Twistor operators in symplectic geometry

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March 26, 2021

Abstract

On a symplectic manifold with a symplectic connection and a metaplectic structure, we define two families of sequences of differential operators, the so called symplectic twistor operators. We prove that if the connection is torsion-free and Weyl-flat, the sequences in these families form complexes.

1 Introduction

Twistor operators on Riemannian spin manifolds are often used in mathematical General relativity and differential geometry (see [18], [3], [5]). They are usually introduced using orthogonal local frames on the manifold and the Clifford multiplication, or using a tensor product decompositions of appropriate spin-modules into irreducible submodules.

Weil, who searched for symmetries of theta functions ([26]) and Shale, who searched for symmetries of quantized Klein–Gordon fields ([19]), discovered a unitary representation of the metaplectic group, a Lie group double cover of the symplectic group. This started the development in the symplectic spin geometry. In the seventies of the last century, Kostant [13] defined a metaplectic structure and enabled a research of symplectic spinor fields, which are sections of bundles that are associated to the representation found by Shale and Weil. Sommen [22] studied these structures on Euclidean spaces from the point of view of supersymmetry and Clifford algebras. In global analysis, the metaplectic structures were investigated with the help of symplectic Dirac operators, defined in the work of Habermann [6]. See also [7], [1], [17].

For a Fedosov connection (see Tondeur [23] or Gelfand, Retakh, Shubin [4]), we consider exterior covariant derivatives (see e.g. Kolář, Michor, Slovák [12]) acting on symplectic spinor fields. We use a decomposition of a tensor product into irreducible submodules over the metaplectic group ([14]) to define two families of sequences of differential operators, which we call *symplectic twistor*

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†The author thanks for financial supports from the founding No. 17-01171S granted by the Czech Science Foundation and from the institutional support program “Progres Q47” granted by the Charles University.

operators. From the point of view of representation theory, these operators are similar to the Riemannian twistor operators. They are compositions of appropriate exterior covariant derivatives with projections onto sections of bundles induced by irreducible modules with a specific highest weight.

Note that Dolbeault operators on an almost complex manifold form complexes if the almost complex structure is integrable. We prove that symplectic twistor operators form complexes if the Fedosov connection is Weyl-flat. This is already known for two sequences of symplectic twistor operators (see Krýsl [16]). We generalize this result by proving that all of the introduced sequences in the two families are complexes under the Weyl-flatness using a formula for the symplectic spinor curvature.

2 Symplectic Spinors

Let (V, ω) be a finite dimensional symplectic vector space over the real numbers. Let us recall that the *symplectic group* $Sp(V)$ of (V, ω) is the Lie subgroup of the general linear group of V consisting of maps preserving ω. Let J be a complex structure on V (linear map satisfying $J^2 = -\mathrm{Id}_V$) such that $g(v, w) = \omega(Jv, w)$, $v, w \in V$, is positive definite. The complex structure determines the unitary group $U(V)$ associated to the triple (V, J, g). From the structure theory of Lie groups, it is known that $U(V)$ is a maximal compact subgroup of $Sp(V)$ (see Knapp [11]). It well known that the homotopy type of $U(V)$ is that of the circle S^1, i.e., the fundamental group of $Sp(V)$ is isomorphic to $\mathbb{Z}$. By the theory of covering spaces, there is a connected two-fold covering of $Sp(V)$. Moreover, for a fixed covering and a choice of an element in the preimage of the neutral element in $Sp(V)$, there is a unique Lie group structure on the covering space so that the covering map is a Lie group homomorphism and the chosen point is the neutral element. The covering space is called the *metaplectic group* and it is denoted by $Mp(V)$. We denote the covering map by λ. It is known that $Mp(V)$ is a non-matrix Lie group, i.e., there is no map of this group, that is a group homomorphism and a topological embedding into the general linear group of a finite dimensional vector space (see [17]).

