Complexes of Hilbert C^*-modules

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Aim of the talk

Give a generalization of the framework for Hodge theory

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Example

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$\operatorname{Ker} \Delta_i$ harmonic elements

Dagger categories

Dagger category = any category $\mathfrak{C}$ with a *dagger functor* $\dagger$ which is a contravariant and idempotent endofunctor in $\mathfrak{C}$ which is identity on objects (i.e., it preserves objects, reverse direction of morphisms and applied twice it is neutral)

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Hilbert spaces: objects = Hilbert spaces, morphisms = linear continuous maps, and the adjoint of maps as $\dagger$

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Hilbert spaces: objects = Hilbert spaces, morphisms = linear continuous maps, and the adjoint of maps as $\dagger$

bord-n category: objects = smooth oriented n-dimensional manifolds, morphisms between objects M, N = any oriented $(n+1)$-manifold L which bords on the objects M and N, and the orientation reversion of L as $\dagger$

TQFT is a specific functor from the cobord-n categories to the monoidal tensor category over a fixed Hilbert space.

Pseudoinverses on additive dagger categories

Let $\mathfrak{C}$ be an additive category (each set of morphism is an abelian group and compositions are bilinear with respect to the abelian structure) with dagger $\dagger$ (no compatibility necessary)

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We say that a complex $(E^i, d_i)_i$ in $\mathfrak{C}$ is *parametrix possessing* if for $\Delta_i = d_i^\dagger d_i + d_{i-1} d_{i-1}^\dagger$, there are morphisms g_i and p_i in $\mathfrak{C}$ such that $Id_{E^i} = g_i \Delta_i + p_i = \Delta_i g_i + p_i$, $d_i p_i = 0$ and $d_{i-1}^\dagger p_i = 0$.

Theorem (Krysl, Ann. Glob. Anal. Geom.): If $(E^i, d_i)_i$ is a parametrix possessing complex in an additive dagger category, then g_i are morphisms of complexes, i.e., $g_{i+1} d_i = d_i g_i$.

Cohomology of complexes in inner product spaces

Theorem (Krysl, Ann. Glob. Anal. Geom.): Let $E^\bullet = (E^i, d_i)_i$ be a parametrix possessing complex with adjointable differentials in the category of inner product spaces. Then $p_{i+1}d_i = 0$ and the cohomology groups of $E^\bullet$ are in a linear set-bijection with $\operatorname{Ker} \Delta_i$.

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We say nothing about the continuity of this bijection.

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Theorem (Hodge, Fredholm, Weyl): Let $(\mathcal{C}^\infty(E^i), d_i)_i$ be an elliptic complex on finite rank vector bundles $(E^i)_i$ over a compact manifold M. Then $(\mathcal{C}^\infty(E^i), d_i)_i$ satisfies the Hodge theory and the cohomology groups $H^i(E^\bullet)(\simeq \mathrm{Ker}\,\Delta_i \subset \mathcal{C}^\infty(E^i))$ are finite dimensional.

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The bijection between cohomology and harmonic element is homeomorphism

Hilbert C^*-modules

1) $(A, || \, ||, *)$ a C^*-algebra, i.e., A is associative algebra, $|| \, ||$ is a norm and $*$ is an linear idempotent map such that $||aa^*|| = ||a||^2$ and $(A, || \, ||)$ is Cauchy complete

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2) $A^+ = \{a | \, a = a^* \text{ and } \mathrm{sp}(a) \subset [0, \infty)\}$ where $\mathrm{sp}(a) = \{\lambda | \, a - \lambda 1 \text{ does not have inverse}\}$

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3) **Hilbert A-module** $= (R, (,))$ is complex vector space which is a right A-module and $(,) : R \times R \to A$ is A-sesquilinear, hermitian ($(u, v) = (v, u)^*$), positive definite $(v, v) \in A^+$, $(v, v) = 0$ implies $v = 0$ and $(R, | \, |)$ is complete where $|v| = \sqrt{||(v, v)||}$

Closed images of Laplacians implies Hausdorff cohomologies

Hilbert C^*-bundles = Banach bundles with fibers a fixed Hilbert C^*-module R and transition maps to the group of C^*- automorph. of fibers - homeomorphisms T such that $T(ra) = [T(r)]a$

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Theorem (S. Krysl, Ann. Glob. Anal. Geom.): Let $(E^i \to M)_i$ be a sequence of finitely generated projective C^*-bundles over compact manifold M and $(\mathcal{C}^\infty(E^i), d_i)_i$ be an elliptic complex of C^*-operators such that the Laplacians have closed images. Then the Hodge theory holds for the complex and the cohomology groups are finitely generated projective Hilbert C^*-modules.

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The assumption on the closed images seems to be hard to verify in specific cases.

Compact operators

Let $A = CH$ be the algebra of compact operators on a Hilbert space, then extension of the image of Δ_i to Sobolev completions is closed, i.e.,

Theorem (S. Krysl, Jour. Geom. Phys.): Let $(E^i \to M)_i$ be a sequence of finitely generated projective CH-bundles over compact manifold M and $(\mathcal{C}^\infty(E^i), d_i)_i$ be an elliptic complex of C^*-operators. Then the Hodge theory holds for them and the cohomology groups of the complex are finitely generated projective Hilbert C^*-modules. Especially,

1) $\mathcal{C}^\infty(E^i) \simeq \text{Im}\, d_{i-1} \oplus \text{Im}\, d_i^* \oplus \text{Ker}\, \Delta_i$
2) $H^i(E^\bullet) \simeq \text{Ker}\, \Delta_i$ is finitely generated over CH
3) $\text{Im}\, \Delta_i \simeq \text{Im}\, d_i^* \oplus \text{Im}\, d_{i-1}$
4) $\text{Ker}\, d_i \simeq \text{Ker}\, \Delta_i \oplus \text{Im}\, d_{i-1}$ and $\text{Ker}\, d_i^* \simeq \text{Ker}\, \Delta_i \oplus \text{Im}\, d_{i+1}^*$

Underlying result [Bakic, Guljas].

An example from bundles induced by representations

(M^{2n}, ω) real symplectic manifold, $Mp(2n, \mathbb{R})$ double cover of the symplectic group, S the complexified Shale–Weil representation on the Hilbert space $H = L^2(\mathbb{C}^n)$, $\mathcal{S}$ the Shale–Weil bundle induced by S over manifold M

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Global trivialization of $\mathcal{S}$, product connection ∇ on $\mathcal{S}$, $\mathcal{S}^i = \bigwedge^i T^*M \otimes \mathcal{S}$, complex $d^\bullet$ of differential operators (twisted deRham ops). Bundles can be viewed as a Hilbert CH-bundle.

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Theorem (S. Krysl): If M is compact than $d^\bullet$ satisfies the Hodge theory. In particular, $\mathcal{C}^\infty(\mathcal{S}^i) \simeq \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^* \oplus \operatorname{Ker} \Delta_i$ and $H^i(d^\bullet) \simeq \operatorname{Ker} \Delta_i$ Further the cohomological groups are CH-isomorphic to $H^k_{deRham}(M) \otimes H$. In particular their rank equals to the Betti numbers.

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