are orthogonally complemented with respect to the topology generated by the natural inner product on the space of smooth sections of the bundle, that we shall call the pre-Hilbert topology (Theorem 11). Notice that in the infinite rank case the kernel of an elliptic operator needn't be a finite dimensional vector space. Indeed, let us consider the product bundle of the unit circle with an infinite dimensional real or complex Hilbert space $V$ and the operator of the directional derivative with respect to an angle coordinate on the circle that acts on smooth $V$-valued maps. It is easy to compute the symbol of this operator and to realize that the operator is elliptic, i.e., that its symbol is an isomorphism outside of the zero section of the cotangent bundle of the circle. Since the kernel of the directional derivative is the infinite dimensional space of constant $V$-valued maps, any extension of the operator to a Banach space is not Fredholm. Obviously, this does not mean that the image of the operator is not closed.

*Context and Applications.* The closed image property of elliptic operators on finite rank bundles plays an important role in the Hodge theory of elliptic complexes (Wells [62] and Hodge [19]) and in the representation theory of Lie groups concerning the Borel–Weil theorem and cohomological induction (Knapp, Vogan [24], Schmid [52] and Wong [64]). Regarding the infinite rank bundles there are articles devoted to consequences of the topological complementability of kernels of elliptic operators with respect to the natural Fréchet topology that is related to the so-called Schwartz kernel theorem. See, e.g., Tréves [58] and [59] (Grothendieck's theorem in Appendix C.1), Poly [44], and Vogt [60]. We refer to Illusie [20] and Röhrl [47] for treatises on topological complexes on Banach and Fréchet bundles. For holomorphic Banach bundles, see Lempert [34] and Kim [23]. Notice that there is an example of a holomorphic Banach fibre bundle on the two dimensional sphere whose first sheaf cohomology is non-Hausdorff, i.e., the image of the zeroth codifferential is not closed in the space of 1-cochains. See Erat [10]. Regarding the analysis on Banach fibre bundles related to non-commutative geometry, quantization, and global analysis on infinite rank bundles, let us mention, e.g., Higson and Roe [17], Maeda and Rosenberg [36], Freed and Lott [15], Larraín-Hubach [33], Krýsl [30] and [29], Fathizadeh and Gabriel [11], and Habermann, Habermann [18]. This reference list shall not be considered as complete.

*Methods for finite rank bundles.* In the case of a *finite rank* vector bundle on a compact manifold, the compact embedding theorem of Rellich and Kondrachov for the Sobolev spaces is usually applied for proving that continuous extensions of elliptic operators to the Sobolev spaces are Fredholm and that consequently, the images of these extensions are closed. See, e.g., Seeley [50] or Palais [42]. However, it is not difficult to see that a straight-forward generalization of this compact embedding theorem does not hold for Sobolev spaces $H^{k,2}(M,V)$ of $V$-valued functions on $M$ if $V$ is infinite dimensional. By such a generalization we mean the assertion that the inclusion $J_{k+1}: H^{k+1,2}(M,V) \to H^{k,2}(M,V)$ is a compact map for a separable Hilbert space $V$. Indeed, let us consider the inclusion map $J^0_{k+1}: H^{k+1,2}(M) \to H^{k,2}(M)$ of the scalar Sobolev spaces for a smooth compact manifold $M$. By Wloka [63], there is a unitary isomorphism $\Phi_k : H^{k,2}(M,V) \to H^{k,2}(M)\widehat{\otimes}_{HS}V$ of Hilbert spaces, where $\widehat{\otimes}_{HS}$ denotes the so-called Hilbert–Schmidt tensor product. The inclusion map $J_{k+1} : H^{k+1,2}(M,V) \to H^{k,2}(M,V)$ is equal to $\Phi_k^{-1} \circ (J^0_{k+1}\widehat{\otimes}_{HS}\mathrm{Id}_V) \circ \Phi_{k+1}$. Since in this case $\mathrm{Id}_V$ is a non-compact operator and $\Phi_k$ and $\Phi_{k+1}$ are unitary, $J_{k+1}$ cannot be compact. As follows by our main result, this does not show that images of elliptic operators on infinite rank Hilbert bundles can not be closed though. If $E$ is a Banach space, we denote the Sobolev-type space of $E$-valued maps on $M$ by $H^k(M,E)$ because we use the Sobolev spaces $H^{k,l}(M,E)$ for $l = 2$ only. We give a definition of these spaces based on [56] in our paper if $E$ is the so-called Hilbert $A$-module.

*$C^*$-elliptic theory of Fomenko and Mishchenko.* We apply parts of the theory of Fomenko and Mishchenko developed in [39] for so-called $C^*$-elliptic operators on Hilbert $C^*$-bundles whose fibres are finitely generated and projective Hilbert $C^*$-modules. From the point of view of topological vector spaces, Hilbert $C^*$-modules are specific Banach spaces that are generalizations of Hilbert spaces as well as of $C^*$-algebras. From the algebraic point of view, they are modules over a $C^*$-algebra. These modules are considered with the topology generated by the so-called induced $C^*$-norm. Let us notice that they are introduced together with pre-Hilbert $C^*$-modules in Paschke [43] and Rieffel [46]. Since the terminology concerning Hilbert $C^*$-modules is non-unique (cf., e.g., Lance [31], Wegge-Olsen [61] and Blackadar [5]), we fix a terminology which we use in our paper (Chapter 2). The space of smooth sections of a smooth Hilbert $C^*$-bundle is a pre-Hilbert $C^*$-module in a natural way and its Sobolev-type completions form Hilbert $C^*$-modules. We call the topology on the smooth sections generated by the induced $C^*$-norm the pre-Hilbert topology though it need not be generated by an inner product. In the theory of Hilbert $C^*$-modules, generalizations of compact and Fredholm operators are defined, which are called $C^*$-compact and $C^*$-Fredholm, respectively. Let us notice that continuous extensions of $C^*$-elliptic operators to Sobolev-type completions are proved to be $C^*$-Fredholm in [39] if the fibres of the bundle are topologically finitely generated and projective Hilbert $C^*$-modules. We cannot use this result since the fibres of the bundles considered in our paper are not topologically finitely generated. Proofs of several theorems in [39] and [56] are only sketched foremost if they are parallel to the proofs for the finite rank bundles. When we need a generalization of a result from [39] or [56], but we find its proof too brief there, we dare to add some details in our article.

For completeness, let us notice that compact perturbations of pseudodifferential operators are handled regarding their Fredholm property also without the assumption on the $C^*$-linearity of these operators. See, e.g., Luke [35] and citations therein. However, we do not get topological properties of the image if we allow perturbations though by compact operators only.

Now we describe the procedure which leads to the proof of the closed image property of elliptic operators on Hilbert bundles. By Burghelea and Kuiper [6] and Moulis [40], an infinite rank Hilbert bundle with fibre the Hilbert space $V$ is $C^\infty$-diffeomorphic to the product Hilbert bundle $M \times V \to M$. It is straightforward to realize that the space $\Gamma^\infty(M,\mathcal{V})$ of smooth sections of $\mathcal{V}$ is linearly homeomorphic to the space $C^\infty(M,V)$ of smooth $V$-valued functions if both of these spaces are equipped with Fréchet topologies or if both are equipped with the natural pre-Hilbert topologies.

*Transfer to compact $CV$-bundles.* The first step we undertake is a transfer to Hilbert $C^*$-bundles and $C^*$-elliptic pseudodifferential operators for the $C^*$-algebra $CV$ of compact operators of a Hilbert space $V$. The vector space of compact operators carries a natural structure of a Hilbert $CV$-module with the $C^*$-product $(,)_{CV}$ given by $(A,B)_{CV} = A^*B$, $A,B \in CV$, whose induced $C^*$-norm equals to the operator norm. We call the Hilbert $CV$-module $CV$ the *compact $CV$-module*, and any Hilbert $C^*$-bundle isomorphic to the product Hilbert $C^*$-bundle $M \times CV \to M$ the *compact $CV$-bundle*. For the transfer procedure we consider the completed injective tensor product of the Fréchet space $C^\infty(M,V)$ with the continuous dual $V^*$ of $V$. The resulting Fréchet space is denoted by $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$. (See Grothendieck [16] or Tréves [58].) By Tréves [58], $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$ and $C^\infty(M,CV)$ are linearly homeomorphic when both of them are equipped with Fréchet topologies. We consider the tensor product of an elliptic operator $D$ with the identity on $V^*$ and extend it continuously to $C^\infty(M,V)\widehat{\otimes}_\epsilon V^* \cong C^\infty(M,CV)$, denoting the resulting operator $D\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$ by $\widehat{D}^\epsilon$. We show that $\widehat{D}^\epsilon$ is $CV$-linear (Lemma 8) and elliptic (Theorem 9). We prove that its image is closed in the Fréchet topology.

*Properties of continuous extensions of $CV$-elliptic operators to Sobolev spaces.* The compact $CV$-module is not topologically finitely generated and thus the appropriate results of Fomenko and Mishchenko in [39] on the $C^*$-compact embedding of the Sobolev-type completions of smooth sections of Hilbert $C^*$-bundles cannot be used. Let us notice that the compact $CV$-module is well known to be projective as a consequence of the results of Magajna [37] and Schweitzer [53]. (See Frank and Paulsen [14].) We derive a $C^*$-compact embedding for compact $CV$-bundles on tori using a theorem of Bakić and Guljaš in [4]. We adapt a classical procedure (Palais [42] or Solovyov and Troitsky [56]) to prove a $CV$-compact embedding for compact $CV$-Hilbert bundles on a compact manifold $M$ (Theorem 7, part a)), i.e., that the inclusion map $H^{k+1}(M,CV) \to H^k(M,CV)$ is $CV$-compact. Then we show that the continuous extension $\mathcal{D}_k$ to the Sobolev-type completions $H^k(M,CV)$ of an elliptic operator $\mathcal{D} : C^\infty(M,CV) \to C^\infty(M,CV)$ are $CV$-Fredholm (Theorem 7 b)). By Lemma 1 the image of the extension is closed in $H^{k-d}(M,CV)$ with respect to the topology generated by the induced $C^*$-norm where $d$ denotes the order of $D$. Let us notice that an operator with a closed image need not be $C^*$-Fredholm for trivial reasons and that on the other hand, an $C^*$-Fredholm operator need not have a closed image (e.g., [26]). However, by [4], for the $C^*$-algebra of compact operators it is known that the $C^*$-Fredholm property implies that the image is closed.

*Properties of $\widehat{D}^\epsilon$.* Since $\widehat{D}^\epsilon = D\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$ is $CV$-elliptic (Theorem 9), its continuous extension $(\widehat{D}^\epsilon)_k$ to $H^k(M,CV)$ is $CV$-Fredholm for any $k \in \mathbb{Z}$ by the mentioned Theorem 7 b) and regular by Theorem 6. We want to prove that the image of $\widehat{D}^\epsilon : C^\infty(M,CV) \to C^\infty(M,CV)$ is closed using Theorem 2 and Corollary 3, which contain assumptions on the regularity not only of the operator but also of the operator's adjoint. In the Hilbert bundle case, the adjoint operators are usually constructed by considering transposed operators defined on the topological duals of the Sobolev-type spaces using the self-duality of Hilbert spaces forming the fibres of the considered bundle. See, e.g., Palais [42] and a parallel construction in Solovyov, Troitsky [56] for Hilbert $C^*$-bundles with the so-called $C^*$-self-dual fibres. However, it is known that the compact $CV$-module is *not $C^*$-self-dual* if $V$ is infinite dimensional. The $C^*$-dual of the compact $CV$-module, i.e., the space of all operators of the Hilbert $C^*$-module $CV$ into the $C^*$-algebra $CV$ contains all bounded operators on $V$ which forms a strict superset of the set of compact operators. We thus have to analyse the adjoints of $CV$-pseudodifferential operators of type $D\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$ on compact $CV$-bundles. This is done carefully in part 2) of the proof of Theorem 9.

*Properties of $D$ and complementability.* At the end of the paper, we return our attention to the investigation of the image of $D$. We prove that the image of $D$ is closed in the inner product space of the smooth sections of the Hilbert bundle. This is done in Theorem 11, in whose proof we use a lemma on a presentation of elements in $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$ (Lemma 10). In the proof of Theorem 11, we make a use of a theorem on the interchange of the limiting and summation processes for vector-valued double series (Scholium 3), based on a result of Antosik in [1]. In Pap et al. [41], there are further results on the convergence of vector-valued double series. Nevertheless, it looks like that no theorems are available that could be used for the purpose of our paper directly. After proving the closed image of $D$ we show that it is complemented in the inner product space of the smooth sections by proving the closed image for $D_k$ in the appropriate Hilbert space-valued Sobolev space and by using the Corollary 3 for the $C^*$-algebra of complex numbers. Notice that this inner product space is not complete if the dimension of the base manifold $M$ is at least 1.

*Organization of the paper.* In the second section, we summarize the terminology on Hilbert $A$-modules and their morphisms and prove two assertions (Theorem 2, and Corollary 3) on pre-Hilbert module morphisms whose continuous extensions are specific Hilbert $A$-module operators with closed images. At the beginning of the third section, we give a definition of a smooth Hilbert $C^*$-bundle based on the notion of the one-point Hilbert $C^*$-module. We prove a lemma in which the pre-Hilbert topology is compared with the Fréchet topology (Lemma 4) and a lemma on adjoints of pseudodifferential operators on Hilbert bundles (Lemma 5). In the main part of the third section, we deal with morphisms of Hilbert $CV$-modules. We prove the regularity for $CV$-elliptic operators on compact manifolds (Theorem 6), an assertion on the $CV$-compact embedding for Sobolev-type spaces on tori (Scholium 1) and on general compact manifolds (Theorem 7). The fourth chapter is devoted to the main theme of the paper, namely to the images of elliptic operators on Hilbert bundles themselves. We recall the completed injective tensor products, prove that continuous extensions to completed injective tensor products of pseudodifferential operators multiplied by the appropriate identity operator are $CV$-linear and that they inherit the ellipticity property (Lemma 8 and Theorem 9). We also prove a lemma on a representation of smooth $CV$-valued maps (Lemma 10). In this part, the theorem is proved on the closed image property of elliptic operators and the complementability of the image with respect to the pre-Hilbert topology (Theorem 11).

The next preamble concerns the notation and conventions used in the paper foremost, regarding the notions of smoothness of maps, smooth Banach manifolds and bundle atlases and their charts. Its parts can be read when the appropriate notions appear in the text.

### 1.1 Preamble

a) We denote the composition of maps $a : Y \to Z$ and $b : X \to Y$ by $a \circ b$ as well as by $ab$. The value of a map $D : X \to Y$ on an element $x \in X$ is denoted by $D(x)$ or by $Dx$.

b) We suppose that a Fréchet topological vector space structure on a vector space $E$ is defined by an ordered countable family of separating seminorms. We equip $E$ with the canonical translation invariant metric induced by this ordered family. The space is complete with respect to this metric.

c) For topological vector spaces $W'$ and $W''$ the symbol $\mathrm{Hom}(W', W'')$ denotes the vector space of continuous linear maps of $W'$ into $W''$, $\mathrm{End}(W)$ denotes the set of all linear continuous maps of $W$ into $W$, and $\mathrm{Aut}(W)$ is the subset of $\mathrm{End}(W)$ consisting of all continuously invertible maps of $W$ onto $W$. The continuous dual of any topological vector space $W$ is denoted by $W^*$.

d) The field $\mathbb{R}$ of real numbers is considered as the image in the field $\mathbb{C}$ of the injection $r \in \mathbb{R} \mapsto r + 0\imath \in \mathbb{C}$. Inner product spaces are considered over real or complex numbers. The inner product is complex anti-linear in the first variable and complex linear in the second variable (physicist's convention). For pre-Hilbert $C^*$-modules, we suppose the same behaviour of the $C^*$-product. The topology of inner product spaces and Hilbert $C^*$-modules is generated by the metric induced by the inner and $C^*$-products, respectively.

e) Continuous maps of topological vector spaces are called $C^k$-differentiable briefly if their $l$th order Fréchet differential is continuous for $l = 0, \ldots, k$. We call them $C^\infty$-differentiable (or smooth) if they are $C^k$-differentiable for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A map is called a $C^\infty$-diffeomorphism if it is smooth, bijective and its inverse is smooth.

f) Any manifold is considered to be a $C^\infty$-differentiable Banach manifold without boundary, i.e., a Hausdorff second countable topological space locally homeomorphic to a fixed Banach space $E$ and equipped with a maximal $C^\infty$-differentiable manifold atlas. Elements of a $C^\infty$-differentiable atlas, called manifold charts, have to be homeomorphisms of open subsets of the manifold into open subsets in $E$. Transition maps of a $C^\infty$-differentiable manifold atlas are demanded to be $C^\infty$-diffeomorphisms of subsets of $E$. For simplicity, we do not consider manifolds as equivalence classes of maximal smooth atlases with respect to $C^\infty$-diffeomorphisms. Therefore a manifold in our sense is what is usually called a $C^\infty$-differentiable structure.

g) Let $\mathcal{W}$ be a manifold, $M$ be a finite dimensional manifold, and $W$ be a Banach space. A Banach fibre bundle $p : \mathcal{W} \to M$ with a fibre the Banach space $W$ is a smooth submersion of manifolds such that for each $m \in M$, the fibre $p^{-1}(m)$ is a Banach space whose normed topology is equal to the subset topology induced by the inclusion $p^{-1}(m) \subseteq \mathcal{W}$. The manifold $\mathcal{W}$ is called the total space and the manifold $M$ is called the base space of $p$. Further, a Banach fibre bundle has to be equipped with a maximal $C^\infty$-differentiable *bundle atlas*. By our convention, elements of the bundle atlas, called bundle charts, have to be $C^\infty$-diffeomorphisms of $U \times W$, $U \subseteq M$ open, onto the open subset $p^{-1}(U)$ of the $C^\infty$-differentiable manifold $\mathcal{W}$, such that its restriction to $\{m\} \times W$ is a linear homeomorphism of Banach spaces onto $p^{-1}(m)$ for each point $m \in M$. A subatlas of an atlas $\mathcal{A}$ is a subset of $\mathcal{A}$ such that the union of the domains of its charts is $M \times W$, i.e., it is still an atlas. Let us notice that by the chain rule for Banach spaces, transition maps of a bundle atlas are smooth maps into the vector space $\mathrm{End}(W)$ considered with the strong operator topology.

   *Comparison to different concepts.* We do not consider Banach fibre bundles as equivalence classes of maximal $C^\infty$-differentiable atlases with respect to $C^\infty$-diffeomorphisms of bundles that cover the identity on the base space. The fibre bundles, which we consider, are smooth analogues of the so-called *coordinate G-bundles* as defined, e.g., in Steenrod [57], where $G = \mathrm{Aut}(W)$ with the strong operator topology. (For details, see the Remark below the Definition 2 in the Section 3.)

h) Pseudodifferential operators on a fibre bundle are, in particular, real or complex linear maps defined on the vector space of *smooth sections* of the fibre bundle.

# 2 Images of pre-Hilbert $CV$-module morphisms

Let $(A, \cdot, |\,|_A, ^*)$ be a $C^*$-algebra and let us denote the closed half-cone of positive elements in $A$ by $A^+$. The spectrum of a positive element in $A$ has to be contained in the set of non-negative real numbers. (See, e.g., [8].) A *right pre-Hilbert A-module* is a complex vector space $W$ on which $A$ acts from the right compatibly with the multiplication by scalars, and that is equipped with a hermitian-symmetric map $(,)_W : W \times W \to A$ that is

i) sesquilinear with respect to the action of the field $\mathbb{C}$ and to the right action of the $C^*$-algebra $A$; and

ii) positive definite in the sense that for each $w \in W$, the element $(w, w)_W$ belongs to $A^+$ and $(w, w)_W = 0$ only if $w = 0$.

