Hodge theory for complexes over C^*-algebras with an application to A-ellipticity

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Abstract

For a class of co-chain complexes in the category of pre-Hilbert A-modules, we prove that their cohomology groups equipped with the canonical quotient topology are pre-Hilbert A-modules, and derive the Hodge theory and, in particular, the Hodge decomposition for them. As an application, we show that A-elliptic complexes of pseudodifferential operators acting on sections of finitely generated projective A-Hilbert bundles over compact manifolds belong to this class if the images of the continuous extensions of their associated Laplace operators are closed. Moreover, we prove that the cohomology groups of these complexes share the structure of the fibers, in the sense that they are also finitely generated projective Hilbert A-modules.

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1 Introduction

The Hodge theory is known to hold for any co-chain complex in the category of finite dimensional vector spaces and linear maps. This theory holds also for *elliptic complexes* of pseudodifferential operators acting between smooth sections of finite rank vector bundles over compact manifolds. See, e.g., Wells [17] or Palais [12] and the references therein. Let us notice that in this case, the considered co-chain complexes consist of spaces of smooth sections of the bundles, which are infinite dimensional if the manifold contains more than a finite number of points.

Let us remark that in connection with renormalization and regularization of certain quantum theories, Hilbert and Banach bundles of infinite rank enjoy an increasing interest. See, e.g., the papers on stochastical quantum mechanics and parallel transport of Prugovečki [13], Drechsler and Tuckey [3], and on spin foams of Denicola, Marcolli and Zainy al-Yasri [1]. This list of references should

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not be considered as complete. The theory of indices and the K-theory are well established for a class of the so-called A-Hilbert bundles, and especially for the subclass consisting of the finitely generated projective ones. See, e.g., Mishchenko, Fomenko [6] and the monograph of Solovyov, Troitsky [15].

One of the reasons for writing of this paper is to separate features that are important for proving the Hodge theory for an algebraically defined and fairly general class of complexes (specified below) from the ones which are specific for A-elliptic complexes appearing in differential geometry and analysis on manifolds. A further reason is to describe also the topological properties of the Hodge isomorphism.

Recall that for a C^*-algebra A, a pre-Hilbert A-module U is a left module over A that is equipped with a map $(,)_U : U \times U \to A$ which is sesquilinear over A and positive definite in the sense that firstly, for any $u \in U$, the inequality $(u, u)_U \geq 0$ holds in A, and secondly, if $(u, u)_U = 0$, then $u = 0$. Let us notice that the product $(,)_U$ induces a norm $|\,|_U$ on U. A pre-Hilbert A-module is called a Hilbert A-module, if it is complete with respect to the norm $|\,|_U$. Hilbert spaces are particular examples of Hilbert A-modules for $A = \mathbb{C}$. An A-Hilbert bundle is, roughly speaking, a Banach bundle whose fibers are Hilbert A-modules.

Let us consider a co-chain complex $d^\bullet = (C^k, d_k)_{k\in\mathbb{Z}}$, where C^k are pre-Hilbert A-modules and the differentials $d_k : C^k \to C^{k+1}$ are A-linear and continuous maps with respect to the induced norms. We suppose that the differentials are adjointable for to may speak about harmonic and co-exact elements. By a *Hodge theory* for a given complex, we mean the Hodge decomposition and the Hodge isomorphism for this complex. The *Hodge decomposition* is an orthogonal sum decomposition (with respect to of $(,)_{C^k}$) of each pre-Hilbert A-module C^k in the complex into the module of harmonic, the module of exact, and the module of co-exact elements. By a *Hodge isomorphism*, one usually means a linear isomorphism of the vector space of harmonic elements and the appropriate cohomology group. Since the cohomology groups of a complex of pre-Hilbert A-modules may not be finite dimensional, we demand the isomorphism to be a homeomorphism. There is one reason more although connected, why we want the isomorphism to have this additional topological feature. Namely, the cohomology groups are quotients by images of the differentials in the complex. Since the images need not be closed, the cohomology groups need not be Hausdorff spaces. Let us notice that the Hausdorff property is well known to be equivalent to the uniqueness of limits of sequences in the considered space and therefore in physical theories, it seems to be reasonable to demand the "Hausdorffness" on each space of measured quantities.

We prove the Hodge theory for the so-called self-adjoint parametrix possessing complexes of pre-Hilbert A-modules. We start dealing with one operator $L : V \to V$ only and prove that the image, $\mathrm{Im}\, L$, is closed and that the decomposition $V = \mathrm{Ker}\, L \oplus \mathrm{Im}\, L$ (no closure) holds if L is self-adjoint parametrix possessing. An endomorphism $L : V \to V$ is called self-adjoint parametrix possessing if there exist maps $g, p : V \to V$ satisfying $1 = gL + p = Lg + p$, $Lp = 0$ and $p = p^*$. After that we handle the case of complexes. To each complex $d^\bullet = (C^k, d_k)_{k\in\mathbb{N}_0}$ of pre-Hilbert A-modules and adjointable differentials, we assign the sequence of self-adjoint endomorphisms $L_i = d_{i-1}d_{i-1}^* + d_i^* d_i : C^i \to C^i$, $i \in \mathbb{N}_0$, called the associated *Laplace operators*. The complexes with self-adjoint parametrix possessing Laplace operators are called *self-adjoint parametrix possessing*. Under the condition that $(C^k, d_k)_{k\in\mathbb{N}_0}$ is self-adjoint parametrix possessing, we show that $C^i = \operatorname{Ker} L_i \oplus \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1}$ (the Hodge decomposition) and that each cohomology group $H^i(d^\bullet, A)$ of $d^\bullet$ is isomorphic to the space $\operatorname{Ker} L_i$ of harmonic elements as a pre-Hilbert A-module (the Hodge isomorphism). In particular, the cohomologies of a self-adjoint parametrix possessing complex are Hausdorff spaces being homeomorphic to kernels of continuous maps. Using these abstract considerations, we prove that the Hodge theory holds also for complexes $D^\bullet = (\Gamma(F^k), D_k)_{k\in\mathbb{N}_0}$ of the so-called A-elliptic operators acting on smooth sections of finitely generated projective A-Hilbert bundles F^k, under an assumption on the image of extensions of the Laplacians $\triangle_k = D_{k-1}D_{k-1}^* + D_k^* D_k$. Supposing that A is unital, we prove that the cohomology groups of these complexes are finitely generated and projective. Let us notice that the theory of parametrix possessing operators is more general then the theory of A-elliptic operators. We demonstrate this fact by giving an explicit example.