Let L be a Lagrangian subspace of the symplectic space (V, ω). On L, we consider the norm induced by g. Let us denote the Hilbert space of square Lebesgue integrable complex valued functions on L modulo equal almost everywhere (ae.) by E, and the unitary group of E by $U(E)$. There is a unitary representation (see e.g. Weil [26], Shale [19] or Wallach [25]) $\rho : Mp(V) \to U(E)$ of $Mp(V)$ on E called the oscillator (Shale, Shale–Weil, Segal–Shale–Weil, metaplectic or symplectic spinor) representation. We shall call it the *oscillator representation*, following Howe [9]. The representation (ρ, E) is a Hilbert space direct sum of two irreducible representations, that we denote by E_+ and E_-, where $E_\pm$ are the spaces of even and odd elements in E (considered modulo ae.), respectively.

Remark: Let I be the two-sided ideal generated by elements $v \otimes w - w \otimes v - \omega(v, w)1$, $v, w \in V$, as a two-sided ideal in the tensor algebra $T(V)$ of V. The quotient $T(V)/I$ is called the *symplectic Clifford algebra* of (V, ω) and we denote it by $Cl_s(V)$. Any associative algebra has a Lie algebra structure defined by the commutator. As in the orthogonal case, we have a Lie algebra monomorphism of the Lie algebra $\mathfrak{sp}(V)$ of $Sp(V)$ into $Cl_s(V)$. This monomorphism makes us able to consider $Cl_s(V)$ as a left $\mathfrak{sp}(V)$-submodule of $Cl_s(V)$. It can be proved that the so-called Harish-Chandra $(\mathfrak{g}, K)$-module of (ρ, E) is a left ideal in $Cl_s(V)$. See Habermann, Habermann [8] and Kirillov [10].

Let λ^* denote the representation on V^* which is dual to the representation λ. Representations λ and λ^* are equivalent as follows by considering the equivariant map $T : V \to V^*$ defined by $T(v)(w) = \omega(v, w)$, where $v, w \in V$. We denote the dual representation by λ as well, and we consider V with the norm induced by g and $\bigwedge^i V$ with the norm induced by the so-called (real) Hodge scalar product. The exterior powers of λ are denoted by λ^i, $\lambda^i : Mp(V) \to GL(\bigwedge^i V)$. Further, let us consider the vector spaces $E^i_\pm = \bigwedge^i V \otimes E_\pm$, $i \in \mathbb{N}_0$, with the Hilbert tensor product topology. No completion is necessary since the exterior powers are finite dimensional. Let $GL(E^i)$ denote the set of all linear homeomorphisms of E^i. The tensor product representations $\rho^i_\pm : Mp(V) \to GL(E^i_\pm)$ are defined by $\rho^i_\pm(g)(\alpha \otimes w) = \lambda^i(g)\alpha \otimes \rho(g)w$ for $g \in Mp(V)$, $\alpha \in V$ and $w \in E_\pm$, and by the linear extension to other elements in the tensor product.

Let $2n$ be the dimension of V. For each $i = 0, \ldots, 2n$, the irreducible decomposition of $\rho^i_\pm$ on $E^i_\pm$ is described in [14].

Theorem 1: For $i = 0, \ldots, 2n$, $j_i = 0, \ldots, k_{n,i} = n - |n - i|$, there are topological vector spaces $E^{ij_i}_\pm$ and irreducible representations $\rho^{ij_i}_\pm$ of $Mp(V)$ on $E^{ij_i}_\pm$ such that $\rho^i_\pm$ is is equivalent to the orthogonal sum $\bigoplus_{j_i=0}^{k_{n,i}} \rho^{ij_i}_\pm$.

Notation: Let us set $\rho^i = \rho^i_+ \oplus \rho^i_-$ for a representation on $E^i_+ \oplus E^i_-$.