Such a pre-Hilbert $A$-module is denoted by $(W, (,)_W)$. The hermitian-symmetric map $(,)_W$ is called the *$C^*$-product* or the *$A$-product* when the $C^*$-algebra is $A$. Since we shall consider only right pre-Hilbert $A$-modules, we omit the word 'right' and call any right Hilbert $A$-module a Hilbert $A$-module only. Let $c \in \mathbb{C}$, $a \in A$, and $v, w \in W$. The compatibility of the $C^*$-algebra action with the multiplication by scalars means that $v \cdot (ca) = (cv) \cdot a = c(v \cdot a)$. The $C^*$-product $(,)$ is hermitian-symmetric, i.e., $(v, w) = (w, v)^*$. By the sesquilinearity we mean that the $C^*$-product $(,)_W$ satisfies $(v \cdot a, w)_W = a^*(v, w)_W$ and that a similar rule holds for the multiplication by complex numbers. The equivariance in the second entry follows from the hermitian-symmetry. The *induced $C^*$-norm* $| \, |_W : W \to \mathbb{R}$ (called also the *induced $A$-norm* if the $C^*$-algebra is $A$) is defined by $|w|_W = \sqrt{|(w, w)_W|_A}$, where $w \in W$. It satisfies $|(v, w)_W|_A \leqslant |v|_W |w|_W$ (Cauchy–Schwartz-type inequality) for all $v, w \in W$. We consider any pre-Hilbert $A$-module as an $A$-module and as a topological vector space with the topology generated by the metric $d(v, w) = |v - w|_W$, $v, w \in W$. (See [31].) Let us notice that a pre-Hilbert $C^*$-module for the $C^*$-algebra of complex numbers is an inner product space. (We shall not use the term 'pre-Hilbert space' for the inner product spaces in the text.)

Objects in the *category of pre-Hilbert $A$-modules* are pre-Hilbert $A$-modules. Let $(W, (,)_W)$ and $(W', (,)_{W'})$ be pre-Hilbert $A$-modules and $L : W \to W'$ be a map. It is called adjointable if there exists a map $L^* : W' \to W$ such that $(Lw, w')_{W'} = (w, L^*w')_W$ for all $w \in W$ and $w' \in W'$. By the non-degeneracy of $(,)_W$ and $(,)_{W'}$, the map $L^*$ (called the adjoint of $L$) is unique if it exists. It is easy to see that an adjointable map is complex- and $A$-linear, i.e., $L(w \cdot a) = L(w) \cdot a$ and $L(cw) = cL(w)$ for all for all $c \in \mathbb{C}$, $a \in A$ and $w \in W$. Morphisms of pre-Hilbert $A$-modules $W$ and $W'$ are all maps of $W$ into $W'$ having the adjoint. We often call a morphism an operator. The composition of morphisms is the composition of maps. This establishes the category of pre-Hilbert $A$-modules for a $C^*$-algebra $A$ since the adjoint of the composition of two operators is the composition of the adjoints of the operators in the reversed order. We denote the set of all adjointable $A$-linear maps of pre-Hilbert $A$-modules $W$ and $W'$ by $\mathrm{Hom}^*_A(W, W')$ and by $\mathrm{End}^*_A(W)$ if $W = W'$. The set of all bijections onto $W$ in $\mathrm{End}^*_A(W)$ is denoted by $\mathrm{Aut}^*_A(W)$ and called the group of pre-Hilbert $A$-module automorphisms of $W$.

We call an adjointable map $L$ self-adjoint if $L = L^*$ and we call it unitary if $L^*L = \mathrm{Id}_W$ and $LL^* = \mathrm{Id}_{W'}$. A pre-Hilbert *$A$-submodule* of $(W, (,)_W)$ is any algebraic $A$-submodule $W_1$ of $W$ equipped with the restriction to $W_1 \times W_1$ of $(,)_W$. A pre-Hilbert $A$-submodule need not be closed in $W$. The orthogonal complement of a subset $W_1 \subseteq W$ is denoted by ${W_1}^\perp = \{w \in W : (w, w_1) = 0 \text{ for all } w_1 \in W_1\}$. The direct sum $W = W_1 \bigoplus W_2$ of pre-Hilbert $A$-modules $W_1$ and $W_2$ is the direct sum of $A$-modules together with the $A$-product defined by

$(w_1 + w_2, w_1' + w_2')_W = (w_1, w_1')_{W_1} + (w_2, w_2')_{W_2}$ for $w_i, w_i' \in W_i$, $i = 1, 2$. It is immediate to see that for any pre-Hilbert $A$-modules $W_1$ and $W_2$, $W_1$ and $W_2$ are closed in the pre-Hilbert $A$-module direct sum $W_1 \oplus W_2$. A pre-Hilbert $A$-submodule $W_1$ of a pre-Hilbert module $W$ is called orthogonally complemented if there exists a pre-Hilbert $A$-submodule $W_2$ of $W$ such that $W$ is isomorphic to $W_1 \oplus W_2$. By isomorphic, we mean that there is a pre-Hilbert $A$-module morphism of $W$ onto $W_1 \oplus W_2$ which has an inverse in the category of pre-Hilbert $A$-modules. In particular, we do not demand the morphism to be unitary. Let us mention that pre-Hilbert $A$-submodules need not be orthogonally complemented even if they are closed. See Lance [31].

A pre-Hilbert $A$-module $(E, (,)_E)$ is called a *Hilbert $A$-module* if it is a complete normed space with respect to the induced $C^*$-norm $||_E$ on $E$, i.e., it is a Banach space. By the *category of Hilbert $A$-modules* we mean the full subcategory of the category of pre-Hilbert $A$-modules whose objects are Hilbert $A$-modules. We denote the set of all Hilbert $A$-module morphisms of Hilbert $A$-modules $E$ and $E'$ by $\mathrm{Hom}_A^*(E, E')$ and by $\mathrm{End}_A^*(E)$ if $E = E'$ as in the pre-Hilbert $A$-module case. Let us remark that each Hilbert $A$-module morphism is continuous and consequently, the elements in $\mathrm{Aut}_A^*(E)$ are homeomorphisms. Notice that if $A$ is the $C^*$-algebra of complex numbers, a Hilbert $A$-module is a complex Hilbert space. A Hilbert $A$-submodule $E_1$ of a Hilbert $A$-module $(E, (,)_E)$ is a pre-Hilbert $A$-submodule of $E$ such that $(E_1, (,)_{E|E_1 \times E_1})$ is a Hilbert $A$-module, i.e., it is a complete normed space. In particular, a Hilbert $A$-submodule $E_1$ is closed in $E$ as a normed space. Hilbert $A$-submodules need not be orthogonally complemented as well. (See [31].) For definiteness, we fix the following definitions. A Hilbert $A$-module $E$ is called finitely generated if $E$ is algebraically generated over $A$ (i.e., by *finite* $A$-linear combinations) by a fixed finite subset of $E$. We call it topologically finitely generated if there exists a dense subspace of $E$ which is finitely generated. For a positive integer $q$, let us consider the Hilbert $A$-module $A^q = \underbrace{A \oplus \ldots \oplus A}_{q-\text{times}}$ with the diagonal right action of $A$ and the Euclidean-type $A$-product given by $((a_1, \ldots, a_q), (b_1, \ldots, b_q)) = \sum_{i=1}^q a_i^* b_i$, where $a_i, b_i \in A$ for $i = 1, \ldots, q$. See Solovyov, Troitsky [56].

*Projectivity and self-duality.* Let us denote the set of *all* continuous complex- and $A$-linear maps of Hilbert $A$-modules $(E, (,)_E)$ and $(F, (,)_F)$ by $\mathrm{Hom}_A(E, F)$. We do not demand the elements of $\mathrm{Hom}_A(E, F)$ to be adjointable. A Hilbert $A$-module $E$ is called *projective* if for each Hilbert $A$-modules $B$ and $C$ and for every surjective $b \in \mathrm{Hom}_A(B, E)$ and every $c \in \mathrm{Hom}_A(C, E)$ there exists an element $d \in \mathrm{Hom}_A(C, B)$ such that $c = b \circ d$. See, e.g., Frank and Paulsen [14] for more details. One calls a Hilbert $A$-module *finitely generated projective* if it is finitely generated and projective. A Hilbert $A$-module $E$ is called *$C^*$-self-dual* (or $A$-self-dual) if it is canonically isomorphic to the $A$-module $E^\star = \mathrm{Hom}_A(E, A)$, that we call the continuous $A$-dual of $E$. See Frank [13]. By the canonical map we mean the map $\phi : E \to E^\star$ given by $\phi(e)(e') = (e, e')_E$, $e, e' \in E$. We consider the action of $A$ on the $A$-dual given by $(f \cdot a)(e) = f(e) \cdot a$, where

$f \in \mathrm{Hom}_A(E, A)$, $a \in A$ and $e \in E$. Let us notice, that in the $C^*$-self-dual case, $\mathrm{Hom}_A(E, A)$ can be equipped with a canonical structure of a Hilbert $A$-module using the map $\phi$. (See Frank [13] or Mishchenko [38] for details.) Notice that if $E$ is $A$-self-dual, $E^\star = \mathrm{Hom}^*_A(E, A)$. If $T : E' \to E''$ is a morphism of Hilbert $A$-modules, its *transpose* $T^t : (E'')^\star \to (E')^\star$ is defined by $T^t(f)(e') = f(T(e'))$ for any $f \in (E'')^\star$ and $e' \in E'$.

*$C^*$-algebras of compact operators and $C^*$-compact operators.* For a complex Hilbert space $(V, h)$, let us consider the $C^*$-algebra consisting of all compact linear operators on $V$, which is equipped with the usual addition, the multiplication by scalars, composition of operators, the adjoint of operators and the operator norm. This algebra is called *the $C^*$-algebra of compact operators* and we denote it by $CV$. Let us recall that a $C^*$-algebra $A$ is called *a $C^*$-algebra of compact operators* if it is a $C^*$-subalgebra of the $C^*$-algebra $CV$ of compact operators on the complex Hilbert space $V$. If $A$ is a $C^*$-algebra, and $E$ and $E'$ are Hilbert $A$-modules, an *$A$-compact operator* of $E$ into $E'$ is the limit in the operator norm topology on $\mathrm{Hom}^*_A(E, E')$ of the so-called $A$-finite rank (or elementary) operators of Hilbert $A$-modules $E$ into $E'$. See Lance [31] or Kasparov [22], p. 789. Let $E, E'$ and $E''$ be Hilbert $A$-modules. If $K \in \mathrm{Hom}^*_A(E, E')$ is an $A$-compact operator and $T \in \mathrm{Hom}^*_A(E', E'')$, the operator $T \circ K$ is an $A$-compact operator and similarly for the composition of $K$ with a Hilbert $A$-module morphism from the right. (See [31], (1.6).) We refer to this property as to the ideal property of $A$-compact operators. It is easy to see that the adjoint of an $A$-compact operator is $A$-compact as well.

**Definition 1:** A morphism $D : X \to Y$ of Hilbert $A$-modules is called an *$A$-Fredholm operator* if there is a Hilbert $A$-module morphism $\check{D} : Y \to X$ such that $K_1 = \check{D}D - \mathrm{Id}_X$ and $K_2 = D\check{D} - \mathrm{Id}_Y$ are $A$-compact operators. We call any operator $\check{D}$ fulfilling this feature *a partial inverse* of $D$.

Thus an $A$-Fredholm operator has a left and right inverse up to an $A$-compact operator.

It is well known that there exist $C^*$-Fredholm operators whose images are not closed. (See, e.g., Krýsl [27].) The next lemma is a straightforward generalization of a theorem of Bakić and Guljaš in [4] (p. 268) on images of $C^*$-Fredholm endomorphisms are closed if the $C^*$-algebra is a $C^*$-algebra of compact operators.

**Lemma 1**: Let $A$ be a $C^*$-algebra of compact operators and $X$ and $Y$ be Hilbert $A$-modules. If $D$ is an $A$-Fredholm operator of $X$ into $Y$, then its image is closed in $F$.

*Proof.* Let $D : X \to Y$ be $A$-Fredholm and $\check{D}, K_1$, and $K_2$ be an appropriate partial inverse and $A$-compact operators, respectively. Thus $\check{D}D = \mathrm{Id}_X + K_1$ and $D\check{D} = \mathrm{Id}_Y + K_2$. Let us consider the following block-wise anti-diagonal element $\mathfrak{D} = \left(\begin{smallmatrix} 0 & D^* \\ D & 0 \end{smallmatrix}\right) \in \mathrm{End}^*_A(X \oplus Y)$. We have

$$\begin{pmatrix} 0 & \breve{D} \\ \breve{D}^* & 0 \end{pmatrix} \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_X + K_1 & 0 \\ 0 & \mathrm{Id}_Y + K_2^* \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{Id}_X & 0 \\ 0 & \mathrm{Id}_Y \end{pmatrix} + \begin{pmatrix} K_1 & 0 \\ 0 & K_2^* \end{pmatrix}.$$

Since the last written matrix is an $A$-compact operator in $\mathrm{End}^*_A(X \oplus Y)$, operator $\mathfrak{D}$ has a left inverse up to an $A$-compact on $X \oplus Y$. The invertibility from the right of $\mathfrak{D}$ up to an $A$-compact operator is proved similarly. Consequently $\mathfrak{D}$ has a partial inverse and thus it is an $A$-Fredholm endomorphism on $X \oplus Y$. According to [4], the image of $\mathfrak{D}$ is closed. This implies that $D$ has a closed image as well because $\mathrm{Im}\,\mathfrak{D} = \mathrm{Im}\,D^* \oplus \mathrm{Im}\,D$ and $X$ and $Y$ are mutually orthogonal in $X \oplus Y$. $\square$

**Remark:** 1) Let $A$ be a $C^*$-algebra of compact operators. A Hilbert $A$-module morphism $D : X \to Y$ is $A$-Fredholm if and only if the image of $D$ is closed and the so-called $A$-dimensions $\dim_A \mathrm{Ker}\,D$ and $\dim_A[(\mathrm{Im}\,D)^\perp]$ are finite (Corollary 5 in Krýsl [28]). The $A$-dimension is defined in [4], where its correctness is proved.

2) Let us notice that the result from [4] used in the above proof is based on the Theorem 2.22 in [3], where the so-called $H^*$-modules are investigated regarding their relation to Hilbert–Schmidt and to compact operators on a Hilbert space $V$. Let us remind the reader that the vector space of Hilbert–Schmidt operators $HS(V)$ on a Hilbert space $V$ is dense in $CV$ and when it is equipped with the so-called Hilbert–Schmidt norm, $HS(V)$ is unitarilly isomorphic to the Hilbert space $V$.

Let $X, Y$ be Hilbert $A$-modules, $Z$ be a pre-Hilbert $A$-module, which is a vector subspace of $X$, and let $\Delta : Z \to Z$ be a pre-Hilbert $A$-module morphism. If $\Delta$ has an adjointable extension $\widetilde{\Delta} : X \to Y$, we denote the adjoint $(\widetilde{\Delta})^*$ by $\widetilde{\Delta}^*$.

In the next theorem we generalize a procedure, which is used to derive the closed image property of an elliptic operator on smooth sections of a finite rank hermitian vector bundle on a compact manifold from the closed image property of the continuous extensions of this operator to Sobolev completions of the smooth sections. See, e.g., Wells [62]. We use it in the proof of Theorems 9 and 11 below. The condition b) in the next theorem is connected to the so-called *regularity* of elliptic operators, that we treat in Section 3. We notice that the second condition in b) below means in the bundle context that the adjoint of the extension is *regular* in the sense that whenever $\widetilde{\Delta}^* f = g$ for a smooth map $g$, the map $f$ is smooth as well. In particular, $Z$ plays a role of the space of smooth sections and $X$ and $Y$ of the Sobolev-type spaces.

**Theorem 2** (partial inverse on pre-Hilbert modules): Let $A$ be a $C^*$-algebra, $(X, (,)_X)$ and $(Y, (,)_Y)$ be Hilbert $A$-modules, and $(Z, (,)_Z)$ be a pre-Hilbert $A$-

submodule of $Y$ and a vector subspace of $X$. Let us consider a self-adjoint map $\Delta \in \mathrm{End}_A^*(Z)$ that has a continuous adjointable extension $\widetilde{\Delta} \in \mathrm{Hom}_A^*(X,Y)$ which satisfies

a) the image of $\widetilde{\Delta}$ is closed in $Y$ and

b) $\widetilde{\Delta}^{-1}(Z), \widetilde{\Delta}^{*-1}(Z) \subseteq Z$.

Then the image of $\Delta$ is closed in $Z$ and $Z = \mathrm{Ker}\,\Delta \oplus \mathrm{Im}\,\Delta$. Moreover, there are self-adjoint pre-Hilbert $A$-module morphisms $\check{\Delta} : Z \to Z$ and $K : Z \to Z$ such that

$$\Delta\check{\Delta} = \check{\Delta}\Delta = K - \mathrm{Id}_Z \text{ and } \Delta K = 0 \textit{ (parametrix equations).}$$

*Proof.* Assumptions in b) imply that $\mathrm{Ker}\,\widetilde{\Delta}, \mathrm{Ker}\,\widetilde{\Delta}^* \subseteq Z$.

1) By the kernel-image theorem of Mishchenko (Theorem 3.2, [31]), the image of $\widetilde{\Delta}^* : Y \to X$ is closed as well, and the following decompositions

$$X = \mathrm{Ker}\,\widetilde{\Delta} \oplus \mathrm{Im}\,\widetilde{\Delta}^* \text{ and } Y = \mathrm{Ker}\,\widetilde{\Delta}^* \oplus \mathrm{Im}\,\widetilde{\Delta}$$

hold.

2) i) If $z \in \mathrm{Ker}\,\widetilde{\Delta}^* \subseteq Y$, it is an element of $Z$ by b). Thus for each $z' \in Z$, we have $(\Delta z, z')_Z = (z, \Delta z')_Z = (z, \Delta z')_Y = (z, \widetilde{\Delta} z')_Y = (\widetilde{\Delta}^* z, z')_X = 0$. This implies that $\Delta z = 0$. Therefore $\mathrm{Ker}\,\widetilde{\Delta}^* \subseteq \mathrm{Ker}\,\Delta$.

ii) Now we show that $Z = \mathrm{Ker}\,\Delta + (\mathrm{Im}\,\widetilde{\Delta} \cap Z)$. For $z \in Z \subseteq Y$ there are elements $z_1 \in \mathrm{Ker}\,\widetilde{\Delta}^*$ and $z_2 \in \mathrm{Im}\,\widetilde{\Delta}$ such that $z = z_1 + z_2$ by the direct sum decomposition of $Y$ given in the item 1. The element $z_1$ is in $\mathrm{Ker}\,\Delta$ since $\mathrm{Ker}\,\widetilde{\Delta}^* \subseteq \mathrm{Ker}\,\Delta$ by the previous paragraph. Since $z, z_1 \in Z$, the element $z_2 = z - z_1 \in Z$. Consequently $Z \subseteq \mathrm{Ker}\,\Delta + (\mathrm{Im}\,\widetilde{\Delta} \cap Z)$. The opposite inclusion is trivial.

iii) Let us take an element in $\mathrm{Im}\,\widetilde{\Delta} \cap Z$ which is in the kernel of $\Delta$, i.e., $z = \widetilde{\Delta} z' \in Z$ for an element $z' \in X$ and $\Delta z = 0$. Thus $z' \in Z$ by the assumption b) and we have $(z,z)_Z = (z, \Delta z')_Z = (\Delta z, z')_Z = 0$. Consequently $z = 0$ and the sum $\mathrm{Ker}\,\Delta + (\mathrm{Im}\,\widetilde{\Delta} \cap Z)$ is orthogonal. We conclude that $Z = \mathrm{Ker}\,\Delta \oplus (\mathrm{Im}\,\widetilde{\Delta} \cap Z)$.

iv) We prove that $\mathrm{Im}\,\widetilde{\Delta} \cap Z = \mathrm{Im}\,\Delta$. Let us suppose that $z = \widetilde{\Delta} z' \in Z$ for an element $z' \in X$. Then $z' \in Z$ by b). Thus $z = \Delta z'$ and $\mathrm{Im}\,\widetilde{\Delta} \cap Z \subseteq \mathrm{Im}\,\Delta$. The opposite inclusion is obvious. Using the item iii), we obtain $Z = \mathrm{Ker}\,\Delta \oplus \mathrm{Im}\,\Delta$. Since the sum is orthogonal, the image of $\Delta$ is closed.