Two properties of C^*-algebras, they share with the complex numbers, appear to be important for proving the Hodge decomposition at the abstract level. Namely, we use that for any non-negative hermitian elements a, b of A, the inequality $|a+b|_A \geq |a|_A$ holds, as well as that $a + b = 0$ implies $a = b = 0$, where $|\,|_A$ denotes the norm in the C^*-algebra A. For these theorems see, e.g., Dixmier [2]. In Krýsl [10], the existence of an A-module isomorphism between the cohomology groups and the space of harmonic elements of the so-called parametrix possessing complexes (Definition 2 in [10]) is proved. However, conditions under which this A-module isomorphism is a homeomorphism are not treated there. Without supposing the self-adjointness, the proof of the existence of this isomorphism as given in [10] is rather intricate. On the contrary, in the present paper, the existence of the isomorphism together with determining its topological character are easy consequences of the Hodge decomposition. Let us notice that A-elliptic complexes are treated also in Troitsky [16] in connection with operator indices and K-theory. In the article of Schick [14], one can find a more geometrically oriented approach to a related subject area (twisted de Rham complexes, connections and curvature). The cohomology groups and their topology are not investigated in the two papers mentioned last.

In the second section, we recall notions related to (pre-)Hilbert modules, and derive several simple properties for projections, orthogonal complementability, and norm topologies on quotients of these modules. Then we prove that for a self-adjoint parametrix possessing endomorphism $L : V \to V$, the decomposition $V = \operatorname{Ker} L \oplus \operatorname{Im} L$ holds (Theorem 3). In the third section, we derive the Hodge decomposition for self-adjoint parametrix possessing complexes (Theorem 5) and the existence of the Hodge isomorphism (Corollary 7). In the fourth section, we recall definitions of A-Hilbert bundles and A-elliptic complexes. In that section, a theorem on the Hodge theory and a specification of the cohomology groups for the mentioned class of A-elliptic complexes is proved (Theorem 8). At the end, we give the example of a self-adjoint parametrix possessing map which is not A-elliptic.

Preamble: All manifolds and bundles (total spaces, base spaces, and bundle projections) are smooth. Base spaces of all bundles are considered to be finite dimensional. The A-pseudodifferential operators are supposed to be of finite order. Further, if an index of a labeled object exceeds its allowed range, it is set to be zero.

2 Parametrix possessing endomorphisms of pre-Hilbert modules

Let A be a unital C^*-algebra. We denote the involution in A, the norm in A, and the partial ordering on hermitian elements in A by *, $|\,|_A$, and $\geq$, respectively.

A *pre-Hilbert A-module* is firstly a complex vector space U on which A acts. We consider that A acts from the left, and denote the action by a dot. Secondly, it has to be equipped with a map $(,)_U : U \times U \to A$ such that for all $a \in A$ and $u, v \in U$, the following relations hold

1) $(a \cdot u, v)_U = a^*(u,v)_U$

2) $(u,v)_U = (v,u)_U^*$

3) $(u,u)_U \geq 0$, and

4) $(u,u)_U = 0$ if and only if $u = 0$.

Any map $(,)_U : U \times U \to A$ with properties 1–4 is called an A-product. If A is the standard normed algebra of complex numbers, properties 3 and 4 are equivalent to the positive definiteness of $(,)_U$. For a pre-Hilbert A-module $(U,(,)_U)$, one defines the norm $|\,|_U : U \to [0,\infty)$ induced by $(,)_U$ by the prescription $U \ni u \mapsto |u|_U = \sqrt{|(u,u)_U|_A}$. By a pre-Hilbert A-submodule U of a pre-Hilbert module V, we mean an A-submodule of V which is also a pre-Hilbert module if equipped with the restriction of the A-product in V to U. In particular, U has to be closed in V with respect to $|\,|_V$. By a *pre-Hilbert A-module homomorphism* L between pre-Hilbert A-modules U and V, we mean an A-linear map, i.e., $L(a \cdot u) = a \cdot L(u)$ for each $a \in A$ and $u \in U$ that is continuous with respect to the norms $|\,|_U$ and $|\,|_V$. We denote the set of pre-Hilbert A-module homomorphisms of U into V by $\mathrm{Hom}_A(U,V)$. As usual, $\mathrm{End}_A(U)$ denotes the space $\mathrm{Hom}_A(U,U)$. An adjoint of a pre-Hilbert A-module homomorphism $L : U \to V$ is a map L^* from V to U satisfying for each $u \in U$ and $v \in V$ the identity $(Lu,v)_V = (u, L^*v)_U$. If the adjoint exists, it is unique, and it is a pre-Hilbert A-module homomorphism as well. See, e.g., Lance [11]. We hope that denoting the adjoint of a homomorphism by the same symbol as the involution in A does not cause a confusion. Quite often in the literature, a pre-Hilbert A-module homomorphism $L : U \to V$ is supposed to be adjointable. We don't follow this convention. Let us notice that when we speak of an A-module, we consider it with its algebraic structure only. Finally, a pre-Hilbert A-module $(U,(,)_U)$ is called a *Hilbert A-module* if it is complete with respect to $|\,|_U$.

Elements $u, v \in U$ are called orthogonal if $(u, v)_U = 0$. When we write a direct sum $V = U \oplus U'$ where U and U' are pre-Hilbert A-submodules of V, we suppose that the summands are mutually orthogonal. For any pre-Hilbert A-submodule U of V, we denote by $U^\perp$ the orthogonal complement of U. It is defined by $U^\perp = \{v \in V \,|(v, u)_V = 0 \text{ for all } u \in U\}$ as one expects. We call U *orthogonally complementable* if there exists a pre-Hilbert A-submodule $U' \subseteq V$ such that $V = U \oplus U'$. It is well known that Hilbert, and consequently pre-Hilbert A-submodules need not be complementable. For it, see, e.g., Lance [11]. It is easy to realize that for any pre-Hilbert A-submodules $U \subseteq V$ of a pre-Hilbert A-module W, the operation of taking the orthogonal complement changes the inclusion sign, i.e.,

$$U^\perp \supseteq V^\perp. \tag{1}$$

An element p in $\mathrm{End}_A(V)$ is called a *projection* if $p^2 = p$. Especially, we do not require a projection to be self-adjoint.

2.1 Complementability and quotients

We start with the following simple observation. Let p be a projection and let us denote the A-submodule $\mathrm{Im}\, p$ by U. For each $z \in U$, there exists $x \in V$ such that $z = px$. Thus, $pz = p^2x$ that implies $pz = px = z$. In other words, if p is a projection onto an A-submodule U, then its restriction to U is the identity on U. Further, if $V = U \oplus U'$ and if we set $p(x_U + x_{U'}) = x_U$, where $x_U \in U$ and $x_{U'} \in U'$, then p is a projection. We call this map a projection onto U along U'. We prove the following simple technical lemma which we will need later.

Lemma 1: Let V be a pre-Hilbert A-module and U be an orthogonally complementable pre-Hilbert A-submodule of V.

1) If $V = U \oplus U'$ holds for a pre-Hilbert A-module U', then $U' = U^\perp$, and the projection p onto U along $U^\perp$ is self-adjoint.

2) If p is a projection in V which is self-adjoint, then $\mathrm{Im}\, p$ is orthogonally complementable by $(\mathrm{Im}\, p)^\perp$ and p is a projection onto $\mathrm{Im}\, p$ along $(\mathrm{Im}\, p)^\perp$. Further, $1 - p$ is a self-adjoint projection onto $(\mathrm{Im}\, p)^\perp$ along $\mathrm{Im}\, p$.

Proof. Because the sum $U \oplus U'$ is orthogonal, $U' \subseteq U^\perp$. Let $x \in U^\perp$ and let us write it according to the decomposition $U \oplus U'$ as $x = x_U + x_{U'}$. We have $(x_U, x_U)_V = (x - x_{U'}, x_U)_V = (x, x_U)_V - (x_{U'}, x_U)_V = 0$ since $x \in U^\perp$ and since U and U' are mutually orthogonal. Thus $x_U = 0$ and consequently, $x \in U'$ which proves the opposite inclusion. Further, for any $x \in V$ and $y = y_U + y_{U'} \in V, y_U \in U, y_{U'} \in U'$, we may write $(px, y)_V = (x_U, y_U + y_{U'})_V = (x_U, y_U)_V = (x, y_U)_V = (x, py)_V$, i.e., p is self-adjoint.

For the second statement, let us set $U = p(V)$ and $U' = (1 - p)(V)$. From $x = px + (x - px)$, which holds for any $x \in V$, we have $V = U + U'$. For $x \in U$ and $y \in U'$, there are $u, v \in V$ such that $x = pu$ and $y = (1 - p)v$. We may write $(x, y)_V = (pu, (1-p)v)_V = (pu, v)_V - (pu, pv)_V = (pu, v)_V - (p^*pu, v)_V =$

$(pu, v)_V - (p^2u, v)_V = 0$. Thus, the above written sum $V = U + U'$ is orthogonal. Due to Lemma 1 item 1, $U' = (\operatorname{Im} p)^\perp$. Since for any $v \in V$, $p(1-p)v = pv - p^2v = pv - pv = 0$, the projection p kills elements from U'. Summing up, p is a projection onto $\operatorname{Im} p$ along $(\operatorname{Im} p)^\perp$. Since $(1-p)^2 = 1 - p - p + p^2 = 1 - p$ and $(1-p)^* = 1 - p^* = 1 - p$, we see that $1 - p$ is a self-adjoint projection. The operator $1 - p$ projects onto U' which equals to $(\operatorname{Im} p)^\perp$ as already mentioned. Further, since $(1-p)pv = pv - p^2v = pv - pv = 0$ for any $v \in V$, $1 - p$ is a projection onto $(\operatorname{Im} p)^\perp$ along $\operatorname{Im} p$. $\Box$

Let us remark that item 1 of the previous lemma expresses the uniqueness for the complements of orthogonally complementable pre-Hilbert A-modules.

Now, we focus our attention to quotients of pre-Hilbert A-modules. Let $U \subseteq V$ be an orthogonally complementable pre-Hilbert A-submodule of a pre-Hilbert A-module V, and p be the projection onto $U^\perp$ along U. When we speak of a quotient V/U, we consider it with the quotient A-module structure, and with the following A-product $(,)_{V/U}$. We set $([u],[v])_{V/U} = (pu, pv)_V$, $u, v \in V$. The map $(,)_{V/U}$ is easily seen to be correctly defined. Firstly, it maps into the set of non-negative elements of A. Secondly, let us suppose that $([u],[u])_{V/U} = 0$ for an element $u \in V$. Then $(pu, pu)_V = 0$ and consequently, $pu = 0$. Thus $u \in U$ and therefore $[u] = 0$ proving that $(,)_{V/U}$ is an A-product. Summing up, in the case of an orthogonally complementable pre-Hilbert A-submodule U of a pre-Hilbert A-module V, we obtain a pre-Hilbert A-module structure on V/U. We shall call this structure the **canonical quotient structure**. However, let notice that for a normed space $(Y, |\,|_Y)$ and its closed subspace X, one usually considers the quotient space Y/X equipped with the norm $|\,|_q : Y/X \to [0, \infty)$ defined by

$$|[y]|_q = \inf\{|y - x|_Y \,|\, x \in X\},$$

where $y \in Y$ and $[y]$ denotes the equivalence class of y in Y/X. We call $|\,|_q$ the quotient norm. It is well known that if Y is a Banach space, the quotient equipped with the quotient norm is a Banach space as well.

The following lemma is often formulated for complementable closed subspaces of Banach spaces. Since we shall need it for pre-Hilbert spaces and in order to stress that the completeness is inessential, we give a detailed proof.

Lemma 2: Let U be an orthogonally complementable pre-Hilbert A-submodule of a pre-Hilbert A-module $(V, (,)_V)$. Then

1) the quotient norm $|\,|_q$ coincides with the norm induced by $(,)_{V/U}$ and

2) V/U and $U^\perp$ are isomorphic as pre-Hilbert A-modules.

Proof. Let $p : V \to V$ be the projection onto $U^\perp$ along U. Then $p' = 1 - p$ is the projection onto U along $U^\perp$ (Lemma 1). For any $v \in V$, we have

$$\begin{aligned} |[v]|_q^2 &= \inf_{u\in U}|v-u|_V^2 \\ &= \inf_{u\in U}|(v-u,v-u)_V|_A \\ &= \inf_{u\in U}|(p'v+pv-u,p'v+pv-u)_V|_A \\ &= \inf_{u\in U}|(p'v-u,p'v+pv-u)_V+(pv,p'v+pv-u)_V|_A \\ &= \inf_{u\in U}|(p'v-u,p'v-u)_V+(pv,pv)_V|_A \\ &= |(pv,pv)_V|_A = |[v]|_{V/U}^2, \end{aligned}$$

where in the second last step, we used the fact that $|a+b|_A \geq |a|_A$ holds for any non-negative elements $a, b \in A$. This is a direct consequence of the well known fact that $\geq$ is compatible with the vector space structure in A. (See, for instance, Dixmier [2], pp. 18.) Thus, the first assertion is proved.

It is easy to check that $\Phi : V/U \to U^\perp$, $\Phi([v]) = pv$, is a well defined A-module homomorphism of V/U into $U^\perp$. Further, let us consider the A-module homomorphism $\Psi : U^\perp \to V/U$ defined by $\Psi(u) = [u]$, $u \in U^\perp$. For any $u \in U^\perp$, we have $\Phi(\Psi(u)) = \Phi([u]) = pu = u$ since p is a projection onto $U^\perp$. For each $[v] \in V/U$, we may write $\Psi(\Phi([v])) = \Psi(pv) = [pv]$. Because the difference of v and pv lies in U, we get $\Psi \circ \Phi = 1_{|V/U}$. Thus, Ψ and Φ are mutually inverse and consequently, V/U and $U^\perp$ are isomorphic as A-modules.