3) Using the orthogonal decomposition $Z = \mathrm{Ker}\,\Delta \oplus \mathrm{Im}\,\Delta$ derived above, we define $K : Z \to Z$ as the projection onto the kernel of $\Delta$ along $\mathrm{Im}\,\Delta$. In particular, $K$ is a self-adjoint morphism of pre-Hilbert $A$-modules.

4) Let us define $\check{\Delta} : Z \to Z$ by

$$\check{\Delta} = \begin{cases} (\Delta_{|\operatorname{Im}\Delta})^{-1} & \text{on } \operatorname{Im}\Delta \\ 0 & \text{on } \operatorname{Ker}\Delta. \end{cases}$$

Using this definition, $\check{\Delta}$ satisfies $\Delta\check{\Delta} = \check{\Delta}\Delta = \mathrm{Id}_Z - K$. Obviously, $\check{\Delta}$ is a self-adjoint pre-Hilbert $A$-module morphism of $Z$ since $\Delta$ is self-adjoint.

$\square$

**Remark**: If $A$ is a $C^*$-algebra of compact operators and $\widetilde{\Delta}$ is $A$-Fredholm, its image is closed by Lemma 1 and thus the condition a) of Theorem 2 is satisfied.

To avoid misunderstanding let us recall that if $W$ is a real or complex topological vector space, the symbol $W'$ denotes neither the dual vector space, nor the topological dual of $W$. The topological dual is denoted by $W^*$ (Preamble a)). The next corollary is a consequence of Theorem 2.

**Corollary 3:** Let $A$ be a $C^*$-algebra and let two triples of $A$-modules $X, Y$ and $Z$ and $X', Y'$ and $Z'$ satisfy the assumptions of the Theorem 2. Further let $D : Z \to Z'$ be a pre-Hilbert $A$-module morphism such that $\Delta = D^*D : Z \to Z$ and $\Delta' = DD^* : Z' \to Z'$ have continuous adjointable extensions $\widetilde{\Delta} \in \mathrm{Hom}^*_A(X, Y)$ and $\widetilde{\Delta'} \in \mathrm{Hom}^*_A(X', Y')$, respectively. If $\widetilde{\Delta}$ and $\widetilde{\Delta'}$ satisfy the assumptions a) and b) of Theorem 2, the images of $D$ and $D^*$ are closed in $Z'$ and $Z$, respectively.

*Proof.* Since $\Delta$ and $\Delta'$ are self-adjoint and since they satisfy assumptions a) and b) of Theorem 2, the decompositions $Z = \operatorname{Ker}\Delta \oplus \operatorname{Im}\Delta$ and $Z' = \operatorname{Ker}\Delta' \oplus \operatorname{Im}\Delta'$ hold. By Theorem 2, we have also the pre-Hilbert $A$-module morphisms $\check{\Delta}$, $K$ and $\check{\Delta'}, K'$ at our disposal. If $a \in \operatorname{Ker}\Delta$, then $(Da, Da)_Z = (a, D^*Da)_Z = (a, \Delta a)_Z = 0$ which implies that $Da = 0$, i.e., $\operatorname{Ker}\Delta \subseteq \operatorname{Ker} D$. The opposite inclusion $\operatorname{Ker} D \subseteq \operatorname{Ker}\Delta$ is obvious. Thus $\operatorname{Ker} D = \operatorname{Ker}\Delta$. We obtain similarly that $\operatorname{Ker} D^* = \operatorname{Ker}\Delta'$.

Now we prove identities concerning the images of $D$ and $D^*$ using the parametrix equations from the Theorem 2. For $b \in \operatorname{Im} D^*$, there is an element $a \in Z'$ such that $b = D^*a$. For $a' = \check{\Delta'}a \in Z'$, we set $b' = D^*a'$ and claim that $b'$ is in the $\Delta$-preimage of $b$. Indeed $\Delta b' = \Delta D^*a' = D^*DD^*a' = D^*DD^*\check{\Delta'}a = D^*\Delta'\check{\Delta'}a = D^*(\mathrm{Id}_{Z'} - K')a$ by Theorem 2. This expression equals to $D^*a - D^*K'a = D^*a = b$ because by Theorem 2, $K'$ maps into $\operatorname{Ker}\Delta'$ which is equal to $\operatorname{Ker} D^*$ due to the previous paragraph. Consequently, $\operatorname{Im} D^* \subseteq \operatorname{Im}\Delta$. Since also $\operatorname{Im}\Delta \subseteq \operatorname{Im} D^*$, we conclude that $\operatorname{Im} D^* = \operatorname{Im}\Delta$. Similarly, we prove that $\operatorname{Im} D = \operatorname{Im}\Delta'$.

Consequently, we have the orthogonal sums $Z = \operatorname{Ker}\Delta \oplus \operatorname{Im}\Delta = \operatorname{Ker} D \oplus \operatorname{Im} D^*$ and similarly $Z' = \operatorname{Ker} D^* \oplus \operatorname{Im} D$. In particular, $\operatorname{Im} D$ and $\operatorname{Im} D^*$ are closed with respect to the topology generated by the $C^*$-norm on $Z'$ and $Z$, respectively. $\square$

# 3 Images of $CV$-elliptic operators on compact $CV$-bundles

Let $p : \mathcal{W} \to M^n$ be a Banach fibre bundle on an $n$-dimensional manifold $M$ with (typical) fibre a Banach space $(W, |\,|_W)$ and a maximal $C^\infty$-differentiable bundle atlas $\mathcal{A}$. In particular, for any $m \in M$

- the fibre $\mathcal{W}_m = p^{-1}(\{m\})$ is a Banach space equipped with a norm, denoted by $|\,|_m$, and
- the topology on $\mathcal{W}_m$ generated by $|\,|_m$ is equal to the subset topology on $\mathcal{W}_m \subseteq \mathcal{W}$.

The space of $C^\infty$-differentiable sections of $p$ is denoted by $\Gamma^\infty(M, \mathcal{W})$ or by $\Gamma^\infty(\mathcal{W})$ when the manifold is known from the context. See the Preamble e), f) and g). Let us recall that a bundle chart of a Banach fibre bundle is in particular a homeomorphism of $U \times W$ onto $p^{-1}(U) \subseteq \mathcal{W}$ for an open subset $U$ of $M$ such that its restriction to $\{m\} \times W$ is a homeomorphism onto $p^{-1}(\{m\})$ for each $m \in U$ which is linear with respect to the vector space structures on $W$ and on $p^{-1}(\{m\})$.

Let $g$ be a Riemannian metric tensor on a compact manifold $M$ and let $\nabla^{\mathcal{W}}$ be a covariant derivative on a Banach fibre bundle $p : \mathcal{W} \to M$. Notice that the existence of a covariant derivative is proven by a choice of a partition of unity on $M$ using the same formula as in the finite rank case. We set for any $l \geqslant 0$ and $s \in \Gamma^\infty(\mathcal{W})$

$$|s|_l^F = \sup\{\big|(\nabla^{\mathcal{W}}_{X_1} \dots \nabla^{\mathcal{W}}_{X_k} s_{|U})(m)\big|_m : X_i \in \Gamma^\infty(U, TU),\, g(X_i, X_i) = 1, \\ i = 1, \dots, k,\, m \in U,\, U \text{ open in } M,\, 0 \leqslant k \leqslant l\}$$

which is easily seen to be a norm on $\Gamma^\infty(\mathcal{W})$, usually called the Fréchet (semi-)norm. For $k = 0$, the expression $|\nabla^{\mathcal{W}}_{X_1} \dots \nabla^{\mathcal{W}}_{X_k} s_{|U}(m)|_m$ means $|s(m)|_m$. We call the topology on $\Gamma^\infty(\mathcal{W})$ generated by the family of these norms the *Fréchet topology*. By the compactness of $M$ the space $\Gamma^\infty(\mathcal{W})$ is complete, i.e., a Fréchet space.

For a Banach fibre bundle $p : \mathcal{W} \to M$, let $\mathcal{W} \times_M \mathcal{W} = \{(w, w') \in \mathcal{W} \times \mathcal{W} |\, p(w) = p(w')\} \subseteq \mathcal{W} \times \mathcal{W}$ be the fibred product of $\mathcal{W}$ with itself. We consider it with the subset topology of the product topology on $\mathcal{W} \times \mathcal{W}$. (See, e.g., [25] for details concerning fibred products.)

A Banach fibre bundle is called a Hilbert fibre bundle if there is a smooth map $(,)_h : \mathcal{W} \times_M \mathcal{W} \to \mathbb{C}$ on the fibred product which is an inner product in each fibre and such that the induced norm $\sqrt{(w, w)_h}$ equals to the Banach norm $|w|_m$ for any $w \in \mathcal{W}_m$ and $m \in M$.

We introduce Hilbert $C^*$-bundles with the help of the next technical notion. Cf., e.g., Schick [51] or Fomenko and Mishchenko [39].

*One-point Hilbert $A$-module.* Let $A$ be a $C^*$-algebra, $(E, (,)_E)$ a Hilbert $A$-module, $M$ a finite dimensional manifold, and $U \subseteq M$ an embedded submanifold of $M$. We consider $\underline{E}_U = U \times E$ with the product topology, bundle projection $p(m,e) = m$, norm $|(m,e)|_m = |e|_E$, fibre-wise addition $(m,e') + (m,e'') = (m, e' + e'')$, and multiplication by scalars $c(m,e) = (m, ce)$, where $m \in U$, $e, e', e'' \in E$, $a \in A$, and $c \in \mathbb{C}$. Obviously the resulting structure is a Banach fibre bundle on $U$ when it is additionally equipped with a maximal smooth atlas. We shall always consider the maximal atlas that contains the identity chart $U \times E \to \underline{E}_U$. Notice that such a bundle atlas exists since the identity chart is global on $\underline{E}_U$, i.e., defined on $U \times E$. Further, we define the fibre-wise $C^*$-product by $((m,e'),(m,e''))_m = (e',e'')_E$ and the right action of $A$ by $(m,e)\cdot a = (m, e\cdot a)$, where at the right-hand side the action of $A$ on $E$ is considered. The space of smooth sections $\Gamma^\infty(\underline{E}_U)$ of $\underline{E}_U$ is linearly isomorphically identified with $C^\infty(U,E)$.

If $U = \{m\} \subseteq M$ is a singleton, the structures on $\underline{E}_{\{m\}}$ introduced above make $\underline{E}_{\{m\}}$ a Hilbert $A$-module, which we call the *one-point Hilbert $A$-module*. In this case, $\underline{E}_{\{m\}}$ is both a Hilbert $A$-module and a Banach fibre bundle on the singleton $\{m\}$.

**Definition 2:** Let $(E, (,)_E)$ be a Hilbert $A$-module and $p : \mathcal{E} \to M$ be a Banach fibre bundle with fibre the Banach space $(E, |\,|_E)$. We call $p$ together with a subatlas $\mathcal{A}$ of the atlas of $p$ a *Hilbert $A$-bundle* with fibre a Hilbert $A$-module $(E, (,)_E)$ if in addition an action $\cdot_{\mathcal{E}} : \mathcal{E} \times A \to \mathcal{E}$ of $A$ and a mapping $(,)_{\mathcal{E}} : \mathcal{E} \times_M \mathcal{E} \to A$ are given that are smooth with respect to $\mathcal{A}$ and such that for each point $m \in M$

i) the action $\cdot_{\mathcal{E}}$ and the map $(,)_{\mathcal{E}}$ restricted to $\mathcal{E}_m \times A$ and to $\mathcal{E}_m \times \mathcal{E}_m$, respectively, make the fibre $\mathcal{E}_m$ a Hilbert $A$-module;

ii) for any chart $\phi : U \times E \to p^{-1}(U)$ in $\mathcal{A}$ such that $m \in U$, the restriction $\phi(m,-) : \{m\} \times E \to \mathcal{E}_m$ is a Hilbert $A$-module morphism of the one-point Hilbert $A$-module $\underline{E}_{\{m\}} = \{m\} \times E$ onto the Hilbert $A$-module $\mathcal{E}_m$; and

iii) $\mathcal{A}$ is maximal with the properties i) and ii).

It is immediate to realize that a complex Hilbert bundle is a Hilbert $A$-bundle if the $C^*$-algebra $A$ is the $C^*$-algebra of complex numbers.

**Remark:** 1) By the chain rule for maps of open subsets of $\mathbb{R}^n$ into normed spaces, transition maps of charts in the atlas of a Hilbert $A$-bundle considered as maps of open subsets of $M$ into the group $G = \mathrm{Aut}^*_A(E)$ are smooth if $\mathrm{Aut}^*_A(E)$ is considered with the subset topology given by the inclusion
$\mathrm{Aut}^*_A(E) \subseteq \mathrm{End}^*_A(E)$, where the space $\mathrm{End}^*_A(E)$ is equipped with the strong operator topology. Since the operator norm topology is coarser than the strong operator topology, the transition maps are smooth also when the space of the Hilbert $A$-module endomorphisms $\mathrm{End}^*_A(E)$ is equipped with the operator norm topology. We consider the bundle $\underline{E}_U$ as the Hilbert $A$-bundle with the maximal smooth atlas containing the identity chart $\mathrm{Id}_{U\times E} : U \times E \to \underline{E}_U$ such that the Definition 2 is fulfilled.

2) To avoid misunderstanding let us mention that there is a notion of a 'bundle of $C^*$-algebras' (e.g., Dixmier [8], Fell [12] and Dupré [9]) which is different from the notion of a Hilbert $C^*$-bundle also in the case when the fibre of the bundle is the Hilbert $C^*$-module $A^1$, i.e., the $C^*$-algebra $A$ with the action given by the multiplication form the right and the Euclidean-type $A$-product as defined above. Nevertheless, we notice that any Hilbert $A$-bundle on a manifold $M$ with a fibre the Hilbert $C^*$-module $A^1$ is a 'bundle of $C^*$-algebras' on $M$.

By an *isomorphism of Hilbert $C^*$-bundles* $p' : \mathcal{E}' \to M$ and $p'' : \mathcal{E}'' \to M$ a $C^\infty$-diffeomorphism $T : \mathcal{E}' \to \mathcal{E}''$ of Banach fibre bundles is meant such that the restriction $T_{|\mathcal{E}'_m}$ is an isomorphism of the Hilbert $C^*$-modules $\mathcal{E}'_m$ and $\mathcal{E}''_m$ for each $m \in M$. In particular, an isomorphism of Hilbert $C^*$-bundles covers the identity on the base manifold, i.e., $p'' \circ T = p'$.

For a Hilbert $A$-bundle $\mathcal{E} \to M$, we define a right action of $A$ on the complex vector space $\Gamma^\infty(\mathcal{E})$ of smooth sections of $\mathcal{E}$ by the formula $(f \cdot a)(m) = f(m) \cdot_{\mathcal{E}} a$, where $f \in \Gamma^\infty(\mathcal{E})$, $a \in A$ and $m \in M$. The action of $A$ restricts to the set $\Gamma^\infty_c(\mathcal{E})$ of compactly supported elements of $\Gamma^\infty(\mathcal{E})$. Further, we define a map $(,)^\sim : \Gamma^\infty(\mathcal{E}) \times \Gamma^\infty(\mathcal{E}) \to C^\infty(M, A)$ by the formula $(f, h)^\sim(m) = (f(m), h(m))_{\mathcal{E}}$ where $f, h \in \Gamma^\infty(\mathcal{E})$ and $m \in M$. For a Riemannian metric tensor $g$ on $M$, we consider its density-form and take the induced Radon measure $\mu$ on the Borel $\sigma$-algebra of $M$. Having done these choices, we define an $A$-valued $A$-sesquilinear map on $\Gamma^\infty_c(\mathcal{E})$ by

$$(f, h) = \int_M (f, h)^\sim \mathrm{d}\mu$$

for each $f, h \in \Gamma^\infty_c(\mathcal{E})$, where we consider the Bochner integral of $A$-valued $\mu$-measurable maps on $M$ for convenience. (See, e.g., Ryan [49].) It is immediate to realize that the pair $(\Gamma^\infty_c(\mathcal{E}), (,))$ is a pre-Hilbert $A$-module. We denote the induced $C^*$-norm by $||$, and we call the topology on $\Gamma^\infty_c(\mathcal{E})$ induced by this norm the *pre-Hilbert topology*. Let us notice that, in general, this topology is not induced by an inner product.

Let us recall that a Hilbert $A$-bundle is called *(topologically) finitely generated*, *finitely generated projective* and *$C^*$-self-dual* if its fibre is (topologically) finitely generated, finitely generated projective and $C^*$-self-dual as a Hilbert $A$-module, respectively.

Let us summarize our notation regarding Hilbert $A$-modules and Hilbert $A$-bundles briefly.

1) The norm on a $C^*$-algebra $A$ is denoted by $||_A$. The $C^*$-product on a (pre-)Hilbert $A$-module $W$ is denoted by $(,)_W$ and the induced $C^*$-norm is denoted by $||_W$.

2) The norm on a fibre of a Banach bundle has the base point of the fibre as its lower index. Thus $||_m$ denotes the norm on the fibre $p^{-1}(m)$. The

Fréchet norms on sections of a Banach fibre bundle on a compact manifold are indexed by non-negative integers and denoted by $|\,|_l^F$.

3) The action of a $C^*$-algebra $A$ on the total space $\mathcal{E}$ of a Hilbert $A$-bundle is denoted by $\cdot_{\mathcal{E}}$ and the appropriate $A$-valued map (Definition 2) on the fibred product $\mathcal{E} \times_M \mathcal{E}$ is denoted by $(,)_{\mathcal{E}}$. The right action of $A$, the $A$-product given by the Bochner integral, and the induced $C^*$-norm on $\Gamma_c^\infty(\mathcal{E})$ have no indices and they are denoted by $\cdot$, $(,)$, and $|\,|$, respectively.

In the next lemma, the Fréchet and the pre-Hilbert topologies are compared.

**Lemma 4:** Let $M$ be a compact manifold and $p : \mathcal{E} \to M$ be a Hilbert $C^*$-bundle. Then the pre-Hilbert topology on $\Gamma^\infty(\mathcal{E})$ is finer than the Fréchet topology.

*Proof.* Since both of the considered topologies are metrisable, they are sequential (see e.g. [48]). Let us consider a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \Gamma^\infty(\mathcal{E})$ that converges in the Fréchet topology (i.e., in all Fréchet norms) to the zero section. By the definition of $|\,|_0^F$, it is obvious that the sequence converges also uniformly to the zero section on $M$. Especially for any $\epsilon > 0$ there is a positive integer $n_0$ such that for each $n > n_0$ and all $m \in M$ we have $|f_n(m)|_m < \epsilon$. The constant function $\epsilon$ defined on $M$ has a finite Lebesgue integral over the compact manifold $M$. Consequently, if $n$ approaches infinity, $\int_M (f_n, f_n)^\sim \mathrm{d}\mu \to 0$ by the dominant convergence of the Bochner integral. Thus $(f_n)_{n\in\mathbb{N}}$ converges to the zero section in the pre-Hilbert topology as well. $\square$

**Sobolev-type spaces for Hilbert $C^*$-bundles**

We generalize the definition of Sobolev-type spaces from Solovyov and Troitsky [56] and Fomenko and Mishchenko [39], in which topologically finitely generated projective Hilbert $C^*$-modules over a unital $C^*$-algebra are considered. Let $A$ be a $C^*$-algebra and $\mathcal{E} \to M$ be a Hilbert $A$-bundle on a compact manifold $M$ with fibre a Hilbert $A$-module $(E, (,)_E)$. In particular, $(E, |\,|_E)$ is a Banach space. We consider the Euclidean space $\mathbb{R}^n$ with the standard scalar product $(,)_{\mathbb{R}^n}$ and the Schwartz space $S(\mathbb{R}^n, E)$ of rapidly decreasing smooth maps on $\mathbb{R}^n$ with values in the Banach space $E$. See Schwartz [54]. It is well known that $S(\mathbb{R}^n, E) \cong S(\mathbb{R}^n, \mathbb{C})\widehat{\otimes}_\epsilon E$ as Fréchet spaces. (See, e.g., [58].) Denoting the Lebesgue measure on $\mathbb{R}^n$ by $\lambda$, we consider that the direct and the inverse Fourier transforms of the scalar-valued function $f \in S(\mathbb{R}^n, \mathbb{C})$ are defined by

$$(\mathcal{F}^{\pm 1} f)(x) = (\mathcal{F}^{\pm 1}_{x,y} f)(x) = \int_{y\in\mathbb{R}^n} f(y) e^{\mp 2\pi i (x,y)_{\mathbb{R}^n}} \mathrm{d}\lambda(y).$$

In both cases, the second lower index in $\mathcal{F}_{x,y}$ points to the variable over which we integrate. The Fourier transforms $(\mathcal{F}^E)^{\pm 1}$ on $S(\mathbb{R}^n, E)$ are defined as the completed injective tensor product $\mathcal{F}^{\pm 1}\widehat{\otimes}_\epsilon \mathrm{Id}_E$ of the Fourier transforms $\mathcal{F}^{\pm 1}$ on $S(\mathbb{R}^n, \mathbb{C})$ with the identity on $E$. See, e.g., Schwartz [54]. The map $(\mathcal{F}^E)^{+1}$ is denoted by $\mathcal{F}^E$ as usual. The inverse Fourier transform $(\mathcal{F}^E)^{-1}$ is the both-sided inverse of $\mathcal{F}^E$.