Since the topology generated by $|\,|_q$ and the one generated by $|\,|_{V/U}$ coincide, and since Ψ is the quotient map, Ψ is continuous with respect to the induced norm topologies on $(U^\perp, (,)_V)$ and $(V/U, (,)_{V/U})$. Further, let $N \subseteq U^\perp$ be an open subset of $U^\perp$. Then $p^{-1}(N)$ is an open set because p is continuous with respect to $|\,|_V$ and with respect to the restriction of $|\,|_V$ to $U^\perp$, being a projection of V onto $U^\perp$ (along U). The set of all $[x] \in V/U$ such that $x \in p^{-1}(N)$ is an open subset of V/U as follows from the definition of the quotient topology and the fact that $|\,|_q = |\,|_{V/U}$. Thus, Φ is continuous as well. Summing up, V/U and $U^\perp$ are isomorphic as pre-Hilbert A-modules. $\Box$

Remark 1: Let U be an orthogonally complementable pre-Hilbert A-module of a pre-Hilbert A-module V. Due to Lemma 2, if $(V/U, |\,|_q)$ is a Banach space, then $(V/U, (,)_{V/U})$ is a Hilbert A-module. Further, if V is a Hilbert A-module, then $(V/U, (,)_{V/U})$ is a Hilbert A-module as well.

2.2 Parametrix possessing endomorphisms

Now, we focus our attention to a relationship of the orthogonal complementability of images of pre-Hilbert A-module endomorphisms and the property described in the following definition.

Definition 1: Let L be an endomorphism of a pre-Hilbert module $(V, (,))_V$. We call L *parametrix possessing* if there exist pre-Hilbert A-module endomorphisms $g, p : V \to V$ such that

$$\begin{aligned} 1 &= gL + p \\ 1 &= Lg + p \text{ and} \\ Lp &= 0, \end{aligned}$$

where 1 denotes the identity on V. We call a parametrix possessing map L *self-adjoint parametrix possessing* if L and p are self-adjoint.

Remark 2: The first two equations in Definition 1 will be referred to as the *parametrix equations* (for L). Notice that there exist pre-Hilbert A-module endomorphisms which are not parametrix possessing (see Example 1) and also such for which the maps g and p are not uniquely determined. Homomorphisms with the latter property exist already for finite dimensional Hilbert spaces ($A = \mathbb{C}$). The name 'parametrix' is borrowed from the theory of partial differential equations where the operator g is often called the *Green function.*

In the next theorem, we derive the following splitting property for the self-adjoint parametrix possessing endomorphisms.

Theorem 3: Let $L : V \to V$ be a self-adjoint parametrix possessing endomorphism of a pre-Hilbert A-module $(V, (,)_V)$ with the corresponding maps denoted by g and p. Then

1) p is a projection onto $\operatorname{Ker} L$ along $(\operatorname{Im} p)^\perp$ and

2) $V = \operatorname{Ker} L \oplus \operatorname{Im} L$.

Proof.

1) Composing the first parametrix equation with p from the right and using the third equation from the definition of a parametrix possessing endomorphism, we get that $p^2 = p$, i.e., p is a projection. Restricting $1 = gL+p$ to $\operatorname{Ker} L$, we get $1_{|\operatorname{Ker} L} = p_{|\operatorname{Ker} L}$ which implies that $\operatorname{Ker} L \subseteq \operatorname{Im} p$. Further, $Lp = 0$ forces $\operatorname{Im} p \subseteq \operatorname{Ker} L$. Thus, $\operatorname{Im} p = \operatorname{Ker} L$. Using Lemma 1 item 2, p is a projection onto $\operatorname{Im} p = \operatorname{Ker} L$ along $(\operatorname{Im} p)^\perp$.

2) Since p is a projection onto $\operatorname{Im} p$ along $(\operatorname{Im} p)^\perp$, we have the orthogonal decomposition $V = \operatorname{Im} p \oplus (\operatorname{Im} p)^\perp$. Using the above derived result $\operatorname{Im} p = \operatorname{Ker} L$, we conclude that $V = \operatorname{Im} p \oplus (\operatorname{Im} p)^\perp = \operatorname{Ker} L \oplus (\operatorname{Ker} L)^\perp$. It is thus sufficient to prove the equality

$$(\operatorname{Ker} L)^\perp = \operatorname{Im} L \tag{2}$$

First, we prove that $\operatorname{Im} L \subseteq (\operatorname{Ker} L)^\perp$. Let $y = Lx$ for an element $x \in V$. For any $z \in \operatorname{Ker} L$, we may write $(y, z)_V = (Lx, z)_V = (x, L^* z)_V = (x, Lz)_V = 0$. Thus, $y \in (\operatorname{Ker} L)^\perp$. Now, we prove that $(\operatorname{Ker} L)^\perp \subseteq \operatorname{Im} L$. Let $x \in (\operatorname{Ker} L)^\perp$. Using the second parametrix equation, we obtain $Lgx = (1-p)x = x$ since $1-p$ is a projection onto $(\operatorname{Ker} L)^\perp$ (Lemma 1 item 2). Therefore $x = Lgx \in \operatorname{Im} L$. Summing up, $\operatorname{Im} L = (\operatorname{Ker} L)^\perp$ and the equation $V = \operatorname{Ker} L \oplus \operatorname{Im} L$ follows.

□

Remark 3: Let us notice that due to Theorem 3, the image of a self-adjoint parametrix possessing endomorphism is closed (see also Equation 2).

Example 1: We give an example of a self-adjoint Hilbert A-module endomorphism which is not self-adjoint parametrix possessing. See, e.g., Lance [11] for this example in a bit different context. Let us consider the commutative C^*-algebra $A = C([0,1])$ equipped with the supremum norm and the complex conjugation as the involution. Take $V = A = C([0,1])$ with the action given by the point-wise multiplication, i.e., $(f \cdot g)(x) = f(x)g(x)$, $x \in [0,1]$, $f, g \in A = V$ and the A-product $(f,g) = fg \in A$. The operator $L : C([0,1]) \to C([0,1])$ is given by $(Lf)(x) = xf(x)$, $x \in [0,1]$, $f \in C([0,1])$, It is obviously self-adjoint, and thus adjointable. If L were self-adjoint parametrix possessing, we would get that $\operatorname{Im} p = \operatorname{Ker} L$ according to item 1 in the proof of Theorem 3. The definition $Lf = xf$ implies that $\operatorname{Ker} L = \{f \in V | f = 0 \text{ on } (0,1]\}$. Since V consists of continuous functions, we see that $\operatorname{Ker} L = \{f \in V | f = 0 \text{ on } [0,1]\} = 0 \in V$. Consequently, $\operatorname{Im} p = 0$ and therefore, p is zero. Now, the parametrix equations imply that L is bijective. On the other hand, any non-zero constant function in V is not in the image of L. This is a contradiction. See also Exel [5] for treating a connected matter in the context of (generalized) pseudoinverses.