For $k \in \mathbb{Z}$ and $f \in S(\mathbb{R}^n, E)$, the so-called $k$th Sobolev-type norm is defined by

$$|f|_k^S = \left( \left| \int_{y \in \mathbb{R}^n} \big( (\mathcal{F}^E f)(y), (\mathcal{F}^E f)(y) \big)_E \, (1 + |y|^2_{\mathbb{R}^n})^k \mathrm{d}\lambda_{\mathbb{R}^n}(y) \right|_A \right)^{1/2}$$

where we consider the Bochner integral (i.e., the strong integral) of $A$-valued maps with respect to the Lebesgue measure.

**Remark:** Note that the Schwartz space $S(\mathbb{R}^n, \mathbb{C})$ is a nuclear space and thus, its completed injective tensor product with $E$ is linearly homeomorphic to the completed projective tensor product with $E$. See [58]. Let us denote the completed projective tensor product by $\widehat{\otimes}_\pi$. Since $S(\mathbb{R}^n, E) \subseteq \big(L^1(\mathbb{R}^n)\widehat{\otimes}_\epsilon E\big) \cap \big(L^1(\mathbb{R}^n)\widehat{\otimes}_\pi E\big)$ the Bochner and the Pettis integral exist on $S(\mathbb{R}^n, E)$ and moreover they are equal. See Ryan [49]. The Fourier transform on $S(\mathbb{R}^n, E)$ can be introduced also by the above formula for the Fouerier transform of scalar-valued functions in which the Pettis or, equivalently, the Bochner integral is used.

The Sobolev-type space $H^k(\mathbb{R}^n, E)$ is defined as the completion of $S(\mathbb{R}^n, E)$ with respect to the norm $|\,|_k^S$. For $f, g \in H^k(\mathbb{R}^n, E)$, we set

$$(f, g)_k^S = \int_{y \in \mathbb{R}^n} \big( (\mathcal{F}^E f)(y), (\mathcal{F}^E g)(y) \big)_E \, (1 + |y|^2_{\mathbb{R}^n})^k \mathrm{d}\lambda_{\mathbb{R}^n}(y) \in A.$$

This sets up a well defined map of $H^k(\mathbb{R}^n, E) \times H^k(\mathbb{R}^n, E)$ into $A$ by the dominant convergence of the Bochner integral and the Cauchy–Schwarz-type inequality for $|(,)_E|_A$ mentioned in Section 2. By the Gelfand–Naimark theorem, $(,)_k^S$ is positive definite. Immediately, we get that it is an $A$-product. The induced $C^*$-norm for this $C^*$-product is the continuous extension of $|\,|_k^S$. We denote it by $|\,|_k^S$ as well. Let us notice that $(,)_0^S$ is an extension to $H^0(\mathbb{R}^n, E) \times H^0(\mathbb{R}^n, E)$ of the $C^*$-product $(,)$ on $\Gamma_c^\infty(\mathbb{R}^n, E)$ introduced above.

For a manifold atlas on a compact manifold $M^n$, a compatible bundle atlas on $\mathcal{E} \to M$, and a subordinated partition of unity on $M^n$, we define the Sobolev-type spaces $(H^k(M, \mathcal{E}), (,)_k^S)$ by a classical procedure using the partition of unity and the Sobolev-type spaces $H^k(\mathbb{R}^n, E)$ on $\mathbb{R}^n$ defined above. See, e.g., [56]. We denote them by $H^k(\mathcal{E})$ if the manifold is known. The Sobolev-type spaces depend on the atlases and on the partition of unity. Let us remark, that for different choices the resulting spaces are isomorphic as Hilbert $C^*$-modules that is proved as in the finite rank case. See, e.g., Palais [42]. If the product bundle $\underline{E}_M = M \times E \to M$ is considered with the canonical Hilbert $A$-bundle structure, we denote the space $H^k(M, \underline{E}_M)$ by $H^k(M, E)$.

**Conventions concerning $C^*$-pseudodifferential operators**

Let $p' : \mathcal{E}' \to M$ and $p'' : \mathcal{E}'' \to M$ be Hilbert $A$-bundles on a compact manifold $M^n$ with fibres $E'$ and $E''$, respectively. We choose a partition of unity on $M$ subordinated to the manifold atlas and to the both bundle atlases. A *symbol* is a map of $T^*M$ that assigns to each cotangent vector $\xi \in T^*_m M$, $m \in M$, a morphism $\sigma(\xi) \in \mathrm{Hom}^*_A(\mathcal{E}'_m, \mathcal{E}''_m) \cong \mathrm{Hom}^*_A(E', E'')$ of Hilbert $A$-modules that satisfies specific growth conditions defined with help of the bundle charts and the partition of the unity. We consider the growth conditions given in [56]. They are generalizations to Hilbert $A$-bundles of the estimates given, e.g., in Seeley [50], Palais [42] and Wells [62], and formulated for smooth complex-valued functions or for smooth sections of finite rank hermitian vector bundles. In particular, symbols are point-wise adjointable Hilbert $A$-module morphisms. The set of symbols forms a $\mathbb{Z}$-filtered vector space, which induces the order of symbols and an associated $\mathbb{Z}$-grading.

For simplicity let us assume that the atlas of $M$ contains a global chart, and denote the image in $\mathbb{R}^n$ of the domain of this chart by $U$. Notice that we have the induced trivialization of $T^*M \cong U \times (\mathbb{R}^n)^* \cong U \times \mathbb{R}^n$ at our disposal. We denote corresponding coordinates on $T^*M$ by the couples $(x, \eta)$, where $x \in U$ and $\eta \in \mathbb{R}^n$. Let $\sigma$ be a symbol and $\sigma^U$ be its coordinate expression with respect to the manifold chart, induced cotangent bundle chart, and to the bundle charts. For each $(x, \eta) \in U \times \mathbb{R}^n$, the map $\sigma^U(x, \eta) : E' \to E''$ is an adjointable map of Hilbert $A$-modules. With respect to the chosen charts, the *$A$-pseudodifferential operator generated by $\sigma$* is defined in coordinates by

$$(Ds)(x) = ((\mathcal{F}^{E''}_{x,\eta})^{-1} \circ \sigma^U(x, \eta) \circ \mathcal{F}^{E'}_{\eta,y})(s)$$

where $(x, \eta) \in U \times \mathbb{R}^n$ and $y \in U$. Below we do not label the Fourier transformation by the lower indices and understand that the order of integrations is set by this formula. In Solovyov, Troitsky [56], p. 104, there is a coordinate expression for an $A$-pseudodifferential operator for the case the manifold or the bundle atlases do not contain a global chart. We apply it in the proof of Thm. 9. Compared to the formula above, it contains a partition of unity on $M$. If $A = \mathbb{C}$, we call an $A$-pseudodifferential operator a pseudodifferential operator.

The order of a $C^*$-pseudodifferential operator is defined as the order of the symbol by which the operator is generated. Let $d$ be the order of $D$. We denote the continuous extension of the $A$-pseudodifferential operator $D : \Gamma^\infty(M, \mathcal{E}') \to \Gamma^\infty(M, \mathcal{E}'')$ to $H^k(M, \mathcal{E}')$ by $D_k$. It is a map into the space $H^{k-d}(M, \mathcal{E}'')$.

If the Hilbert $C^*$-bundles $\mathcal{E}'$ and $\mathcal{E}''$ are $C^*$-self-dual, $D$ is adjointable as a morphism of the pre-Hilbert $A$-modules $\Gamma^\infty(M, \mathcal{E}')$ and $\Gamma^\infty(M, \mathcal{E}'')$. This is proved by considering the transposed operators (see Section 1) of the continuous extensions of $D$ to elements in the chain $(H^k(M, \mathcal{E}'))_{k \in \mathbb{Z}}$ of Hilbert $A$-modules. The construction of the adjoint of $D$ is based on the fact that $H^k(M, \mathcal{E})^\star \cong H^{-k}(M, \mathcal{E})$ if $\mathcal{E}$ is an $A$-self-dual bundle. See [56] or Palais [42] in which the case of Hilbert bundles is treated.

**Definition 3:** An $A$-pseudodifferential operator of order $d$ on Hilbert $A$-bundles $p' : \mathcal{E}' \to M$ and $p'' : \mathcal{E}'' \to M$ on a manifold $M$ of dimension $n \geqslant$

1 is called $A$-*elliptic* if its symbol $\sigma(\xi)$ in $\xi \in T^*M$ is a Hilbert $A$-module isomorphism of $(\mathcal{E}'_m, |\,|'_m)$ onto $(\mathcal{E}''_m, |\,|''_m)$ for any non-zero element $\xi \in T^*_m M$ and any $m \in M$.

**Remark:** 1) If we allow null dimensional base manifolds, i.e., countable sets with the discrete topology, in the definition above, the set of non-zero cotangent vectors is empty, and therefore any $A$-pseudodifferential operator would be $A$-elliptic in this "trivial" case. We exclude the case of zero dimensional manifolds from the definition of the $C^*$-ellipticity. See the remark on $C^*$-compact embeddings for null dimensional manifolds below the Scholium 1.

2) If $A = \mathbb{C}$, an $A$-elliptic operator is called elliptic.

**Smooth trivializations of Hilbert bundles**

Let $p : \mathcal{V} \to M$ be an infinite rank Hilbert bundle on a manifold $M^n$ with a complex Hilbert space $(V, h)$ as the fibre and with a maximal $C^\infty$-differentiable atlas. If the dimension of $V$ is infinite, the unitary group of $V$ equipped with the strong operator topology is *continuously* contractible. See Dixmier, Douady [7]. Consequently, an infinite rank Hilbert bundle is continuously trivializable, i.e., there is a homeomorphism of $\mathcal{V}$ onto the product Hilbert bundle $\underline{V}_M = M \times V \to M$ that covers the identity on $M$. See, e.g., [45] for a cohomology approach to a proof of this fact. See also Schottenloher [55] for a treatise on the norm and strong operator topologies on the unitary group of a Hilbert space and on continuous Hilbert fibre bundles.

Nevertheless, the fact that the unitary group of the infinite dimensional Hilbert space is continuously contractible is not sufficient for our purpose since we shall consider pseudodifferential operators on Hilbert bundles. By results of Burghelea, Kuiper [6] and Moulis in [40], it is possible to approximate a trivializing bundle homeomorphism by a fibre bundle $C^\infty$-diffeomorphism. Consequently, $p$ is also *smoothly trivializable.*

*Trivializing construction.* Let $M$ be compact and $a : \mathcal{V} \to \underline{V}_M = M \times V$ be a trivializing $C^\infty$-diffeomeorphism. It induces a linear isomorphism $\alpha : \Gamma^\infty(\mathcal{V}) \to \Gamma^\infty(\underline{V}_M)$ by $\alpha(s) = a \circ s$, where $s \in \Gamma^\infty(\mathcal{V})$. whose inverse is given by a similar formula where $a^{-1}$ is used instead of $a$. Recall that $\Gamma^\infty(\underline{V}_M)$ is already linearly homeomorphically identified with $C^\infty(M, V)$.

We want to show that $\alpha$ is a homeomorphism of topological vector spaces for both the introduced topologies on the section spaces. For each $m \in M$, $a_m : \mathcal{V}_m \to V$ is defined by $a_m(v) = a(v)$ for any $v \in \mathcal{V}_m$. It is the restriction of $a$ to the singleton $\{m\}$. Let us denote the operator norm of continuous linear maps of the Hilbert space $\mathcal{V}_m$ into the Hilbert space $V$ by $||\,||_m$. In the case of the pre-Hilbert topology, we use the dominant convergence of the Bochner integral, similarly as in the proof of Lemma 4, and the fact that $m \in M \mapsto ||a_m||_m$ and $m \mapsto ||(a_m)^{-1}||$ are continuous functions defined on the compact set $M$, and thus bounded by a constant that we denote by $c$. Indeed, let $f_n \to 0$ in the pre-Hilbert topology on $\Gamma^\infty(\mathcal{V})$ if $n$ tends to infinity. Then

$$
\begin{aligned}
|\alpha(f_n)|^2 &= |\int_{m\in M} h\left(a(f_n(m)), a(f_n(m))\right) \mathrm{d}\mu(m)| \\
&= \int_{m\in M} |a_m((f_n)(m))|_V^2 \mathrm{d}\mu(m) \leqslant \int_{m\in M} ||a_m||_m^2 |f_n(m)|_V^2 \mathrm{d}\mu(m) \\
&\leqslant c^2 \int_{m\in M} |f_n(m)|_V^2 \mathrm{d}\mu(m)
\end{aligned}
$$

that converges to null by the assumption on $(f_n)_n$, proving that $\alpha$ is continuous. The continuity of $\alpha^{-1}$ is proved by using the formula $(\alpha^{-1}s)(m) = a^{-1}(s(m))$ and similar estimates as above. In the case of the Fréchet topology, we assume w.l.o.g. that the Fréchet topology on $\Gamma^\infty(\mathcal{V})$ is given by the pull-back $\nabla^{\mathcal{V}}$ by the bundle map $a$ of the connection $\nabla$ on $\underline{V}_M$ that is used to define a Fréchet topology on $\Gamma^\infty(\underline{V}_M)$. Let $X$ be a local unit-length vector field on $M$. Since $a$ is $C^\infty$-differentiable, its covariant derivatives with respect to the induced connection $\nabla_X^{\mathrm{Hom}}$ on the homomorphism bundle $\mathrm{Hom}(\mathcal{V}, \underline{V}_M)$ are bounded on $M$ with respect to the operator norm topology by the compactness of $M$. The Fréchet norms on $\Gamma^\infty(\mathcal{V})$ are bounded from above by the Fréchet norms on $\Gamma^\infty(\underline{V}_M)$ using the formula $\nabla_X^{\mathcal{V}} s = a^{-1} \circ \left((\nabla_X^{\mathrm{Hom}} a)(s)\right) - a^{-1} \circ (\nabla_X(a \circ s))$, where $X$ is a local vector field on $M$ and $s$ is a smooth section of $\mathcal{V}$. This proves that $\alpha^{-1}$ is continuous. The continuity of $\alpha$ is derived by the open map theorem for Fréchet spaces or using a similar formula as above but expressing $\nabla_X$ with help of $\nabla_X^{\mathcal{V}}$.

If $D : \Gamma^\infty(\mathcal{V}') \to \Gamma^\infty(\mathcal{V}'')$ is a pseudodifferential operator on Hilbert bundles $\mathcal{V}'$ and $\mathcal{V}''$, we define the pseudodifferential operator $\widetilde{D} : C^\infty(M, V') \to C^\infty(M, V'')$ by $\widetilde{D} = \alpha'' \circ D \circ {\alpha'}^{-1}$, where $\alpha'$ is the induced complex linear homeomorphism of $\Gamma^\infty(M, \mathcal{V}')$ with the Fréchet and the pre-Hilbert topology onto $C^\infty(M, V')$ with the Fréchet and the pre-Hilbert topology, respectively, and similarly for $\alpha''$. We often consider $\widetilde{D}$ instead of $D$ without mentioning it explicitly. The image of $\widetilde{D}$ is closed if and only if the image is closed $D$ with respect to the corresponding topologies. Notice that already for finite rank bundles, $D$ need not be continuous with respect to the pre-Hilbert topology.

*Extensions.* For a $C^*$-algebra $A$, let $A^e$ denote the unitalization of $A$. In particular, $A^e = A \oplus \mathbb{C}$ as a complex vector space. Any (pre-)Hilbert $A$-module $W$ is turned into a (pre-)Hilbert $A^e$-module by setting $w \cdot (B, c) = w \cdot B + cw$, where $B \in A$, $w \in W$ and $c \in \mathbb{C}$, and by keeping the $A$-product unchanged. We call the resulting (pre-)Hilbert $A^e$-module the *extended (pre-)Hilbert $A$-module.* It is easy to see that any morphism of (pre-)Hilbert $A$-modules is also a morphism of the corresponding extended pre-Hilbert $A^e$-modules. Note that a (pre-)Hilbert $A$-module and its extension are equal as topological vector spaces.

We use the assertion on the so-called smooth embedding of Sobolev-type spaces for Hilbert $A$-bundles on compact manifolds, derived as Lemma 5 in Krýsl [26]. The assumption on the unitality on $A$ is not used in the proof of this lemma. The assumption can also be removed using the extended modules described above. Namely, we consider the bundles as $A^e$-bundles and embed the Sobolev-type spaces into the smooth sections spaces by the mentioned Lemma 5 in [26]. The embedding is a morphism of pre-Hilbert $A^e$-modules. Since the considered pre-Hilbert $A^e$-modules are pre-Hilbert $A$-modules as well (i.e., the induced $C^*$-products maps into $A$), the embedding is a morphism of pre-Hilbert $A$-modules as well. Let us notice that a smooth embedding for Sobolev-type spaces is proved in Fomenko and Mishchenko [39] for unital $C^*$-algebras and topologically finitely generated projective Hilbert $C^*$-bundles. Nevertheless, in that proof neither the unitality of $A$, nor the assumptions are used that the fibres are topologically finitely generated and projective.

**Lemma 5:** Let $p' : \mathcal{V}' \to M$ and $p'' : \mathcal{V}'' \to M$ be Hilbert bundles (i.e., Hilbert $\mathbb{C}$-bundles) on a compact manifold $M$. Then any $\mathbb{C}$-pseudodifferential operator $D : \Gamma^\infty(M, \mathcal{V}') \to \Gamma^\infty(M, \mathcal{V}'')$ has an adjoint as a linear map of inner product spaces. For each $\xi \in T^*M$ if $\sigma(\xi)$ is the symbol of $D$ in $\xi$, the adjoint operator $\sigma(\xi)^*$ is the symbol of $D^*$ evaluated in $\xi$. Moreover, if $D$ is elliptic, $D^*$ is elliptic as well.

*Proof.* Let $V'$ and $V''$ be the fibres of $p'$ and $p''$, respectively. By Wloka [63], $H^k(M, V) \cong H^k(M)\widehat{\otimes}_{HS} V$, where $\widehat{\otimes}_{HS}$ denotes the Hilbert–Schmidt tensor product and $V$ is a Hilbert space. Consequently, $H^k(M, V)$ are Hilbert spaces. In particular, they are $\mathbb{C}$-self-dual by the Riesz representation theorem. Let us consider the pseudodifferential operator $D$, its continuous extensions $D_k : H^k(M, V') \to H^{k-d}(M, V'')$ for $k \in \mathbb{Z}$, and suppose that $d$ is the order of $D$. Since any continuous linear map of Hilbert spaces is adjointable, we have the operator

$$(D_k)^* : H^{k-d}(M, V'') \to H^k(M, V')$$

and its restriction to $C^\infty(M, V'')$ at our disposal. By the mentioned Sobolev-type smooth embedding (Lemma 5, [26]), ${(D_k)^*}_{|C^\infty(M,V'')}$ maps into $C^\infty(M, V')$. By the uniqueness of adjoints, we get that $D^* = {(D_d)^*}_{|C^\infty(M,V'')}$ as in the finite rank case, i.e., if we identify $H^l(M, V')$ with $H^{-l}(M, V')$ and similarly for $H^l(M, V'')$. See Palais [42].

Since any continuous linear map of Hilbert spaces has an adjoint, the adjoint $\sigma(\xi)^* : V'' \to V'$ of $\sigma(\xi) : V' \to V''$ exists for all $\xi \in T^*M$. The facts that $\xi \mapsto \sigma(\xi)^*$ satisfies the correct growth conditions and that it is a symbol of the adjoint $D^*$ follow by the same lines as in the finite rank case. See, e.g., the proof of Theorem 3.16 in Wells [62].

If $D$ is elliptic, its symbol $\sigma(\xi)$ is a linear homeomorphism for any $0 \neq \xi \in T^*M$. It is immediate to check that the inverse of the adjoint $\sigma(\xi)^*$ is $\left(\sigma(\xi)^{-1}\right)^*$. Consequently for each $\xi \neq 0$, the symbol $\sigma(\xi)^*$ of $D^*$ is a linear homeomorphism and thus $D^*$ is elliptic. □

**Remark:** In the above proof, Theorem 3.16 from [62] is used that relies on the Theorem 3.10 of the ibid citation on so-called generalized symbols. In its proof the mean value theorem is applied to scalar components of the symbols on finite rank vector bundles. If the fibres $V'$ and $V''$ of $\mathcal{V}'$ and $\mathcal{V}''$, respectively, are Hilbert spaces, we use Thm. 4.2 in Lang [32], where a generalization to Banach spaces is given of the mean value theorem. We use it for the space $\mathrm{Hom}(V', V'')$ equipped with the operator norm topology.

Let us suppose that $A$ is a unital $C^*$-algebra, $p' : \mathcal{E}' \to M$ and $p'' : \mathcal{E}'' \to M$ are topologically finitely generated projective Hilbert $A$-bundles on a compact manifold $M$, and $D : \Gamma^\infty(\mathcal{E}') \to \Gamma^\infty(\mathcal{E}'')$ is an $A$-elliptic operator. It is known that if $s \in H^k(\mathcal{E}')$ satisfies the equation $D_k s = f$ for a smooth section $f \in \Gamma^\infty(\mathcal{E}'')$, then $s$ is smooth. This property of $D$ is called the *elliptic regularity*. See Mishchenko and Fomenko [39] where the regularity is treated for this case. We prove the elliptic regularity for Hilbert $CV$-bundles, the fibres of which need not be topologically finitely generated projective. We use the Lemma 5 from [26] on the smooth embedding of Sobolev-type spaces mentioned above.

Recall that an $A$-pseudodifferential operator is called a *smoothing operator* if its continuous extension to $H^k(\mathcal{E}')$ is a map into the space $H^{k+1}(\mathcal{E}'')$ for all sufficiently big integers $k$.

**Theorem 6** (elliptic regularity): Let $A$ be a $C^*$-algebra, $p' : \mathcal{E}' \to M$ and $p'' : \mathcal{E}'' \to M$ be Hilbert $A$-bundles on a compact manifold $M$, and $D : \Gamma^\infty(\mathcal{E}') \to \Gamma^\infty(\mathcal{E}'')$ be an $A$-elliptic operator of order $d$. If $f = D_k s \in \Gamma^\infty(\mathcal{E}'')$ for a map $s \in H^k(\mathcal{E}')$, then $s \in \Gamma^\infty(\mathcal{E}')$.

*Proof.* Since $D$ is $A$-elliptic, we may construct a partial inverse $\breve{D}_{k-d} : H^{k-d}(\mathcal{E}'') \to H^k(\mathcal{E}')$ of $D_k : H^k(\mathcal{E}') \to H^{k-d}(\mathcal{E}'')$ as in [39], i.e., by taking the Fourier transform of the inverse of the symbol of $D$ (out of the zero section) multiplied by a cut-off function and extending the resulting $A$-pseudodifferential operator to the appropriate Sobolev-type completion. It follows by this construction, that the operator $N_k = \breve{D}_{k-d} D_k - \mathrm{Id}_{H^k(\mathcal{E}')}$ maps $H^k(\mathcal{E}')$ into $H^{k+1}(\mathcal{E}')$. For any map $s$ satisfying the assumptions of the theorem, we get $s = \breve{D}_{k-d} D_k s - (\breve{D}_{k-d} D_k - \mathrm{Id}_{H^k(\mathcal{E}')}) s = \breve{D}_{k-d} f - N_k s \in H^{k+1}(\mathcal{E}')$ because $\breve{D}_{k-d} f \in \Gamma^\infty(\mathcal{E}')$. By mathematical induction $s \in \bigcap_{l=k}^\infty H^l(\mathcal{E}')$ which equals to $\Gamma^\infty(\mathcal{E}')$ by the Lemma 5 in Krýsl [26]. $\square$

**Remark:** 1) For details on the construction of $\breve{D}_{k-d}$ sketched above, see, e.g., the proof of Thm. 3.4 in [39]. Notice, that the assumption is not used in the proof in [39] that the Hilbert $C^*$-bundle is topologically finitely generated and projective and that the $C^*$-algebra is unital.

2) Using the notation in the proof of the Theorem 6, let us set $\breve{D} = (\breve{D}_k)_{|\Gamma(\mathcal{E}')}$. Since $N_k$ is the continuous extension to $H^k(\mathcal{E}')$ of $N = \breve{D} D - \mathrm{Id}_{\Gamma(\mathcal{E}')}$ and $N_k$ maps into $H^{k+1}(\mathcal{E}')$, $N$ is smoothing.

## 3.1 Compact $CV$-modules and compact $CV$-bundles

The complex vector space $CV$ of compact operators on the Hilbert space $(V, h)$ is a right $CV$-module with respect to the action $CV \times CV \ni (B, C) \mapsto B \cdot C = B \circ C \in CV$. Setting $(B, C)_{CV} = B^* \circ C$ for $B, C \in CV$, we get a $C^*$-product on $CV$. Since $CV$ is a $C^*$-algebra, the induced $C^*$-norm is equal to the operator norm on $CV$. The right action of $CV$ and the $C^*$-product define a structure of a Hilbert $CV$-module on the space of compact operators. We call this Hilbert module the *compact $CV$-module.* It is isomorphic to the Hilbert $C^*$-module $CV^1$ defined above. The compact $CV$-module is not topologically finitely generated if the dimension of $V$ is infinite. See, e.g., Wegge-Olsen [61]. (In [61], the term 'finitely generated' means 'topologically finitely generated' in our sense.)

We mention the following definition from [4].

**Definition 4:** Let $(E, (,)_E)$ be a Hilbert $A$-module. We call a subset $(v_j)_{j \in J} \subseteq E$ an *orthonormal basis* of $(E, (,)_E)$ if

i) the set $(v_j)_{j \in J}$ generates (by taking finite right $A$-linear combinations) a dense $A$-submodule of the Hilbert $A$-module $E$;

ii) $(v_j, v_{j'})_E = 0$ whenever $j \neq j'$;

iii) for each $j \in J$, the element $\xi_j = (v_j, v_j)_E \in A$ is an orthogonal projection, i.e., a non-zero hermitian-symmetric idempotent in $A$; and

iv) $\xi_j A \xi_j = \mathbb{C}\xi_j$ for all $j \in J$ (minimality).

Let us consider a Hilbert basis $(e_j)_{j \in \mathbb{N}}$ of a separable Hilbert space $(V, h)$, and denote the dual basis by $(\epsilon^j)_{j \in \mathbb{N}} \subseteq V^*$. For each $i \in \mathbb{N}$, we set

$$v_i = e_i \otimes \epsilon^1 \in CV$$

where for $v \in V$ and $\alpha \in V^*$, the elementary tensor $v \otimes \alpha$ is defined by the formula $(v \otimes \alpha)(w) = \alpha(w)v$, $w \in V$. It is easy to see that $(v_i)_{i \in \mathbb{N}}$ is an orthonormal basis of the compact Hilbert $CV$-module. See also Bakić, Guljaš [4].

*Musical isomorphisms $\flat$ and $\sharp$:* Let $(V, h)$ be a Hilbert space. We define the map $\flat : V \to V^*$ by $[\flat(v)](w) = h(v, w)$, where $v, w \in V$. On the continuous dual $V^*$, we consider the action of complex scalars given by $(c\alpha)(v) = c\alpha(v)$ for each $\alpha \in V^*$, $v \in V$ and $c \in \mathbb{C}$. Also we have the map $\sharp : V^* \to V$ defined by $h(\sharp(\alpha), v) = \alpha(v)$ for all $v \in V$. The element $\sharp(\alpha)$ exists by the Riesz representation theorem for Hilbert spaces. This makes us able to set $h^*(\alpha, \beta) = h(\sharp(\beta), \sharp(\alpha))$, where $\alpha, \beta \in V^*$. It is immediate to see that the hermitian-symmetric sesquilinear form $h^*$ is an inner product on $V^*$ and $(V^*, h^*)$ is a Hilbert space. Since $\sharp$ and $\flat$ are mutually inverse, $\flat$ is onto $V^*$ and thus $(V, h)$ is $\mathbb{C}$-self-dual. Notice also that $\flat$ and $\sharp$ are complex anti-linear homeomorphisms. We use the convenient notation $\alpha^\sharp = \sharp(\alpha)$ and $v^\flat = \flat(v)$.

Let $V'$ an $V''$ be Hilbert spaces and $T : V' \to V''$ be a continuous linear map. We remark that using the introduced notation, the adjoint of $T$ can be computed by the transposed map $T^t : V''^* \to V'^*$ as $T^*(v'') = [T^t(v''^\flat)]^\sharp$, where $v'' \in V''$ and $\sharp$ and $\flat$ are defined with respect to the inner products on $V'$ and $V''$, respectively. See Section 1 for the transposed maps of morphisms of Hilbert $A$-modules. Notice that in the theory of pseudodifferential operators, $T^*$ us usually called the transposed operator. However, in this case, we follow conventions used in pre-Hilbert $C^*$-modules and in inner product spaces.

We shall prove a $C^*$-compact embedding for Hilbert $C^*$-bundles that satisfy the following definition.

**Definition 5:** Let $M$ be a smooth manifold and $V$ be a Hilbert space. A Hilbert $CV$-bundle on $M$ is called a *compact $CV$-bundle* on $M$ if it is isomorphic as a Hilbert $C^*$-bundle to the product Hilbert $CV$-bundle $\underline{CV}_M = M \times CV \to M$ equipped with the Hilbert $CV$-bundle structure introduced below the Definition 2, in which we set $U = M$.

*Construction of embeddings for tori.* Because $CV$ is not topologically finitely generated over the $C^*$-algebra $CV$, we shall give a construction which replaces the formally similar construction in the proof of the Lemma 3.3 in Fomenko, Mishchenko [39] done for topologically finitely generated Hilbert $A$-modules over a unital $C^*$-algebra $A$. Let us consider the $n$-dimensional torus $T^n$ as the quotient $\mathbb{R}^n/(2\pi\mathbb{Z})^n$ of the standard manifold structures on $\mathbb{R}^n$ and $(2\pi\mathbb{Z})^n$, equip it with the flat Riemannian metric induced by the Euclidean inner product $(,)_{\mathbb{R}^n}$ on $\mathbb{R}^n$, and denote the corresponding norm on $\mathbb{R}^n$ by $|\,|_{\mathbb{R}^n}$. Further let $\mu_{T^n}$ be the Radon measure on $T^n$ induced by the volume density-form for the chosen Riemannian metric. For $i, j \in \mathbb{N}$, $\vec{m} \in \mathbb{Z}^n$ and $\vec{\theta} \in \mathbb{R}^n$, we define the $CV$-valued maps on the torus ${\phi^{\vec{m}}}_i^j : T^n \to CV$ by

$${\phi^{\vec{m}}}_i^j([\vec{\theta}]) = e^{\imath(\vec{m},\vec{\theta})_{\mathbb{R}^n}} e_i \otimes \epsilon^j$$

where $[\vec{\theta}]$ denotes the equivalence class of $\vec{\theta} \in \mathbb{R}^n$ in the quotient $\mathbb{R}^n/(2\pi\mathbb{Z})^n$, $(e_i)_{i\in\mathbb{N}}$ is a Hilbert basis of $V$, and $(\epsilon^i)_{i\in\mathbb{N}} \subseteq V^*$ is the dual Hilbert basis.

Let us consider the (positive semi-definite) Laplace operator $\Delta = -\sum_{i=1}^n \partial^2_{\theta^i}$ defined on smooth functions on the Euclidean space $(\mathbb{R}^n, (,)_{\mathbb{R}^n})$, and the appropriate differential operator $\Delta^{CV}$ that acts on smooth $CV$-valued functions defined on $\mathbb{R}^n$ as well. (The partial derivative $\partial_{\theta^i}$ denotes the Gateaux derivative in the direction of the $i$-th vector of the standard basis of $\mathbb{R}^n$.) By its translational invariance, the operator $\Delta^{CV}$ descends to a differential operator on the smooth $CV$-valued functions on the torus. Due to the smoothness of the action of $CV$ on the total space of the compact $CV$-bundle, the resulting operator is right $CV$-linear. We denote it by $\Delta^{CV}_{T^n} : C^\infty(T^n, CV) \to C^\infty(T^n, CV)$. It is convenient to use the $CV$-products defined by the powers of the Laplacian operator, that are equal to the $C^*$-products $(,)^S_k$ up to a constant multiple. See

Palais [42]. We denote them by $(,)_k^S$ as well. For $\vec{m}_1, \vec{m}_2 \in \mathbb{Z}^n$ and $i, j, l, p \in \mathbb{N}$, we have

$$\begin{aligned}(\phi^{\vec{m}_1}{}^j_i, \phi^{\vec{m}_2}{}^l_p)_k^S &= \int_{T^n} \left(\phi^{\vec{m}_1}{}^j_i\right)^* \circ \left(\left(\mathrm{Id}_{C^\infty(T^n,CV)} + \Delta_{T^n}^{CV}\right)^k \phi^{\vec{m}_2}{}^l_p\right) \mathrm{d}\mu_{T^n} \\ &= (1 + |\vec{m}_2|^2_{\mathbb{R}^n})^k \delta^{\vec{m}_1,\vec{m}_2} \delta_{ip} e_j \otimes \epsilon^l\end{aligned}$$

where $\delta^{\vec{m}_1,\vec{m}_2} = 0$ or 1 iff $\vec{m}_1 \neq \vec{m}_2$ or $\vec{m}_1 = \vec{m}_2$, respectively; the Kronecker symbol $\delta_{ip}$ has its classical meaning; the $k$th power of the operator means the $k$-fold composition of the operator with itself; and the integral is the Bochner integral of maps defined on the torus that have values in the Banach space $CV$. The above computation is based on the observations that $\int_0^{2\pi} e^{iny} \mathrm{d}\lambda_{\mathbb{R}}(y) = \delta_{0n}$ for each $n \in \mathbb{Z}$ and $-\partial^2_{\theta^j}(e^{\imath(\vec{m},\vec{\theta})_{\mathbb{R}^n}} e_p \otimes \epsilon^l) = m_j^2 e^{\imath(\vec{m},\vec{\theta})_{\mathbb{R}^n}} e_p \otimes \epsilon^l$, where $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $j = 1, \ldots, n$. For $k \in \mathbb{Z}$, let us set $\psi^{\vec{m},k}{}^j_i = (1 + |\vec{m}|^2_{\mathbb{R}^n})^{-k/2} \phi^{\vec{m}}{}^j_i$. Due to the above computation, $(\psi^{\vec{m},k}{}^1_i)_{i \in \mathbb{N}, \vec{m} \in \mathbb{Z}^n}$ is an orthonormal basis of the Hilbert $CV$-module $(H^k(T^n, CV), (,)_k^S)$.

Let us consider the canonical inclusion

$$I_{k+1} : H^{k+1}(T^n, CV) \to H^k(T^n, CV)$$

and compute the limit

$$\begin{aligned}\lim_{(\vec{m},i)\to\infty} |I_{k+1}(\psi^{\vec{m},k+1}{}^1_i)|_k^S &= \lim_{(\vec{m},i)\to\infty} |\psi^{\vec{m},k+1}{}^1_i|_k^S \\ &= \lim_{(\vec{m},i)\to\infty} |(1 + |\vec{m}|^2_{\mathbb{R}^n})^{-\frac{k+1}{2}} \phi^{\vec{m}}{}^1_i|_k^S \\ &= \lim_{(\vec{m},i)\to\infty} |(1 + |\vec{m}|^2_{\mathbb{R}^n})^{-\frac{1}{2}} (1 + |\vec{m}|^2_{\mathbb{R}^n})^{-\frac{k}{2}} \phi^{\vec{m}}{}^1_i|_k^S \\ &= \lim_{(\vec{m},i)\to\infty} (1 + |\vec{m}|^2_{\mathbb{R}^n})^{-\frac{1}{2}} |\psi^{\vec{m},k}{}^1_i|_k^S \\ &= \lim_{(\vec{m},i)\to\infty} (1 + |\vec{m}|^2_{\mathbb{R}^n})^{-\frac{1}{2}} = 0\end{aligned}$$

if $n \geqslant 1$.

**Scholium 1:** If $n \geqslant 1$, the map $I_{k+1}$ is $CV$-compact.

*Proof.* Since $\lim_{(\vec{m},i)\to\infty} |I_{k+1}(\psi^{\vec{m},k+1}{}^1_i)|_k^S = 0$ as computed above, $I_{k+1}$ is $CV$-compact by [4] (Theorem 9 (ii)). $\square$

**Remark:** For $n = 0$, $T^0 = \mathbb{R}^0/(2\pi\mathbb{Z})^0$ is just the singleton $\{0\}$. In this case each $I_{k+1}$ is identified with the identity map on $CV$. In particular, it is not a compact map. By the above computation for $n = 0$, $\lim_{(\vec{m},i)\to\infty} |I_{k+1}(\psi^{\vec{m},k+1}{}^1_i)|_k^S = 1$ because $|\vec{m}|_{\mathbb{R}^n} = 0$. Thus the map $I_{k+1}$ is even not $CV$-compact as follows from the mentioned Theorem 9 in [4].

## 3.2 Closed images of $CV$-elliptic operators on compact $CV$-bundles

For $k \in \mathbb{Z}$, unital $C^*$-algebra $A$, compact manifold $M$, Hilbert $A$-bundle $\mathcal{E} \to M$ with fibre a Hilbert $A$-module $(E, (,)_E)$, each subordinated partition of unity on $M$, and for a convenient integer $l$, the surjective Hilbert $A$-module morphism $P_{[k]} : \bigoplus_{i=1}^{l} H^k(T^n, CV) \to H^k(M, CV)$ is defined in Solovyov, Troitsky (Construction 2.1.76, [56]). The construction is parallel to the finite rank case ([42]). It is easy to realize that it may be applied at least for any Hilbert $C^*$-module $E$ which is orthogonally complemented in $A^q$ for a suitable $q$, regardless whether $A$ is unital or not. Let us notice that the compact $CV$-module $E = CV$ is orthogonally complemented in $CV$ for trivial reasons. Thus we may define $P_{[k]}$ for a compact $CV$-bundle on $M$ by the same formula as in [56], getting a Hilbert $CV$-module morphism as well. Let us notice that for $k' \leqslant k$, $P_{[k']}$ is the continuous extension of $P_{[k]}$ since all of the maps $P_{[k]}$ are defined as the continuous extensions to the Sobolev-type spaces of a single map $P : \bigoplus_{i=1}^{l} C^\infty(T^n, E) \to \Gamma^\infty(M, \mathcal{E})$, whose domain is dense in the direct sum $\bigoplus_{i=1}^{l} H^k(T^n, E)$ of the Sobolev-type completions for each $k \in \mathbb{Z}$.