3 Hodge theory for self-adjoint parametrix possessing complexes

In this section, we focus our attention to co-chain complexes $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ of pre-Hilbert A-modules and adjointable pre-Hilbert A-module homomorphisms, i.e., for each $k \in \mathbb{N}_0$, the morphism $d_k : C^k \to C^{k+1}$ is supposed to be an adjointable pre-Hilbert A-module homomorphism, and $d_{k+1}d_k = 0$. Let us consider the sequence of *Laplace operators* $L_k = d_k^* d_k + d_{k-1} d_{k-1}^*$, $k \in \mathbb{N}_0$, associated to $d^\bullet$. Notice that in concordance with the preamble, L_0 equals $d_0^* d_0$.

Lemma 4: Let $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ be a co-chain complex of pre-Hilbert A-modules and adjointable pre-Hilbert A-module homomorphisms. Then

$$\operatorname{Ker} L_k = \operatorname{Ker} d_k \cap \operatorname{Ker} d_{k-1}^*.$$

Proof. The inclusion $\operatorname{Ker} L_k \supseteq \operatorname{Ker} d_k \cap \operatorname{Ker} d_{k-1}^*$ follows directly from the definition of the Laplace operator L_k. To prove the opposite one, let us consider an element $x \in \operatorname{Ker} L_k$, and let us write $0 = (x, L_k x)_{C^k} = (x, d_k^* d_k x + d_{k-1} d_{k-1}^* x)_{C^k} = (d_k x, d_k x)_{C^{k+1}} + (d_{k-1}^* x, d_{k-1}^* x)_{C^{k-1}}$. It is known that the intersection of the cone of non-negative hermitian elements in A with the opposite cone consists only of the zero element. See, e.g., Dixmier [2], Proposition 1.6.1., pp. 15 and 16. Thus, $(d_k x, d_k x)_{C^{k+1}} = 0$ and $(d_{k-1}^* x, d_{k-1}^* x)_{C^{k-1}} = 0$, and consequently, $d_k x = d_{k-1}^* x = 0$ due to the positive definiteness of the A-products in C^{k+1} and C^{k-1}, respectively. □

As announced earlier, we prove the Hodge theory for complexes introduced in the next definition.

Definition 2: Let $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ be a co-chain complex of pre-Hilbert A-modules and adjointable pre-Hilbert A-module homomorphisms. We call $d^\bullet$ a *parametrix possessing complex* if for each $k \in \mathbb{N}_0$, the associated Laplace operator L_k is a parametrix possessing pre-Hilbert A-module endomorphism of C^k. We call $d^\bullet$ a *self-adjoint parametrix possessing complex* if the operators L_k are self-adjoint parametrix possessing pre-Hilbert A-module endomorphisms for all $k \in \mathbb{N}_0$.

Since we suppose that the differentials are pre-Hilbert A-module homomorphisms, the associated Laplace operators are pre-Hilbert A-module endomorphisms as well. Because the associated Laplace operators L_k are self-adjoint by their definitions, we could have demanded the maps L_k to be parametrix possessing and p_k to be self-adjoint in the previous definition only.

In the next theorem, the "abstract" Hodge decomposition is formulated. We use Theorem 3 in its proof.

Theorem 5: Let $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ be a self-adjoint parametrix possessing complex. Then for any $k \in \mathbb{N}_0$, we have the decomposition

$$C^k = \operatorname{Ker} L_k \oplus \operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1}.$$

Proof.

1) Due to Lemma 4, we have $\operatorname{Ker} L_k \subseteq \operatorname{Ker} d_{k-1}^*$. Therefore using the formulas (1) and (2), we get $(\operatorname{Ker} d_{k-1}^*)^\perp \subseteq (\operatorname{Ker} L_k)^\perp = \operatorname{Im} L_k$. Further, due to Lemma 4 again, we have $\operatorname{Ker} L_k \subseteq \operatorname{Ker} d_k$. Using (1) and (2), we get $(\operatorname{Ker} d_k)^\perp \subseteq (\operatorname{Ker} L_k)^\perp = \operatorname{Im} L_k$. Summing up, $(\operatorname{Ker} d_{k-1}^*)^\perp + (\operatorname{Ker} d_k)^\perp \subseteq \operatorname{Im} L_k$.

2) The inclusion $\operatorname{Im} d_{k-1} \subseteq (\operatorname{Ker} d_{k-1}^*)^\perp$ holds since for any $x \in C^{k-1}$ and $y \in \operatorname{Ker} d_{k-1}^*$, we have $(d_{k-1}x, y)_{C^k} = (x, d_{k-1}^* y)_{C^{k-1}} = 0$. Similarly, $\operatorname{Im} d_k^* \subseteq (\operatorname{Ker} d_k)^\perp$. Combining these two facts with the result of item 1 of this proof, we get $\operatorname{Im} d_{k-1} + \operatorname{Im} d_k^* \subseteq (\operatorname{Ker} d_{k-1}^*)^\perp + (\operatorname{Ker} d_k)^\perp \subseteq \operatorname{Im} L_k$. Now, we show that the sum $\operatorname{Im} d_k^* + \operatorname{Im} d_{k-1}$ is orthogonal. Let us take two elements $d_k^* x$ and $d_{k-1} z$ (for $x \in C^{k+1}$ and $z \in C^{k-1}$) from $\operatorname{Im} d_k^*$ and $\operatorname{Im} d_{k-1}$, respectively. The computation $(d_k^* x, d_{k-1} z)_{C^k} = (x, d_k d_{k-1} z)_{C^{k+1}} = 0$ shows that $\operatorname{Im} d_k^*$ and $\operatorname{Im} d_{k-1}^*$ are mutually orthogonal. Summing up, $\operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1} \subseteq \operatorname{Im} L_k$.

3) It is easy to prove that $\operatorname{Im} L_k \subseteq \operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1}$. Indeed, for any $y \in \operatorname{Im} L_k$, there exists $x \in C^k$ such that $y = L_k x = d_k^* d_k x + d_{k-1} d_{k-1}^* x = d_k^*(d_k x) + d_{k-1}(d_{k-1}^* x) \in \operatorname{Im} d_k^* + \operatorname{Im} d_{k-1}$. This observation together with item 2 proves that $\operatorname{Im} L_k = \operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1}$.