The existence of a right inverse $\gamma_{[k]}$ to $P_{[k]}$ is stated in Theorem 2.1.77 in [56] without a precise reference to a proof. However the proof may proceed in the same way as in Palais [42] (Theorem 2 in Paragraph 4, Chapter X). The appropriate Hilbert $A$-bundle analogue of $\gamma_{[k]}$ is $A$-linear since it is constructed by scalar-valued functions derived from the partition of unity on $M$ and Hilbert $A$-bundle charts, which are fibre-wise Hilbert $A$-module morphisms by Definition 2. Consequently, $\gamma_{[k]}$ is a Hilbert $A$-module morphism. Further for $k \geqslant k'$, the map $\gamma_{[k']}$ is the continuous extension of the map $\gamma_{[k]}$. We remark that the proof of this statement proceeds as in the Corollary 1 of Theorem 2, Paragraph 4, Chapter X in Palais [42].

Let us mention (cf., e.g., 2.1.28 in [56]) that a *Rellich $A$-chain* is a descending chain of Hilbert $A$-modules $(X_k)_{k\in\mathbb{Z}}$ such that the inclusion maps $X_{k+1} \hookrightarrow X_k$ are $A$-compact for each integer $k$.

Below we prove a theorem whose second part is targeted to the images of the continuous extensions to Sobolev-type spaces of $CV$-elliptic operators. For simplicity, we suppose that the finite orthogonal sum $\bigoplus_{i=1}^{l} H^k(T^n, E)$ contains one element only, i.e., $l = 1$. In the proof of the first part, we proceed similarly as in the proof of the Thm. 3, Paragraph 4, Chapter X in [42].

**Theorem 7**: Let $M$ be a compact manifold of dimension $n$ and

$$\mathcal{D} : C^\infty(M, CV) \to C^\infty(M, CV)$$

be an adjointable $CV$-elliptic operator of order $d$ on the compact $CV$-bundle $\underline{CV}_M = M \times CV$ on $M$. Then

a) the inclusion map $J_{k+1} : H^{k+1}(M, CV) \to H^k(M, CV)$ is $CV$-compact and

b) the continuous extension $\mathcal{D}_k$ of $\mathcal{D}$ to $H^k(M, CV)$ is $CV$-Fredholm and its image is closed in $H^{k-d}(M, CV)$ with respect to the topology given by the Sobolev-type norm $|\,|^S_{k-d}$.

*Proof.* 1) We have sequences $(H^k(T^n, CV))_{k\in\mathbb{Z}}$ and $(H^k(M, CV))_{k\in\mathbb{Z}}$ of Hilbert $A$-modules at our disposal. The sequence $(H^k(T^n, CV))_{k\in\mathbb{Z}}$ is a Rellich $CV$-chain because the inclusions

$$I_{k+1} : H^{k+1}(T^n, CV) \to H^k(T^n, CV)$$

are $CV$-compact by Scholium 1. Let us consider the inclusion map $J_{k+1} : H^{k+1}(M, CV) \to H^k(M, CV)$. The diagram

$$\begin{tikzcd} H^{k+1}(T^n, CV) \arrow[r, "I_{k+1}"] & H^k(T^n, CV) \arrow[d, "P_{[k]}"] \\ H^{k+1}(M, CV) \arrow[u, "\gamma_{[k+1]}"] \arrow[r, "J_{k+1}"] & H^k(M, CV) \end{tikzcd}$$

is commutative since $P_{[k]}I_{k+1}\gamma_{[k+1]}s = P_{[k]}\gamma_{[k+1]}s = P_{[k]}\gamma_{[k]}s = s = J_{k+1}s$ for all $s \in H^{k+1}(M, CV)$. Since $I_{k+1}$ is $CV$-compact (Scholium 1), $J_{k+1}$ is $CV$-compact by the ideal property of $C^*$-compact operators (see the Remark above the Definition 1). Consequently $(H^k(M, CV))_{k\in\mathbb{Z}}$ is a Rellich $CV$-chain as well and the corresponding inclusions are $CV$-compact homomorphisms. Thus a) is proved.

2) Let $k \in \mathbb{Z}$. Since $\mathcal{D}$ is $CV$-elliptic, there exists a partial inverse of $\mathcal{D}_k$, denoted by $\check{\mathcal{D}}_{k-d}$, such that the operator $N_k = \check{\mathcal{D}}_{k-d}\mathcal{D}_k - \mathrm{Id}_{H^k(M,CV)}$ maps $H^k(M, CV)$ into $H^{k+1}(M, CV) \subseteq H^k(M, CV)$, i.e., it is smoothing. Recall that $\check{\mathcal{D}}_{k-d}$ is constructed by inverting the symbol of $\mathcal{D}$ out of the image of the zero section of $T^*M$, using the already mentioned construction. Since $N_k = J_{k+1} \circ N_k$ and $J_{k+1}$ is $CV$-compact by a), $N_k$ is $CV$-compact as a map of $H^k(M, CV)$ into $H^k(M, CV)$ by the ideal property for $C^*$-compact operators. Similarly, we proceed for the opposite composition, i.e., of $\mathcal{D}_k$ with $\check{\mathcal{D}}_{k-d}$ in this order.

Consequently $\mathcal{D}_k$ is $CV$-Fredholm. By Lemma 1, the image of $\mathcal{D}_k$ is closed in $H^{k-d}(M, CV)$ and thus b) follows. □

# 4 Images of elliptic operators

In this chapter, we investigate topological properties of images of elliptic operators defined on smooth sections of Hilbert bundles on compact manifolds, not assuming additionally the invariance of these operators with respect to a $C^*$-algebra of compact operators other than the $C^*$-algebra of complex numbers.

## 4.1 Injective completion of tensor products

Let $\widehat{X}$ be the unique completion of a metric space $X$ up to isometry. We suppose that it is defined by taking Cauchy sequences in $X$ and by considering two of such sequences equivalent if they differ by a sequence converging to zero in $X$ that is usually called the null-sequence. The completion $\widehat{X}$ is equipped with a metric induced canonically by the metric on $X$. The element in $\widehat{X}$ determined by a Cauchy sequence $(a_i)_{i\in\mathbb{N}} \subseteq X$ is denoted by $[(a_i)_i]$ or by $\lim_i a_i$. We always consider $X$ to be isometrically embedded into $\widehat{X}$ by the map that takes an element $a \in X$ to the equivalence class containing the constant sequence $(a)_{i\in\mathbb{N}}$.

Let $X_1$ and $X_2$ be both real or both complex vector spaces. We denote their (algebraic) tensor product over the ring of real or complex numbers, respectively, by $X_1 \otimes X_2$. If $X_1$ and $X_2$ are vector spaces equipped with countable families of seminorms, we denote the tensor product $X_1 \otimes X_2$ by $X_1 \otimes_\epsilon X_2$, when we consider it with the so-called injective family of seminorms (see Tréves [58]) which is induced by the seminorms on $X_1$ and $X_2$. We denote the completion of the tensor product $X_1 \otimes_\epsilon X_2$ with respect to the metric generated by the injective family of seminorms by $X_1 \widehat{\otimes}_\epsilon X_2$ and call it the *completed injective tensor product* or simply the injective completion. If $X_1$ and $X_2$ are Fréchet topological vector spaces, it is well known that the completion is a Fréchet topological vector space as well. See [58].

Let $Z$ be a Fréchet topological vector space and $(V, h)$ be a Hilbert space. For a homogeneous element $C = f \otimes \alpha \in Z \otimes V^*$ and an element $B \in CV$, we consider the right action of $B$ on $C$ defined by $C \cdot B = f \otimes (\alpha \circ B)$, and extend it linearly to the tensor product $X \otimes V^*$. We can think of the action on $Z \otimes V^*$ by a fixed element $B \in CV$ as of the map $\mathrm{Id}_Z \otimes P_B$, where $P_B(\alpha) = \alpha \circ B$ for each $\alpha \in V^*$. The action by $B$ on elements of the completed injective tensor product $Z \widehat{\otimes}_\epsilon V^*$ is defined as the unique continuous extension of the map $\mathrm{Id}_Z \otimes P_B$ to $Z \widehat{\otimes}_\epsilon V^*$ denoted by $\mathrm{Id}_Z \widehat{\otimes}_\epsilon P_B$. It is easy to see that the resulting action is continuous as a map $(Z \widehat{\otimes}_\epsilon V^*) \times CV \to Z \widehat{\otimes}_\epsilon V^*$ if $CV$ is considered with the operator norm topology and also if it is considered with the strong operator topology. The continuity is verified for each of the Fréchet norms defining the metric on $Z$ separately.

For any continuous map $D : X \to Y$ of Fréchet spaces $X$ and $Y$, we consider the continuous map $D^\epsilon = D \otimes \mathrm{Id}_{V^*} : X \otimes_\epsilon V^* \to Y \otimes_\epsilon V^*$ and its unique continuous extension $\widehat{D}^\epsilon : X \widehat{\otimes}_\epsilon V^* \to Y \widehat{\otimes}_\epsilon V^*$ to the completed injective tensor product $X \widehat{\otimes}_\epsilon V^*$. See Tréves [58].

**Lemma 8:** Let $D : X \to Y$ be a continuous linear map of Fréchet topological vector spaces $X$ and $Y$. Then $\widehat{D}^\epsilon$ is $CV$-linear from the right.

*Proof.*

1) Each element $c \in X \otimes V^*$ can be written as $c = \sum_{i=1}^{l} f_i \otimes \alpha_i$, where $l$ is an integer, $f_i \in X$ and $\alpha_i \in V^*$ for each $i = 1, \dots, l$. For $B \in CV$, we have $c \cdot B = \sum_{i=1}^{l} f_i \otimes (\alpha_i \circ B)$ and consequently $(D^\epsilon(c)) \cdot B = ((D \otimes \mathrm{Id}_{V^*})(c)) \cdot B = \left((D \otimes \mathrm{Id}_{V^*}) \sum_{i=1}^{l} f_i \otimes \alpha_i\right) \cdot B = \left(\sum_{i=1}^{l} (Df_i \otimes \alpha_i)\right) \cdot B = \sum_{i=1}^{l} Df_i \otimes (\alpha_i \circ B) = (D \otimes \mathrm{Id}_{V^*})(c \cdot B) = D^\epsilon(c \cdot B)$. Thus $D^\epsilon$ is $CV$-linear from the right.

2) For $c \in X \widehat{\otimes}_\epsilon V^*$, let us consider a Cauchy sequence $(c_i)_{i \in \mathbb{N}}$ in $X \otimes_\epsilon V^*$ that converges to $c$ in the completed injective tensor product $X \widehat{\otimes}_\epsilon V^*$. Using the fact that $P_B$ and $D \widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$ are continuous, the $CV$-linearity from the right of $D^\epsilon = D \otimes \mathrm{Id}_{V^*}$ on $X \otimes V^*$ proved in the paragraph 1) above, and the continuity of $P_B$ again, we obtain $\widehat{D}^\epsilon(c \cdot B) = \widehat{D}^\epsilon((\lim_i c_i) \cdot B) = \widehat{D}^\epsilon(\lim_i (c_i \cdot B)) =$

$$= \lim_i (D^\epsilon(c_i \cdot B)) = \lim_i ((D^\epsilon(c_i)) \cdot B) = \left(\lim_i (D^\epsilon(c_i))\right) \cdot B =$$
$$= \left(\widehat{D}^\epsilon(\lim_i c_i)\right) \cdot B = \left(\widehat{D}^\epsilon(c)\right) \cdot B.$$

Consequently, $\widehat{D}^\epsilon$ is right $CV$-linear as a map of $X \widehat{\otimes}_\epsilon V^*$ into $Y \widehat{\otimes}_\epsilon V^*$. □

Let $V$ and $V'$ be Banach spaces and let us consider the operator norm topology on the continuous dual $V^*$ of $V$. The following isometric isomorphism of Fréchet topological vector spaces

$$C^\infty(M, V') \widehat{\otimes}_\epsilon V^* \cong C^\infty(M, V' \widehat{\otimes}_\epsilon V^*)$$

is proved in [58] (Theorem 44.1). Moreover, if $V$ and $V'$ are unitarilly isomorphic Hilbert spaces, we have $C^\infty(M, V' \widehat{\otimes}_\epsilon V^*) \cong C^\infty(M, CV)$, where the space $CV$ of compact operators on $V$ is considered with the operator norm topology. See, e.g., [58], Theorem 48.3.

**Remark:** Let us denote the $\mathbb{C}$-product on $C^\infty(M, V)$ induced by the inner product $h$ on $V$ by $(,)$, the $CV$-product on $C^\infty(M, CV)$ by $(,)'$, and the measure induced by the volume density-form of a Riemannian metric tensor on $M$ by $\mu$. For $f, g \in C^\infty(M, V)$ and $\alpha, \beta \in V^*$, we thus have

$$\begin{aligned}
(f \otimes \alpha, g \otimes \beta)' &= \int_{y \in M} (f(y) \otimes \alpha, g(y) \otimes \beta)_{CV} \mathrm{d}\mu(y) \\
&= \int_{y \in M} ((f(y) \otimes \alpha)^* \circ (g(y) \otimes \beta)) \mathrm{d}\mu(y) \\
&= \int_{y \in M} \left((\alpha^\sharp \otimes f(y)^\flat) \circ (g(y) \otimes \beta)\right) \mathrm{d}\mu(y) \\
&= \left(\int_{y \in M} h(f(y), g(y)) \mathrm{d}\mu(y)\right) \alpha^\sharp \otimes \beta \\
&= (f, g) \alpha^\sharp \otimes \beta
\end{aligned}$$

where $f \otimes \alpha$ is defined by $(f \otimes \alpha)(y) = f(y) \otimes \alpha$ for $y \in M$ and similarly for $g$ and $\beta$.

We denote the Sobolev-type product on the Hilbert $CV$-module $H^k(M, CV)$ by $(,)'^S_k$ in order to distinguish it from the Sobolev-type product $(,)^S_k$ on the Hilbert $\mathbb{C}$-module $H^k(M, V)$. Note that the same multiplicative factor $(1 + |y|^2_{\mathbb{R}^n})^k$ appears in the local coordinate expressions for $(,)^S_k$ as well as for $(,)'^S_k$. Substituting $\mathcal{F}^V f$ and $\mathcal{F}^V g$ for $f$ and $g$, respectively, in the above integral formulas for $(,)'$ considered in local coordinates, i.e., on $\mathbb{R}^n$, and adding the multiplicative factor $(1 + |y|^2_{\mathbb{R}^n})^k$ under all integral signs, we get that for $f, g \in H^k(M, V) \otimes V^*$ and $\alpha, \beta \in V^*$

$$(f \otimes \alpha, g \otimes \beta)'^S_k = (f, g)^S_k \, \alpha^\sharp \otimes \beta \tag{1}$$

In the next theorem, we analyse the $C^*$-ellipticity of the $CV$-linear operator $\widehat{D}^\epsilon = D \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$ if $D$ is elliptic.

**Theorem 9:** Let $D$ be an elliptic operator on the product Hilbert bundle $p : \underline{V}_M = M \times V \to M^n$ on the compact manifold $M$ with fibre a Hilbert space $V$. Then the operator $\widehat{D}^\epsilon : C^\infty(M, CV) \to C^\infty(M, CV)$ is a $CV$-elliptic operator on the compact $CV$-bundle $\underline{CV}_M = M \times CV \to M$ whose image is closed in the pre-Hilbert topology on $C^\infty(M, CV)$.

*Proof.* 1) Let $\sigma$ denote the symbol of $D$ and let us consider the map $\sigma' = \sigma \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$ of $T^*M$ by which we mean that $\sigma'(\xi) = \widehat{\sigma}^\epsilon(\xi) = \sigma(\xi) \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$ for each $\xi \in T^*_m M$ and $m \in M$. In particular, $\sigma'(\xi) : V \widehat{\otimes}_\epsilon V^* \to V \widehat{\otimes}_\epsilon V^*$. By Lemma 8 used for $\sigma'(\xi)$, $\sigma'(\xi)$ is $CV$-linear. In particular, $\sigma'(\xi) \in \mathrm{End}_{CV}(CV)$. This map satisfies the growth conditions for symbols of the same order because $\mathrm{Id}_{V*}$ does not depend on $\xi$ and thus it does not change the defining growth estimates for symbols as given in [56]. The operator $\sigma'(\xi)$ is adjointable with the adjoint $\sigma(\xi)^* \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$. Consequently, $\sigma'$ is a symbol.

Since $M$ is compact, there is a finite set $K$ and a partition of unity $(U_j, \chi_j)_{j \in K}$ subordinated to the manifold and to the bundle atlases. Let $\psi_j : U_j \to \mathbb{R}^n$ be a chart in the atlas of $M$, $j \in K$, and let us denote the coordinate expression of $\sigma$ restricted to $T^*U_j$ by $\sigma_j$. It maps an open subset in $\mathbb{R}^{2n}$ into $\mathrm{End}(V)$. The coordinate expression of $\sigma'$ restricted to $T^*U_j$ is denoted by $\sigma'_j$. It maps an open set in $\mathbb{R}^{2n}$ into $\mathrm{End}^*_{CV}(CV)$. We have $\sigma'_j(\xi) = \sigma_j(\xi) \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$, where the cotangent vector $\xi$ is considered as an element of $(\mathbb{R}^{2n})^*$. For a manifold chart $(U_j, \psi_j)$, we denote the push-forward $\chi_j \circ \psi_j^{-1}$ of $\chi_j$ by $\chi_j^\psi$, $j \in K$.

Using the associativity of the tensor product and the equation $\mathrm{Id}_{CV} = \mathrm{Id}_V \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$, we get $\mathcal{F}^{CV} = \mathcal{F} \widehat{\otimes}_\epsilon \mathrm{Id}_{CV} = \mathcal{F} \widehat{\otimes}_\epsilon (\mathrm{Id}_V \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}) = \mathcal{F}^V \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$. In the next computation, $\widehat{D}^\epsilon$ denotes the coordinate expression of the corresponding operator with respect to the chosen charts and to the chosen partition of unity. We use the equations $\mathrm{Id}_{V*} = \mathrm{Id}_{V*} \otimes \mathrm{Id}_{V*} = \mathrm{Id}^{-1}_{V*} \circ \mathrm{Id}_{V*}$ and the bifunctoriality of the tensor product with respect to composition of maps. We thus have

$$
\begin{aligned}
\widehat{D}^{\epsilon} = D\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*} &= \sum_{j\in K}\left[(\mathcal{F}^V)^{-1}\circ\chi_j^{\psi}\sigma_j\circ\mathcal{F}^V\right]\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*}\\
&= \sum_{j\in K}\left[\left((\mathcal{F}^V)^{-1}\circ\chi_j^{\psi}\sigma_j\right)\circ\mathcal{F}^V\right]\widehat{\otimes}_{\epsilon}(\mathrm{Id}_{V*}\circ\mathrm{Id}_{V*})\\
&= \sum_{j\in K}\left[\left((\mathcal{F}^V)^{-1}\circ\chi_j^{\psi}\sigma_j\right)\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*}\right]\circ\left(\mathcal{F}^V\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*}\right)\\
&= \sum_{j\in K}\left[\left((\mathcal{F}^V)^{-1}\circ\chi_j^{\psi}\sigma_j\right)\widehat{\otimes}_{\epsilon}(\mathrm{Id}_{V*}^{-1}\circ\mathrm{Id}_{V*})\right]\circ\left(\mathcal{F}^V\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*}\right)\\
&= \sum_{j\in K}\left[((\mathcal{F}^V)^{-1}\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*}^{-1})\circ(\chi_j^{\psi}\sigma_j\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*})\right]\circ\mathcal{F}^{CV}\\
&= \sum_{j\in K}\left[\left(\mathcal{F}^V\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*}\right)^{-1}\circ(\chi_j^{\psi}\sigma_j\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*})\right]\circ\mathcal{F}^{CV}\\
&= \sum_{j\in K}(\mathcal{F}^{CV})^{-1}\circ\chi_j^{\psi}\sigma_j'\circ\mathcal{F}^{CV}
\end{aligned}
$$

i.e., $\widehat{D}^{\epsilon}$ is generated by the symbol $\sigma' = \sigma\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*}$. In particular, $\widehat{D}^{\epsilon}$ is a $CV$-pseudodifferential operator.

Since $D$ is elliptic, $\sigma(\xi)$ is a linear homeomorphism for any $\xi \neq 0$. It is immediate to realize that the continuous $CV$-linear map $\sigma(\xi)^{-1}\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*}$ of $CV$ is the inverse of $\sigma'(\xi)$. Thus $\sigma'(\xi) = \sigma(\xi)\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*}$ is a Hilbert $CV$-module automorphism for any $\xi \neq 0$ and consequently, the operator $\widehat{D}^{\epsilon}$ is $CV$-elliptic.

2) Notice that $D$ is adjointable by the Lemma 5. It can be easily realized from the definition of the adjoint, that the adjoint of $\widehat{D}^{\epsilon}$ as a morphism of the pre-Hilbert $C^*$-module $C^{\infty}(M, CV)$ is $D^*\widehat{\otimes}_{\epsilon}\mathrm{Id}_{V*}$. For proving the closed image property of $\widehat{D}^{\epsilon}$ with respect to the pre-Hilbert topology on $C^{\infty}(M, CV)$ we use the Corollary 3. We set $A = CV$, $X = X' = H^{2d}(M, CV)$, $Y = Y' = H^0(M, CV)$, $Z = Z' = C^{\infty}(M, CV)$, $\Delta = (\widehat{D}^{\epsilon})^*\widehat{D}^{\epsilon} : Z \to Z$, and $\Delta' = \widehat{D}^{\epsilon}(\widehat{D}^{\epsilon})^* : Z \to Z$, where $d$ is the order of $D$. We verify the assumptions of the corollary below.

i) The $CV$-pseudodifferential operators $\Delta$ and $\Delta'$ have continuous extensions $\widetilde{\Delta} = \Delta_{2d}$ and $\widetilde{\Delta'} = (\Delta')_{2d}$ to $X$ that map the space $X$ into the space $Y$. These extensions are adjointable since any continuous $CV$-linear map of Hilbert $CV$-modules is adjointable. See Remark 5 (b) in [4].

ii) Further conditions in Corollary 3 are the assumptions a) and b) of Theorem 2. Let us set $\mathcal{D} = D^*D : C^{\infty}(M, V) \to C^{\infty}(M, V)$. Since $D$ is elliptic, $D^*$ is elliptic by the Lemma 5. By the Theorem 2.1.116 [56] on the symbol of the composition and Lemma 5 used once more, the symbol $\sigma(\mathcal{D})(\xi)$ of the composition $D^*D$ in $\xi$ equals to $\sigma(D^*)(\xi) \circ \sigma(D)(\xi) = \sigma(D)(\xi)^* \circ \sigma(D)(\xi)$ which is easy to see to be an isomorphism of $V$ for each $\xi \neq 0$. By the paragraph 1) above used for $\widehat{\mathcal{D}}^\epsilon$, $\widehat{\mathcal{D}}^\epsilon$ is $CV$-pseudodifferential and its symbol in $\xi \in T^*M$ is $\sigma(\mathcal{D})(\xi)\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$ which is an automorphism of $CV$ for $\xi \neq 0$. Thus $\Delta = (D^*\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}) \circ (D\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}) = D^*D\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*} = \mathcal{D}\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$ is $CV$-elliptic. The operator $\Delta'$ is elliptic by interchanging the roles of $D$ and $D^*$. By the Theorem 7, $\widetilde{\Delta}$ and $\widetilde{\Delta'}$ are $CV$-Fredholm and thus by Lemma 1, their images are closed, i.e., the condition a) of Theorem 2 is satisfied.

iii) Since $\Delta$ and $\Delta'$ are $CV$-elliptic, they are regular by Theorem 6, and thus the first part of the condition b) of Thm. 2 is satisfied. We verify the second part of b), i.e., the regularity for $\widetilde{\Delta}^*$ and $\widetilde{\Delta'}^*$. By Wloka [63], the spaces $H^k(M,V)$ are Hilbert spaces. In particular they are self-dual by the Riesz representation theorem. In this case the adjoint $(\mathcal{D}_{2d})^* : H^0(M,V) \to H^{2d}(M,V)$ is identified with $(\mathcal{D}^*)_0 = \mathcal{D}_0 : H^0(M,V) \to H^{-2d}(M,V)$ by the standard procedure based on the Sobolev smooth embedding and the uniqueness of adjoints. See Palais [42] or Solovyov, Troitsky [56], p. 84. Denoting $\Gamma = \mathcal{C}^\infty(M,V)$, we thus have

$$\mathcal{D} = (\mathcal{D}_{2d})^*{}_{|\Gamma} \tag{2}$$

Let us return to the regularity question of $\widetilde{\Delta}^* = (\Delta_{2d})^*$. Using the equations (1) and (2), we get for $f, g \in C^\infty(M,V)$ and $\alpha, \beta \in V^*$ that

$$\begin{aligned}
&(f\otimes\alpha, \Delta_{2d}^*(g\otimes\beta))'^S_{2d} = (\Delta_{2d}(f\otimes\alpha), g\otimes\beta)'^S_0 = (\Delta(f\otimes\alpha), g\otimes\beta)'^S_0 \\
&= (\mathcal{D}f\otimes\alpha, g\otimes\beta)'^S_0 = (\mathcal{D}_{2d}f\otimes\alpha, g\otimes\beta)'^S_0 = (\mathcal{D}_{2d}f, g)^S_0\alpha^\sharp\otimes\beta \\
&= (f, (\mathcal{D}_{2d})^*g)^S_{2d}\alpha^\sharp\otimes\beta = (f, \mathcal{D}g)^S_{2d}\alpha^\sharp\otimes\beta = (f\otimes\alpha, \mathcal{D}g\otimes\beta)'^S_{2d} \\
&= (f\otimes\alpha, \Delta(g\otimes\beta))'^S_{2d}
\end{aligned}$$

where $(,)'^S_k$ denotes the Sobolev-type product on $H^k(M,CV)$. Denoting $\Gamma' = C^\infty(M,V)\otimes V^*$ and comparing the first and the last term of the equalities above, we get $(\Delta_{2d})^*{}_{|\Gamma'} = \Delta_{|\Gamma'}$. Since $C^\infty(M,V)\otimes V^*$ is dense in $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$ with respect to the Fréchet topology and $C^\infty(M,V)\widehat{\otimes}_\epsilon V^* \cong C^\infty(M,CV)$ both considered in the Fréchet topology, $\Delta$ and $(\Delta_{2d})^*{}_{|C^\infty(M,CV)}$ are equal. Since $\Delta$ is $CV$-elliptic by ii), $(\Delta_{2d})^*{}_{|C^\infty(M,CV)}$ is $CV$-elliptic as well. By Theorem 6, $\widetilde{\Delta}^*$ is regular and therefore the condition b) of Theorem 2 is satisfied for the operator $(\widetilde{\Delta}^*)_{|C^\infty(M,CV)}$. For the operator $\widetilde{\Delta'}^*$ the condition b) is verified by interchanging the roles of $D$ and $D^*$ in the composition defining $\mathcal{D}$.

By Corollary 3, the images of $\widehat{D}^\epsilon$ and $(\widehat{D}^\epsilon)^*$ are closed with respect to the pre-Hilbert topology on $C^\infty(M,CV)$. □

**Remark**: 1) Let $V$ be an infinite dimensional Hilbert space. In the part 2) of the above proof, we do not construct the adjoints of the extensions $\Delta_k$ and $\Delta'_k$

using the transposed operators since $H^k(M,CV)$ are not a $CV$-self-dual Hilbert $CV$-modules in general. Note that even if $M$ is a single point, $H^k(M,CV)$ is isomorphic $CV$ that is not $CV$-self-dual. Its $CV$-dual is the space of all linear bounded operators on $V$. (See Jensen, Thomsen [21] (E 1.1.4) and Frank [13] for more information.) Moreover and for different reasons, $H^k(M,CV)$ is not linearly homeomorphic to $H^k(M,V)\widehat{\otimes}_\epsilon V^*$ if the dimension of $M$ is at least 1 and $V$ is infinite dimensional. (Private communication with D. Vogt.)

2) The previous lemma has an appropriate generalization for an elliptic operator $D$ on smooth sections of Hilbert bundles $p' : \underline{V'}_M \to M$ and $p'' : \underline{V''}_M \to M$ with fibres the Hilbert spaces $V'$ and $V''$, respectively. For a Hilbert space $V$, we consider the $CV$-linear operator $\widehat{D}^\epsilon = D\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*} : C^\infty(M, V'\widehat{\otimes}_\epsilon V^*) \to C^\infty(M, V''\widehat{\otimes}_\epsilon V^*)$ whose symbol in $\xi$ is $\sigma(\xi)\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$, where $\sigma$ denotes the symbol of $D$. The inverse of the symbol of $\widehat{D}^\epsilon$ is $\sigma(\xi)^{-1}\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$ for any non-zero $\xi$. Thus $\widehat{D}^\epsilon$ is $CV$-elliptic as well. The growth condition for symbols are satisfied since the identity does not change the appropriate estimates.

3) The adjontability of $\sigma'(\xi) = \sigma(\xi)\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$ from the proof of the above theorem follows from the assertion in Remark 5 (b) of [4] as well since $\sigma'(\xi)$ is continuous.

Next we prove a lemma on a representation of smooth sections of the compact Hilbert $CV$-bundle $\underline{CV}_M$ on a compact $M$. Let us recall that we consider only separable Hilbert spaces (Preamble, item d)). Nevertheless, it is easy to see that the next lemma holds for non-separable Hilbert spaces as well, that is proved by taking nets instead of sequences.

**Lemma 10**: For an arbitrary element $\widehat{f'}$ in $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$ and all positive integers $i,j \in \mathbb{N}$, there exists a smooth $V$-valued function $\phi_{ij} \in C^\infty(M,V)$ such that $\widehat{f'}$ is the equivalence class of the Cauchy sequence in $C^\infty(M,V) \otimes_\epsilon V^*$ with elements $f'_i = \sum_{j=1}^\infty \phi_{ij} \otimes \epsilon^j$, $i \in \mathbb{N}$, i.e., $\widehat{f'} = \lim_i f'_i$ with respect to the Fréchet topology on $C^\infty(M,V)\widehat{\otimes}_\epsilon V^* \cong C^\infty(M,CV)$. The equality $\widehat{f'} = \lim_i f'_i$ also holds with respect to the pre-Hilbert topology on $C^\infty(M,CV)$. Moreover, for each $j \in \mathbb{N}$ the limit $\lim_i \phi_{ij}$ exists in both of these topologies.

*Proof.* Let $(f'_i)_{i\in\mathbb{N}} \subseteq C^\infty(M,V) \otimes_\epsilon V^*$ be a Cauchy sequence representing $\widehat{f'}$, i.e., $\widehat{f'} = \lim_i f'_i$. By the definition of the algebraic tensor product of vector spaces, we have that for each $i \in \mathbb{N}$ there is a positive integer $m_i \in \mathbb{N}$, and for each $k = 1,\ldots,m_i$, there is a function $f_{ik} \in C^\infty(M,V)$ and a continuous functional $\alpha_k \in V^*$ such that $f'_i = \sum_{k=1}^{m_i} f_{ik} \otimes \alpha_k$. For any $j \in \mathbb{N}$, there exist complex numbers $\theta_{kj}$, $k = 1,\ldots,m_i$, such that $\alpha_k = \sum_{j=1}^\infty \theta_{kj}\epsilon^j$, where $(\epsilon^j)_{j\in\mathbb{N}}$ is a Hilbert basis of the separable Hilbert space $(V,h)$. Consequently, $f'_i = \sum_{k=1}^{m_i} f_{ik} \otimes \sum_{j=1}^\infty \theta_{kj}\epsilon^j$ for each $i \in \mathbb{N}$. Thus $f'_i = \sum_{j=1}^\infty \sum_{k=1}^{m_i} f_{ik} \otimes \theta_{kj}\epsilon^j = \sum_{j=1}^\infty \phi_{ij} \otimes \epsilon^j$, where $\phi_{ij} = \sum_{k=1}^{m_i} \theta_{kj} f_{ik}$ for each $i,j \in \mathbb{N}$. By the isomorphism $C^\infty(M,V)\widehat{\otimes}_\epsilon V^* \cong C^\infty(M,CV)$ of Fréchet topological vector spaces mentioned above, we consider $f'_i = \sum_{j=1}^\infty \phi_{ij} \otimes \epsilon^j$ as an element of $C^\infty(M,CV)$ by $f'_i(m) = \sum_{j=1}^\infty \phi_{ij}(m) \otimes \epsilon^j \in CV$, $m \in M$. Since the pre-Hilbert topology on $C^\infty(M,CV)$

is finer than the Fréchet topology (Lemma 4), the equality $\widehat{f}' = \lim_i f'_i$ holds also in the pre-Hilbert topology on $C^\infty(M, CV)$.

For $v \in V$, let us consider the map $\mathrm{ev}_v : C^\infty(M, CV) \to C^\infty(M, V)$ defined by $(\mathrm{ev}_v f)(m) = (f(m))(v)$, where $f \in C^\infty(M, CV)$ and $m \in M$. It is easy to realize that this map is continuous for any $v \in V$ with respect to the Fréchet topologies. Namely for $l \geqslant 0$ the operator norm $|| \, ||^F_{l,l}$ of $\mathrm{ev}_v$ with respect to the $l$th Fréchet seminorm on $C^\infty(M, CV)$ and the $l$th seminorm on $C^\infty(M, V)$ is bounded by the constant $|v|_V$ as is easily seen by the inequality $|A(v)|_V \leqslant |A|_{CV} |v|_V$ where $A \in CV$. Due to $f'_i = \sum_{j=1}^\infty \phi_{ij} \otimes \epsilon^j$ and the mentioned continuity of $\mathrm{ev}_v$, we have $\phi_{ij} = \mathrm{ev}_{e_j}(f'_i)$. Taking the limit of the expression with respect to $i$, we get $\lim_i \phi_{ij} = \mathrm{ev}_{e_j} \lim_i (f'_i) = \mathrm{ev}_{e_j} \widehat{f}'$. In particular, $\lim_i \phi_{ij}$ exists. Since the pre-Hilbert topology is finer than the Fréchet topology (Lemma 4), the limit $\lim_i \phi_{ij}$ exists for each $j$ also with respect to the pre-Hilbert topology. □

We derive the following scholium.

**Scholium 2**: Let $D$ be a pseudodifferential operator on the Hilbert bundle $\underline{V}_M$ on a compact manifold $M$ with fibre a Hilbert space $V$. Let $(a_j)_{j \in \mathbb{N}} \subseteq C^\infty(M, V) \otimes_\epsilon V^*$ be a sequence such that the series $\sum_{j=1}^\infty a_j$ converges with respect to the Fréchet topology on $C^\infty(M, V) \widehat{\otimes}_\epsilon V^*$. Then

$$\widehat{D}^\epsilon \left( \sum_{j=1}^\infty a_j \right) = \sum_{j=1}^\infty D^\epsilon a_j$$

with respect to the Fréchet topology.

*Proof.* The series $\sum_{j=1}^\infty a_j$ converges in the Fréchet topology by the assumption. Since $\widehat{D}^\epsilon$ is a pseudodifferential operator (Theorem 9), we have for the Fréchet norms $| \sum_{j=1}^k \widehat{D}^\epsilon a_j - \widehat{D}^\epsilon \sum_{j=1}^\infty a_j |^F_l \leqslant || \widehat{D}^\epsilon ||^F_{l,l-d} | \sum_{j=1}^k a_j - \sum_{j=1}^\infty a_j |^F_l$, where $d$ denotes the order of $D$ and $|| \, ||^F_{l,l-d}$ denotes the operator norm of continuous linear maps between the normed vector spaces $(C^\infty(M, CV), | \, |^F_l)$ and $(C^\infty(M, CV), | \, |^F_{l-d})$. Therefore $\sum_{j=1}^\infty \widehat{D}^\epsilon a_j$ converges in the Fréchet topology to $\widehat{D}^\epsilon \left( \sum_{j=1}^\infty a_j \right)$. □

## 4.2 Closed images of elliptic operators on Hilbert bundles

Let $p' : \mathcal{V}' \to M$ and $p'' : \mathcal{V}'' \to M$ be infinite rank Hilbert fibre bundles on a compact manifold $M$ with fibres the separable Hilbert spaces $(V', h')$ and $(V'', h'')$, respectively. Let $D : \Gamma^\infty(\mathcal{V}') \to \Gamma^\infty(\mathcal{V}'')$ be a pseudodifferential operator. We consider the pseudodifferential operator $D$ in a global smooth trivialization as described in the Section 3, part Trivializing construction. Thus we have a map $\widetilde{D} : C^\infty(M, V') \to C^\infty(M, V'')$ defined by $\widetilde{D} = \alpha'' \circ D \circ {\alpha'}^{-1}$, where $\alpha'$ and $\alpha''$ are defined in the Trivializing construction. For our purpose, we may identify $D$ with $\widetilde{D}$ and consider that $D : C^\infty(M, V') \to C^\infty(M, V'')$. Let us notice that in the case of the pre-Hilbert topologies on the spaces of smooth maps, neither $D$ nor $\widetilde{D}$ have to be continuous. Since we shall investigate elliptic operators, we suppose that $(V', h')$ and $(V'', h'')$ are linearly homeomorphic. By polar decomposition for continuous linear maps of Hilbert spaces, the Hilbert spaces $(V', h')$ and $(V'', h'')$ are also unitarilly isomorphic. We identify these spaces and denote them by $(V, h)$. We keep denoting the differential operator by the same symbol, i.e., we consider $D : C^\infty(M, V) \to C^\infty(M, V)$.

Let us recall that we have the operator $D^\epsilon = D \otimes \mathrm{Id}_{V*} : C^\infty(M, V) \otimes V^* \to C^\infty(M, V) \otimes V^*$ and its continuous extension

$$\widehat{D}^\epsilon : C^\infty(M, V)\widehat{\otimes}_\epsilon V^* \to C^\infty(M, V)\widehat{\otimes}_\epsilon V^*$$

to the injectively completed tensor product at our disposal. This operator is continuous as a map on $C^\infty(M, CV)$ considered with the Fréchet topology. Operator $\widehat{D}^\epsilon$ is $CV$-linear by Lemma 8, $CV$-pseudodifferential, $CV$-elliptic and its image is closed in the pre-Hilbert topology (Theorem 9).

For each $l \geqslant 0$, the space $(C^\infty(M, CV), |\,|_l^F)$ is a normed abelian group with respect to the point-wise addition of vector-valued maps and thus, by Antosik [1], it satisfies the so-called "FLYUS" convergence conditions of [1], p. 369. Consequently, the Theorem 2 in [1] can be used for this space, which is a normed abelian group with a convergence structure from the point of view of [1].

**Scholium 3** (limit and sum interchange): Let us consider a double sequence of smooth maps $(\phi_{ij})_{i,j\in\mathbb{N}} \subseteq C^\infty(M, V)$, and suppose that for each $i \in \mathbb{N}$ the series $L_i = \sum_{j=1}^\infty \phi_{ij} \otimes_\epsilon \epsilon^j$ converges, and that the limit $L = \lim_i L_i$ of the sequence $(L_i)_{i\in\mathbb{N}}$ exists in $C^\infty(M, V)\widehat{\otimes}_\epsilon V^* (\cong C^\infty(M, CV))$ with respect to the Fréchet topology. Then $\sum_{j=1}^\infty \lim_i \phi_{ij} \otimes_\epsilon \epsilon^j$ exists with respect to this topology and it is equal to $L$.

*Proof.* For determining the topology on smooth sections of $CV$-valued maps on $M$, let $g$ be a Riemannian metric on $M$ and $\nabla'$ be a covariant derivative on the product bundle $\underline{V}_M = M \times V \to M$, e.g., the one defined by the Cartesian product structure of this bundle. For each vector field $X$ on $M$, let us consider the operator $\nabla_X$ on $\Gamma^\infty(\underline{CV}_M) \cong C^\infty(M, CV) \cong C^\infty(M, V)\widehat{\otimes}_\epsilon V^*$ defined by $\nabla_X = \nabla'_X\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$. It is easy to see that $\nabla$ is a covariant derivative on $\underline{CV}_M$. We suppose that $g$ and $\nabla$ determine the Fréchet norms on $C^\infty(M, CV) \cong C^\infty(M, V)\widehat{\otimes}_\epsilon V^*$, described in the Section 3.