4) Because L_k is a self-adjoint parametrix possessing pre-Hilbert A-module endomorphism of C^k, we get the equality $C^k = \operatorname{Im} L_k \oplus \operatorname{Ker} L_k$ due to Theorem 3. Substituting for $\operatorname{Im} L_k$ from item 3 of this proof, we obtain the decomposition from the statement of the theorem.

□

Remark 4:

1) In item 3 of the proof of the previous theorem, we obtained for a self-adjoint parametrix possessing complex $d^\bullet$ the decomposition

$$\operatorname{Im} L_k = \operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1}.$$

2) Notice that if $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ is a co-chain complex, then its adjoint $(C^{k+1}, d_k^*)_{k \in \mathbb{N}_0}$ is a chain complex as follows from $d_k^* d_{k+1}^* = (d_{k+1} d_k)^*$.

Theorem 6: Let $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ be a self-adjoint parametrix possessing complex. Then for any $k \in \mathbb{N}_0$,

$$\begin{aligned} \operatorname{Ker} d_k &= \operatorname{Ker} L_k \oplus \operatorname{Im} d_{k-1} \text{ and} \\ \operatorname{Ker} d_k^* &= \operatorname{Ker} L_{k+1} \oplus \operatorname{Im} d_{k+1}^*. \end{aligned}$$

Proof. Due to Theorem 5, we know that the sums at the right hand side in both rows are orthogonal.

The inclusion $\operatorname{Ker} L_k \oplus \operatorname{Im} d_{k-1} \subseteq \operatorname{Ker} d_k$ is an immediate consequence of the definition of a co-chain complex and of Lemma 4. To prove the opposite inclusion, let us consider an element $y \in \operatorname{Ker} d_k$. Due to Theorem 5, there exist elements $y_1 \in \operatorname{Ker} L_k$, $y_2 \in \operatorname{Im} d_{k-1}$, and $y_3 \in \operatorname{Im} d_k^*$ such that $y = y_1 + y_2 + y_3$. It is sufficient to prove that $y_3 = 0$. Let $z_3 \in C^{k+1}$ be such that $y_3 = d_k^* z_3$. We have $0 = (d_k y, z_3) = (d_k y_1 + d_k y_2 + d_k y_3, z_3) = (d_k y_3, z_3) = (y_3, d_k^* z_3) = (y_3, y_3)$ which implies $y_3 = 0$. Thus, the first equality follows.

The inclusion $\operatorname{Ker} L_{k+1} \oplus \operatorname{Im} d_{k+1}^* \subseteq \operatorname{Ker} d_k^*$ follows from Lemma 4 and from item 2 of Remark 4. To prove the inclusion $\operatorname{Ker} d_k^* \subseteq \operatorname{Ker} L_{k+1} \oplus \operatorname{Im} d_{k+1}^*$, we proceed similarly as in the previous paragraph. For $y \in \operatorname{Ker} d_k^*$, there exist $y_1 \in \operatorname{Ker} L_{k+1}$, $y_2 \in \operatorname{Im} d_k$, and $y_3 \in \operatorname{Im} d_{k+1}^*$ such that $y = y_1 + y_2 + y_3$ (Theorem 5). Let us consider an element $z_2 \in C^k$ for which $y_2 = d_k z_2$. We have $0 = (d_k^* y, z_2) = (d_k^* y_1 + d_k^* y_2 + d_k^* y_3, z_2) = (d_k^* y_2, z_2) = (y_2, y_2)$. Thus $y_2 = 0$ which proves the equation in the second row. $\square$

Now, for a complex $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ of pre-Hilbert A-modules, we consider the cohomology groups

$$H^i(d^\bullet, A) = \frac{\operatorname{Ker}(d_i : C^i \to C^{i+1})}{\operatorname{Im}(d_{i-1} : C^{i-1} \to C^i)},$$

$i \in \mathbb{N}_0$. Notice that in general, the A-module $Z^i(d^\bullet, A) = \operatorname{Im}(d_{i-1} : C^{i-1} \to C^i)$ of co-boundaries need not be orthogonally complementable or even not a closed subspace of the pre-Hilbert A-module of boundaries $B^i(d^\bullet, A) = \operatorname{Ker} d_i$. Consequently, the appropriate cohomology group need not be a Hausdorff space (with respect to the quotient topology). Nevertheless, for self-adjoint parametrix possessing complexes, we derive the following corollary.

Corollary 7: If $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ is a self-adjoint parametrix possessing complex of pre-Hilbert A-modules, then for each i the cohomology group $H^i(d^\bullet, A)$ and the space $\operatorname{Ker} L_i \subseteq C^i$ are isomorphic as pre-Hilbert A-modules. If $d^\bullet$ is a self-adjoint parametrix possessing complex of Hilbert A-modules, then for each i, the cohomology group $H^i(d^\bullet, A)$ is a Hilbert A-module and in particular, a Banach space.

Proof. Because of Theorem 6, $U = \operatorname{Im} d_{i-1}$ is an orthogonally complementable submodule of $V = \operatorname{Ker} d_i$. Thus we may use Lemma 2 item 2 to conclude that the cohomology group $H^i(d^\bullet, A) = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$ equipped with the canonical quotient structure is a pre-Hilbert A-module isomorphic to the orthogonal complement of $\operatorname{Im} d_{i-1}$ in $\operatorname{Ker} d_i$. This complement equals $\operatorname{Ker} L_i$ thanks to Theorem 6 and the uniqueness for orthogonal complements (Lemma 1 item 1). The second statement follows in the same way using Remark 1. □

Remark 5: The isomorphism $H^i(d^\bullet, A) \cong \operatorname{Ker} L_i$ is the Hodge isomorphism mentioned in the Introduction.

4 Application to A-elliptic complexes

Let M be a finite dimensional manifold and $p : F \to M$ be a Banach bundle over M with a differentiable bundle structure $\mathfrak{S}$. Recall that each Banach bundle has to be equipped with a Banach structure $|| \, || : F \to [0, +\infty)$. As it is standard, we denote the fiber $p^{-1}(m)$ in m by F_m and the restriction of $|| \, ||$ to F_m by $|| \, ||_m$. A Banach structure is a smooth map from F to $\mathbb{R}_0^+$ such that for each $m \in M$, $(F_m, || \, ||_m)$ is a Banach space.

We call a Banach bundle $p : F \to M$ with a differentiable bundle structure $\mathfrak{S}$ an *A-Hilbert bundle* if there exists a Hilbert A-module $(S, (,)_S)$ and a bundle atlas $\mathcal{A}$ in the differentiable bundle structure $\mathfrak{S}$ such that

1) for each $m \in M$, the fiber F_m is equipped with a Hilbert A-product, denoted by $(,)_m$, such that the Banach spaces $(F_m, | \, |_m)$ and $(F_m, || \, ||_m)$ are isomorphic as normed spaces,

2) for each $m \in M$ and each chart $(\phi_U, U) \in \mathcal{A}$, $M \supset U \ni m$, the map $\phi_{U|F_m} : (F_m, (,)_m) \to (S, (,)_S)$ is a Hilbert A-module isomorphism, and

3) the transition maps between all charts in the bundle atlas $\mathcal{A}$ are maps into the group $\operatorname{Aut}_A(S)$ of Hilbert A-module automorphisms of S.

The first condition is set in order the norm $| \, |_m$ varies smoothly with respect to $m \in M$ as the Banach structure $|| \, ||$ has to do due to its definition.

Let us recall that for two bundle charts $\phi_U : p^{-1}(U) \to U \times S$ and $\phi_V : p^{-1}(V) \to V \times S$, their transition map $\phi_{UV} : U \cap V \to \operatorname{Aut}(S)$ (the group of homeomorphisms of $(S, | \, |_S)$) is defined by the formula $(\phi_U \circ \phi_V^{-1})(m, v) = (m, \phi_{UV}(m)v)$, where $m \in U \cap V$ and $v \in S$. A *homomorphism of A-Hilbert bundles* $p_1 : F_1 \to M$ and $p_2 : F_2 \to M$ is a map $R : F_1 \to F_2$ between the total spaces of p_1 and p_2, such that $p_2 \circ R = p_1$ and such that R is a Hilbert A-module homomorphism in each fiber, i.e., for any $m \in M$, $R_{|p_1^{-1}(m)} : (F_1)_m \to (F_2)_m$ is a Hilbert A-module homomorphism. An A-Hilbert bundle is called *finitely generated projective* if the typical fiber, the Hilbert A-module $(S, (,)_S)$, is a finitely generated and projective Hilbert A-module. See, e.g., Solovyov, Troitsky [15] for these notions.

The space $\Gamma(F)$ of smooth sections of an A-Hilbert bundle $p : F \to M$ carries a left A-module structure given by $(a \cdot s)(m) = a \cdot (s(m))$ for $a \in A$, $s \in \Gamma(F)$ and $m \in M$. From now on, let us suppose that M is compact and equipped with a Riemannian metric g. We choose a volume element $|\mathrm{vol}_g|$ on the Riemannian manifold (M, g). For each $t \in \mathbb{N}_0$, one then defines an A-product $(,)_t$ of Sobolev type on $\Gamma(F)$. The Sobolev completion $W^t(F)$ is the completion of the space of smooth sections $\Gamma(F)$ of F with respect to the norm induced by $(,)_t$. The Sobolev completion together with the continuous extension of $(,)_t$ form a Hilbert A-module. See Solovyov, Troitsky [15] or Fomenko, Mishchenko [6] for these constructions. For a different metric or a different choice of the volume element, one may get different Sobolev completions. However, they are isomorphic as Hilbert A-modules (see Schick [14]). By definition, the A-product $(,)_{\Gamma(F)}$ on $\Gamma(F)$ equals to the restriction of the Hilbert A-product $(,)_0$ on $W^0(F)$ to $\Gamma(F)$.

For a definition of an A-pseudodifferential operator we refer to Solovyov, Troitsky [15], pp. 79 and 80. For any A-pseudodifferential operator $D : \Gamma(F_1) \to \Gamma(F_2)$, we have the order $\mathrm{ord}(D) \in \mathbb{Z}$ of D, the adjoint $D^* : \Gamma(F_2) \to \Gamma(F_1)$ of D (Theorem 2.1.37 in [15]), and the continuous extension $D_t : W^t(F_1) \to W^{t-\mathrm{ord}(D)}(F_2)$ of D (Theorem 2.1.60, p. 89 in [15]) at our disposal. Only finite order A-pseudodifferential operators are considered. Note that the adjoint is an A-pseudodifferential operator and a pre-Hilbert A-module homomorphism, and that the continuous extension D_t is a Hilbert A-module homomorphism.

Let us denote the cotangent bundle $T^*M \to M$ by π. For an A-pseudodifferential operator D, one defines the notion of its symbol $\sigma(D) : \pi^*(F_1) \to F_2$. See Solovyov, Troitsky [15] pp. 79 and 80 for a definition which generalizes the classical one. Notice that the cotangent bundle T^*M is considered with the trivial A-Hilbert bundle structure, i.e., we set $a \cdot \alpha_m = \alpha_m$ for each $a \in A$, $\alpha_m \in T^*_m M$, and $m \in M$. It is known that $\sigma(D) : \pi^*(F_1) \to F_2$ is an adjointable A-Hilbert bundle homomorphism.

Let $(p_k : F^k \to M)_{k \in \mathbb{N}_0}$ be a sequence of A-Hilbert bundles over M and let $D^\bullet = (\Gamma(F^k), D_k)_{k \in \mathbb{N}_0}$ be a complex of A-pseudodifferential operators in F^k, i.e., $D_k : \Gamma(F^k) \to \Gamma(F^{k+1})$ is an A-pseudodifferential operator and $D_{k+1} D_k = 0$, $k \in \mathbb{N}_0$. For each $\xi \in T^*M$, the sequence $\sigma^\bullet(\xi) = (F^k, \sigma(D_k)(\xi, -))_{k \in \mathbb{N}_0}$ is easily seen to be a complex in the category of A-Hilbert bundles.

Definition 3: A complex $D^\bullet = (\Gamma(F^k), D_k)_{k \in \mathbb{N}_0}$ of A-pseudodifferential operators in A-Hilbert bundles is called *A-elliptic* if $\sigma^\bullet(\xi)$ is an exact complex in the category of A-Hilbert bundles for each $\xi \in T^*M \setminus \{(m, 0) \in T^*M | \, m \in M\}$, i.e., outside the image of the zero section of T^*M.

In accordance with classical conventions, we denote the Laplace operators L_k associated to a complex $D^\bullet = (\Gamma(F^k), D_k)_{k \in \mathbb{N}_0}$ of A-pseudodifferential operators by $\triangle_k$. Their orders, $\mathrm{ord}(\triangle_k)$, will be denoted by r_k for brevity.

Remark 6:

1) A single A-pseudodifferential operator $D : \Gamma(E) \to \Gamma(F)$ may be considered as the complex
$$0 \to \Gamma(E) \xrightarrow{D} \Gamma(F) \to 0.$$

In this case, the definition of an A-elliptic complex coincides with the definition of an A-elliptic operator as given, e.g., in Solovyov, Troitsky [15].

2) If $D^\bullet$ is an A-elliptic complex, then for each $i \in \mathbb{N}_0$, the Laplace operator $\triangle_i$ is an A-elliptic operator. See Corollary 10 in Krýsl [10] for a proof.

Next, we prove that certain specified A-elliptic complexes are self-adjoint parametrix possessing and that, consequently, the Hodge theory holds for them. We use results from Section 3 and Theorems 8 and 11 from [10] in the proof.