Let $l$ be an arbitrary non-negative integer. By Theorem 2 in [1], the series $\sum_{j=1}^\infty \lim_i \phi_{ij} \otimes \epsilon^j$ converges with respect to the norm $|\,|_l^F$ and equals to $L$ if a) $\lim_i \phi_{ij} \otimes \epsilon^j$ exists for all $j$, b) $\lim_j \phi_{ij} \otimes \epsilon^j$ exists for all $i$, and c) the series $L_i = \sum_{j=1}^\infty \phi_{ij} \otimes \epsilon^j$ is subseries convergent in $(C^\infty(M, CV), |\,|_l^F)$ for all $i$. We verify the conditions a), b) and c).

a) Recall that $(\mathrm{ev}_v f)(m) = (f(m))(v)$, where $m \in M$ and $v \in V$. As realized in the item b) of the proof of the Lemma 10, $\mathrm{ev}_v : C^\infty(M, CV) \to C^\infty(M, V)$ is continuous with respect to the Fréchet topologies for each $v \in V$. Thus we have

$$\begin{aligned}
\mathrm{ev}_{e_j} L &= \mathrm{ev}_{e_j} \left( \lim_i \sum_{k=1}^{\infty} \phi_{ik} \otimes \epsilon^k \right) = \lim_i \sum_{k=1}^{\infty} \mathrm{ev}_{e_j} (\phi_{ik} \otimes \epsilon^k) \\
&= \lim_i \sum_{k=1}^{\infty} \phi_{ik} \mathrm{ev}_{e_j}(\epsilon^k) = \lim_i \sum_{k=1}^{\infty} \phi_{ik} \delta_{kj} = \lim_i \phi_{ij}.
\end{aligned}$$

Consequently, $\lim_i (\phi_{ij} \otimes \epsilon^j) = (\lim_i \phi_{ij}) \otimes \epsilon^j = (\mathrm{ev}_{e_j} L) \otimes \epsilon^j$ exists for all $j$.

b) For the limit with respect to $j$, it is sufficient to realize that $\phi_{ij} \otimes \epsilon^j = \sum_{k=1}^{j} \phi_{ik} \otimes \epsilon^k - \sum_{k=1}^{j-1} \phi_{ik} \otimes \epsilon^k$ and that if $j \to \infty$, both of the sums at the right-hand side converge by the assumption. Thus the limit of $\phi_{ij} \otimes \epsilon^j$ with respect to $j$ is zero.

c) In the rest of the proof, we verify the condition on the subseries convergence. Let us consider an increasing sequence of positive integers $\nu : \mathbb{N} \to \mathbb{N}$, with help we choose a subseries. For each $i \in \mathbb{N}$, let us set $L_i^\nu(k,r) = \sum_{j=k}^{r} \phi_{i\nu(j)} \otimes \epsilon^{\nu(j)}$ and $L_i(k,r) = L_i^{\mathrm{Id}}(k,r) = \sum_{j=k}^{r} \phi_{ij} \otimes \epsilon^j$, where Id denotes the identity sequence $\mathrm{Id}(j) = j$, $j \in \mathbb{N}$, and $1 \leqslant k < r$ are arbitrary integers. We prove that for each $l \in \mathbb{N}_0$, $|L_i^\nu(k,r)|_l^F \leqslant |L_i(k,r')|_l^F$ for an integer $r' \geqslant k$ that may depend on $l$.

c.i) First, let us suppose that $l = 0$. For a fixed $i \in \mathbb{N}$, we define $\Psi^\nu(m,k,r) = \sum_{j=k}^{r} \phi_{i\nu(j)}(m) \otimes \epsilon^{\nu(j)}$ and $\Psi(m,k,r) = \Psi^{\mathrm{Id}}(m,k,r)$, where $m \in M$. Let us set $P^\nu(m,k,r) = \sum_{j=k}^{r} e_{\nu(j)} \otimes \epsilon^{\nu(j)}$, which is a map on $M$ whose values are orthogonal-projections in $V$. For each $m \in M$, $P^\nu(m,k,r)$ is of finite rank, thus it is an element of $CV$. It is easy to verify that

$$\Psi^\nu(m,k,r) = \Psi(m,k,\nu(r)) \circ P^\nu(m,k,r).$$

Since $|\,|_{CV}$ is submultiplicative and $|P^\nu(m,k,r)|_{CV} = 1$ for each $m \in M$, we have $|\Psi^\nu(m,k,r)|_{CV} \leqslant |\Psi(m,k,\nu(r))|_{CV}$. Taking the supremum over $m \in M$ of both the sides of this inequality, we get for each $i \in \mathbb{N}$ that

$$|L_i^\nu(k,r)|_0^F \leqslant |L_i(k,\nu(r))|_0^F \tag{3}$$

c.ii) Let $l \geqslant 1$ and let us consider local unit length tangent vector fields $X_i$ on subsets of $M$, $g(X_i,X_i) = 1$, $i = 1,\ldots,l$. For deriving the appropriate estimates, we replace the $CV$-valued maps $\phi_{ij} \otimes \epsilon^j$ in the item c.i) above by the $CV$-valued maps $\nabla_{X_1} \ldots \nabla_{X_l} (\phi_{ij} \otimes \epsilon^j) = (\nabla'_{X_1} \ldots \nabla'_{X_l} \phi_{ij}) \otimes \epsilon^j$. The inequality (3) transforms into

$$|L_i^\nu(k,r)|_l^F \leqslant |L_i(k,\nu(r))|_l^F \tag{4}$$

because $P^\nu(-,k,r)$ is constant for each fixed $k$ and $r \in \mathbb{N}$, i.e., with respect to $M$. Thus we see that it is sufficient to consider $r' = \nu(r)$.

Using the derived inequalities (3) and (4), we show routinely that $L_i$ is subseries convergent for each $i \in \mathbb{N}$. Let us recall that $C^\infty(M, CV)$ with the Fréchet topology is Cauchy complete. Since for each $i$, the original series $L_i$ is convergent in $C^\infty(M, CV)$ by the assumption, it is Cauchy with respect to $|\,|_{l'}^F$ for each $l' \geqslant 0$. Thus for each $\epsilon$, there is $k_0 \geqslant 0$ (dependent possibly on $i$ and $l'$) such that for all $k \geqslant k_0$ and all $p' \geqslant 0$ we have $|L_i(k, k+p')|_{l'}^F = |\sum_{j=k}^{k+p'} \phi_{ij} \otimes \epsilon^j|_{l'}^F < \epsilon$. Let $k \geqslant k_0$ and let us consider an arbitrary $p \geqslant 0$. By inequalities (3) and (4), we obtain $|L_i^\nu(k, k+p)|_{l'}^F \leqslant |L_i(k, k+(\nu(k+p)-k))|_{l'}^F$. Thus taking $p' = \nu(k+p)-k$ in the inequality $|L_i(k, k+p')|_{l'}^F < \epsilon$, we get $|L_i^\nu(k, k+p)|_{l'}^F < \epsilon$ for all $p \geqslant 0$. This shows that for each increasing $\nu$, the sequence $(L_i^\nu)_{i \in \mathbb{N}}$ is Cauchy with respect to $|\,|_{l'}^F$ for all $l'$. This means that $(L_i^\nu)_{i\in\mathbb{N}}$ is Cauchy in the space $C^\infty(M, CV)$ equipped with the Fréchet topology, and thus convergent. Consequently $(L_i^\nu)_{i\in\mathbb{N}}$ is convergent in the chosen $(C^\infty(M, CV), |\,|_l^F)$, and thus $(L_i)_{i\in\mathbb{N}}$ is subseries convergent in this normed space.

In particular the assumptions of the Theorem 2 in Antosik [1] are satisfied and $\sum_{j=1}^\infty \lim_i \phi_{ij} \otimes \epsilon^j \to L$ in $(C^\infty(M, CV), |\,|_l^F)$ if $i \to \infty$. Since $l$ is arbitrary, the sequence $(L_i)_{i\in\mathbb{N}}$ converges to $L$ with respect to the Fréchet topology on $C^\infty(M, CV)$ as well. $\square$

**Remark:** 1) Theorem 2 of [1] used in the above proof is a generalization of a theorem on a sum and limit interchange of Schur. See Pap et al. [41]. It relies on the so-called Antosik–Mikusiński basic matrix theorem. See [2]. Let us remark that pages 371 and 372 should be swapped and renumbered in Antosik [1].

2) It is not difficult to see that each of the series $L_i = \sum_{j=1}^\infty \frac{e_i}{ij} \otimes \epsilon^j = \frac{e_i}{i} \otimes \sum_{j=1}^\infty \frac{\epsilon^j}{j} \in CV$, $i \geqslant 1$, is subseries convergent and that $\lim_i \frac{e_i}{ij} \otimes \epsilon^j = 0$ for each $j$. Thus $\lim_i L_i = 0$ by the above Scholium. In this case, we cannot apply dominant convergence criteria since the series are even not absolutely convergent.

Now we prove the main result, i.e., the closed image property of elliptic operators on Hilbert bundles on compact manifolds. We derive it from the closed image property of $\widehat{D}^\epsilon$ proved in Theorem 9. Although fibres are supposed to be separable Hilbert spaces in our article (Preamble, item d)), we recall this assumption in the theorem.

**Theorem 11:** Let $D : \Gamma^\infty(\mathcal{V}') \to \Gamma^\infty(\mathcal{V}'')$ be an elliptic operator on sections of infinite rank separable Hilbert bundles $\mathcal{V}'$ and $\mathcal{V}''$ on a compact manifold $M$. Then the image of $D$ is closed in $\Gamma^\infty(\mathcal{V}'')$ with respect to the pre-Hilbert topology. Moreover, we have the orthogonal decompositions $\Gamma^\infty(\mathcal{V}'') = \operatorname{Ker} D^* \oplus \operatorname{Im} D$ and $\Gamma^\infty(\mathcal{V}') = \operatorname{Ker} D \oplus \operatorname{Im} D^*$ with respect to the pre-Hilbert topology.

*Proof.* Since the symbol of $D$ is a linear homeomorphism for any non-zero cotangent vector on $M$, we identify the fibre $V'$ of $\mathcal{V}'$ with the fibre $V''$ of $\mathcal{V}''$ and denote them by $V$. The operator $D$ is considered as a map of $C^\infty(M, V)$ into $C^\infty(M, V)$ as explained at the beginning of this section.

1) Let $(g_i)_{i\in\mathbb{N}} \subseteq C^\infty(M,V)$ be a sequence in the image of $D$ that converges to an element $g \in C^\infty(M,V)$ with respect to the pre-Hilbert topology, induced by an appropriate measure denoted by $\mu$. Let $f_i$ be an element in the $D$-preimage of $g_i$, i.e., for each $i \in \mathbb{N}$, $Df_i = g_i$. Since $\widehat{D}^\epsilon(f_i \otimes \epsilon^1) = Df_i \otimes \epsilon^1 = g_i \otimes \epsilon^1$, the sequence $(g_i \otimes \epsilon^1)_i$ is in $\operatorname{Im}\widehat{D}^\epsilon$. By the definition of the Bochner integral by step functions, we get that $\int_M (g_i \otimes \epsilon^1 - g \otimes \epsilon^1)\mathrm{d}\mu = \left(\int_M (g_i - g)\mathrm{d}\mu\right) \otimes \epsilon^1$ which tends to 0 if $i \to \infty$ since $g_i \to g$ in the pre-Hilbert topology. Consequently $g_i \otimes \epsilon^1$ converges to $g \otimes \epsilon^1$.

Since the image of $\widehat{D}^\epsilon$ is closed in $C^\infty(M,CV)$ with respect to the pre-Hilbert topology (Theorem 9), $g \otimes \epsilon^1 \in \operatorname{Im}\widehat{D}^\epsilon$. Let us choose a $\widehat{D}^\epsilon$-preimage of $g \otimes \epsilon^1$ and denote it by $\widehat{f}'$. Thus

$$\widehat{D}^\epsilon \widehat{f}' = g \otimes \epsilon^1 \tag{5}$$

As an equivalence class in $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$, the element $\widehat{f}'$ is represented by a Cauchy sequence $(f_i')_{i\in\mathbb{N}} \subseteq C^\infty(M,V) \otimes_\epsilon V^*$, where $C^\infty(M,V)$ is considered with the Fréchet topology. By Lemma 10, for each $i \in \mathbb{N}$ there exists a family of smooth functions $(\phi_{ij})_{j\in\mathbb{N}} \subseteq C^\infty(M,V)$ such that $f_i' = \sum_{j=1}^\infty \phi_{ij} \otimes \epsilon^j$ with respect to the Fréchet topology on $C^\infty(M,CV)$.

Using the continuity of $\widehat{D}^\epsilon$ with respect to the Fréchet topology, we have $\widehat{D}^\epsilon \widehat{f}' = \widehat{D}^\epsilon[(f_i')_i] = [(D^\epsilon f_i')_i] = [(D^\epsilon(\sum_{j=1}^\infty \phi_{ij} \otimes \epsilon^j))_i] = [(\sum_{j=1}^\infty D\phi_{ij} \otimes \epsilon^j)_i]$, where the last equality follows from Scholium 2. Comparing this result with (5), we obtain that the sequence $(\sum_{j=1}^\infty D\phi_{ij} \otimes \epsilon^j)_{i\in\mathbb{N}}$ differs from the constant sequence $(g \otimes \epsilon^1)_i$ by a null-sequence in the Fréchet topology, i.e., $g \otimes \epsilon^1 = \lim_i \left(\sum_{j=1}^\infty D\phi_{ij} \otimes \epsilon^j\right)$. By Scholium 3 on the sum and limit interchange, we get that $g \otimes \epsilon^1 = \sum_{j=1}^\infty (\lim_i D\phi_{ij}) \otimes \epsilon^j$. Since $(\epsilon^i)_i$ is a Hilbert basis of $V^*$, we obtain from the last equality that $\lim_i D\phi_{ij} = 0$ for all $j \in \mathbb{N}\backslash\{1\}$. The existence of $\lim_i \phi_{i1}$ is due to the Lemma 10. Setting $\phi = \lim_i \phi_{i1}$, we get

$$\begin{aligned} g \otimes \epsilon^1 &= \sum_{j=1}^\infty (\lim_i D\phi_{ij}) \otimes \epsilon^j = \lim_i (D\phi_{i1}) \otimes \epsilon^1 \\ &= (D \lim_i \phi_{i1}) \otimes \epsilon^1 = D\phi \otimes \epsilon^1 \end{aligned}$$

by the continuity of $D$ with respect to the Fréchet topology on $C^\infty(M,V)$. Consequently, we have $g = D\phi$ and especially $g \in \operatorname{Im} D$. Thus we find that the limit point of a sequence in the image of $D$ converging in the pre-Hilbert topology on $C^\infty(M,V)$ belongs to $\operatorname{Im} D$, which means that the image of $D$ is closed in $C^\infty(M,V)$ with respect to the pre-Hilbert topology.

2) We prove the part of the assertion on the complementability of the images.

a) For Corollary 3, we consider that $A$ is the $C^*$-algebra of complex numbers $\mathbb{C}$, $X = H^{2d}(M,V)$, $Y = H^0(M,V)$, $Z = C^\infty(M,V)$, $X' = H^{2d}(M,V)$, $Y' = H^0(M,V)$, and $Z' = C^\infty(M,V)$, where $d$ denotes the order of $D$. Since $D : Z \to Z'$ is a $\mathbb{C}$-pseudodifferential operator, it is adjointable by the Lemma 5 and thus we have the morphisms $\Delta = D^*D : Z \to Z$ and $\Delta' = DD^* : Z' \to Z'$ of the pre-Hilbert $\mathbb{C}$-modules of smooth sections of the appropriate bundles, both of which are self-adjoint as maps of these inner product spaces. Let $U$ be a non-empty open subset of $\operatorname{Im} \Delta_{2d}$ where $d$ is the order of $D$. Since $\Delta_{2d} : H^{2d}(M,V) \to H^0(M,V)$ is continuous, the non-empty set $\Delta_{2d}^{-1}(U)$ is an open subset of $H^{2d}(M,V)$. Since $C^\infty(M,V)$ is dense in $H^0(M,V)$, the set $\Delta_{2d}^{-1}(U) \cap C^\infty(M,V) \neq \varnothing$. Taking an element $g$ in this intersection, we get that $\Delta_{2d} g = \Delta g \in U \cap \operatorname{Im} D$, and thus $U \cap \operatorname{Im} \Delta \neq \varnothing$. Consequently $\operatorname{Im} \Delta$ is dense in $\operatorname{Im} \Delta_{2d}$.

b) Let $(f_i)_{i\in\mathbb{N}}$ is an arbitrary sequence in $\operatorname{Im} \Delta_{2d}$ converging to the element $f$ in $H^0(M,V)$. Thus for any $\epsilon > 0$ there is an integer $i_0$, such that $|f_i - f| < \epsilon$ for all $i \geqslant i_0$. Since $\operatorname{Im} \Delta$ is dense in $\operatorname{Im} \Delta_{2d}$, for each $i \geqslant 1$ there exists a sequence $(f_i^j)_{j\in\mathbb{N}} \subseteq \operatorname{Im} D$ and an integer $j_0(i) \in \mathbb{N}$ such that $|f_i^j - f_i|_0^S < \epsilon/2^i$ for all $j \geqslant j_0(i)$. Setting $\widetilde{f_i} = f_i^{j_0(i)}$ for $i \geqslant 1$, we obtain a sequence $(\widetilde{f_i})_{i\in\mathbb{N}}$ in $\operatorname{Im} \Delta$. For all $i \geqslant i_0$ we have $|\widetilde{f_i} - f|_0^S = |\widetilde{f_i} - f_i + f_i - f|_0^S \leqslant |\widetilde{f_i} - f_i|_0^S + |f_i - f|_0^S = |f_i^{j_0(i)} - f_i|_0^S + |f_i - f|_0^S \leqslant \epsilon/2^i + \epsilon \leqslant 2\epsilon$. Consequently $(\widetilde{f_i})_{i\in\mathbb{N}}$ converges to $f$ in the pre-Hilbert topology. Since $(\widetilde{f_i})_{i\in\mathbb{N}} \subseteq \operatorname{Im} \Delta$ and $\operatorname{Im} \Delta$ is closed in $C^\infty(M,V)$ by the item 1) of this proof, $f$ belongs to $\operatorname{Im} \Delta$ and we may write $f = \Delta g$ for an element $g$ in $C^\infty(M,V)$. Therefore $f = \Delta_{2d} g$ and thus $f$ is in the image of $\Delta_{2d}$ proving that the image of $\Delta_{2d}$ is closed. Similarly, we proceed in the case of $\Delta'$.

c) Operators $\Delta$ and $\Delta'$ are self-adjoint. They are elliptic since the symbol of the composition is the composition of the symbols by [56] and since $D^*$ is elliptic by Lemma 5. By the Theorem 6, they are regular. Moreover, $(\Delta_{2d})^*{}_{|C^\infty(M,V)} = \Delta$ by the same arguments as in the proof of the Theorem 9 which we use for the self-dual fibre $V$ and $A = \mathbb{C}$. Similarly we get $(\Delta'_{2d})^*{}_{|C^\infty(M,V)} = \Delta'$. Since $\Delta$ and $\Delta'$ are regular, $(\Delta_{2d})^*$ and $(\Delta'_{2d})^*$ are regular as well. Therefore the assumptions of the Corollary 3 are satisfied. Using the Corollary, $\Gamma(\mathcal{V}') = \operatorname{Ker} D \bigoplus \operatorname{Im} D^*$ and $\Gamma(\mathcal{V}'') = \operatorname{Ker} D^* \bigoplus \operatorname{Im} D$.

$\square$

*Question:* We ask whether the statement of the Theorem 11 on the closed image property of elliptic operators holds for Hilbert bundles that have non-separable fibres. Let us notice that the separability is not assumed in the results on a smooth trivialization of infinite rank Hilbert bundles of Burghelea and Kuiper [6] and Moulis [40]. However, the basic matrix theorem from Antosik and Swartz [2] is formulated for sequences and not for arbitrary nets.

*Future aim:* It is seems interesting to study consequences of the closed image property for elliptic complexes on Hilbert bundles.

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