Theorem 8: Let A be a unital C^*-algebra and $D^\bullet = (\Gamma(F^k), D_k)_{k\in\mathbb{N}_0}$ be an A-elliptic complex in finitely generated projective A-Hilbert bundles F^k over a compact manifold M. Let us suppose that for each $k \in \mathbb{N}_0$, the image of the continuous extension $(\triangle_k)_{r_k} : W^{r_k}(F^k) \to W^0(F^k)$ of the Laplace operator $\triangle_k$ is closed in $W^0(F^k)$. Then for any $i \in \mathbb{N}_0$

1) $H^i(D^\bullet, A)$ is a finitely generated projective Hilbert A-module isomorphic to $\operatorname{Ker} \triangle_i$ as a Hilbert A-module

2) $\Gamma(F^i) = \operatorname{Ker} \triangle_i \oplus \operatorname{Im} D_i \oplus \operatorname{Im} D^*_{i-1}$

3) $\operatorname{Ker} D_i = \operatorname{Ker} \triangle_i \oplus \operatorname{Im} D^*_i$, and

4) $\operatorname{Ker} D^*_i = \operatorname{Ker} \triangle_{i+1} \oplus \operatorname{Im} D_i$.

Proof. For a self-adjoint A-elliptic operator $K : \Gamma(F) \to \Gamma(F)$ of order r such that $\operatorname{Im} K_r$ is closed in $W^0(F)$, two maps denoted by G and P are constructed in the proof of Theorem 8 in Krýsl [10]. They satisfy the parametrix equations (for K) and the equation $KP = 0$. In the terminology of the current paper, K is a parametrix possessing pre-Hilbert A-module endomorphism of the pre-Hilbert A-module $(\Gamma(F), (,)_{\Gamma(F)})$. The construction of P goes as follows. For $K_r : W^r(F) \to W^0(F)$, one considers the adjoint $(K_r)^* : W^0(F) \to W^r(F)$ and the projection $p_{\operatorname{Ker}(K_r)^*}$ of $W^0(F)$ onto the kernel $\operatorname{Ker}(K_r)^*$ along the closed Hilbert A-module $\operatorname{Im} K_r$. Thus, according to Lemma 1 item 2, the projection $p_{\operatorname{Ker}(K_r)^*}$ is self-adjoint. The operator P is defined as the restriction of $p_{\operatorname{Ker}(K_r)^*}$ to $\Gamma(F) \subseteq W^0(F)$. Restricting $p_{\operatorname{Ker}(K_r)^*}$ to $\Gamma(F)$ does not change its property of being an idempotent and keeps the operator self-adjoint because the A-product $(,)_{\Gamma(F)}$ coincides with the restriction of $(,)_0$ to $\Gamma(F)$. Summing up, P is a projection and a self-adjoint pre-Hilbert A-module endomorphism. Since K is supposed to be self-adjoint, it is a self-adjoint parametrix possessing pre-Hilbert A-module endomorphism according to Definition 1.

Now, we prove the theorem. Since $\triangle_i = D_{i-1}D^*_{i-1} + D^*_i D_i$ is self-adjoint and A-elliptic (Remark 6 item 2) and since we suppose that $\operatorname{Im}(\triangle_i)_{r_i}$ is closed in $W^0(F^i)$, we may use the conclusion of the previous paragraph for $K = \triangle_i$, $F = F^i$ and $r = r_i$. Thus, $\triangle_i$ is a self-adjoint parametrix possessing pre-Hilbert A-module endomorphism. Consequently, $D^\bullet$ is a self-adjoint parametrix possessing complex (Definition 2). Using Theorems 5 and 6, one obtains the statements in parts 2, 3 and 4.

Due to Corollary 7, the cohomology group $H^i(D^\bullet, A)$ is a pre-Hilbert A-module isomorphic to the kernel of the Laplace operator $\triangle_i$. According to Theorem 11 in [10], $H^i(D^\bullet, A)$ is a finitely generated A-module and a Banach space (with respect to the quotient norm $|\,|_q$). Consequently (Remark 1), $H^i(D^\bullet, A)$ equipped with the canonical quotient structure is a Hilbert A-module. It is known that a finitely generated Hilbert A-module over a unital C^*-algebra is projective. For it see Theorem 5.9 in Frank, Larson [7]. Thus, also item 1 is proved. $\Box$

Remark 7: Notice that the decompositions and the adjoints of the maps contained in items 2, 3 and 4 of the previous theorem are meant with respect to the A-product $(,)_{\Gamma(F^i)}$ on the pre-Hilbert A-module $\Gamma(F^i)$. Instead for pre-Hilbert modules we could have formulated Sections 2 and 3 for Hilbert A-modules only and then derive a theorem parallel to Theorem 8 for the spaces $W^0(F^k)$ and for the appropriate "L^2-cohomology" groups.

Remark 8: Let us remark that there are holomorphic Banach bundles whose Čech cohomology groups are known to be non-Hausdorff. See Erat [4]. We should mention that the fact that the Čech cohomology groups are considered in that text makes the situation different from the case of cohomology of complexes which we study.

In the future, we would like to find a convenient class of Hilbert A-modules and A-pseudodifferential operators for which the condition on the image of (the extension of) $\triangle_k$ in Theorem 8 is automatically satisfied.

Remark 9: Non-elliptic and parametrix possessing operator

In this example we show that the notion of a self-adjoint parametrix possessing operator is more general than the one of an A-elliptic operator. (We will not always indicate that we speak about homomorphisms or endomorphisms of Hilbert A-modules and omit the expression "Hilbert A-module".) Let U be an infinite dimensional separable Hilbert space considered as a Hilbert A-module for $A = \mathbb{C}$ and let $l : U \to U$ be the orthogonal projection onto a finite dimensional subspace V of U. For a compact manifold M, we consider the trivial A-Hilbert bundle $q : \mathcal{U} = M \times U \to M$. The projection l can be lifted to the operator L in the space of smooth sections $\Gamma(\mathcal{U}) : L(s)(m) = (m, l(s(m)))$, where $s \in \Gamma(\mathcal{U})$ and $m \in M$. It is of order zero, and thus it equals to its symbol. More precisely, its symbol is the map $\pi^*(\mathcal{U}) \ni (\xi, \tau) \mapsto (q(\tau), l(\mathrm{pr}_2 \tau))$, where $\mathrm{pr}_2 : M \times U \to U$ is the projection onto the second component of the product and $\xi \in T^*_{q(\tau)}M$. This map is obviously not an isomorphism (in any fiber) of $\mathcal{U}$ (out of the zero section of T^*M). We set $g = L$ on $\Gamma(\mathcal{U})$ and $(ps)(m) = (m, (1-l)(s(m)))$. It is trivial to verify that $1 = Lg + p$, $1 = gL + p$, and $p = p^*$.

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