Remark:

1. Spaces E^{ij} are endowed with the topology inherited by the inclusion $E^{ij} \subseteq E^i$.
2. Each ρ^i is multiplicity-free, i.e., when $(\rho', E'), (\rho'', E'')$ are different irreducible subrepresentations of ρ^i, they are not equivalent (see [14]).
3. Representation $\rho^{ij}_\pm$ is equivalent to $\rho^{i+l,j}_\mp$, $0 \le i \le 2n$, $0 \le i + l \le 2n$ $1 \le j \le \min\{k_{n,i}, k_{n,i+l}\}$ (see [14]).
4. The highest weights of the Harish-Chandra $(\mathfrak{g}, K)$-modules of $\rho^{ij}_\pm : Mp(V) \to GL(E^{ij}_\pm)$ are discribed in Krýsl [14].
5. For $j < 0$ and for $j > k_{n,i}$, we set $\rho^{ij}_\pm = 0$, $\rho^{ij} = 0$, $E^{ij}_\pm = 0$ and $E^{ij} = 0$.

$$\begin{array}{ccccccc}
E^0 & E^1 & E^2 & E^3 & E^4 & E^5 & E^6 \\
E_{00} & E_{10} & E_{20} & E_{30} & E_{40} & E_{50} & E_{60} \\
 & E_{11} & E_{21} & E_{31} & E_{41} & E_{51} & \\
 & & E_{22} & E_{32} & E_{42} & & \\
 & & & E_{33} & & &
\end{array}$$

Pic. 1. Decomposition for $n = 3$.

Notation: For each $i = 0, \ldots, n$, and $j_i = 0, \ldots, k_{n,i}$, let $p_{\pm}^{ij_i}$ denote the unique $Mp(V)$-equivariant projection of $E^i_{\pm}$ onto $E^{ij_i}_{\pm}$. The correctness of the definition of the projections follows from the previous theorem (multiplicity freeness; see the item 2 of the Remark above), Schur lemma for weighted representations (Dixmier [2]) and a globalization theorem for Harish-Chandra modules (see e.g. Schmid [20]).

3 Symplectic twistor operators and Complexes

Let (M, ω) be a symplectic manifold of dimension $2n$, $n \in \mathbb{N}$, and (V, ω_0) be a symplectic vector space over the real numbers of the same dimension. We consider the set of symplectic frames $\mathcal{Q} = \{A : V \to T_mM \,|\, \omega(Av, Aw) = \omega_0(v, w), v, w \in V, m \in M\}$ and the map $p_Q : \mathcal{Q} \to M$ defined by $p_Q(A) = m$ if and only if $A : V \to T_mM$. The topology on $\mathcal{Q}$ is given by considering the so-called frame topology, which is the final topology for the set of the inverses of canonical charts (see Sternberg [21]). It is well known that the canonical charts define also a smooth bundle atlas. Let us consider the right action of $Sp(V)$ on $\mathcal{Q}$ given by the map composition from the right. Then $p_Q : \mathcal{Q} \to M$ is a principal $Sp(V)$-bundle on M. We call a principal $Mp(V)$-bundle $p_P : \mathcal{P} \to M$ on M and a morphism of fibre bundles $\Lambda : \mathcal{P} \to \mathcal{Q}$ a *metaplectic structure* on (M, ω) if and only if $\Lambda(Ag) = \Lambda(A)\lambda(g)$ for any $A \in \mathcal{P}$ and $g \in Mp(V)$. Thus those fibre bundle morphisms Λ are allowed for which the next diagram commutes, where the horizontal arrows represent actions of the appropriate groups.

$$\begin{tikzcd}
\mathcal{P} \times Mp(V) \arrow[r] \arrow[dd, "\Lambda\times\lambda"] & \mathcal{P} \arrow[dr, "\pi_P"] \arrow[dd, "\Lambda"] & \\
 & & M \\
\mathcal{Q} \times Sp(V) \arrow[r] & \mathcal{Q} \arrow[ur, "\pi_Q"'] &
\end{tikzcd}$$

Pic. 2. Metaplectic structure.

Definition: An affine connection ∇ is called a *symplectic connection* if $\nabla\omega = 0$. It is called a *Fedosov connection* if it is symplectic and torsion-free.

Remark: Let us remark that in contrast to the Riemannian connection, there are infinitely many Fedosov connections if $n \geq 1$. Moreover, these connections form an infinite dimensional affine space if $n \geq 1$ (see Gelfand, Retakh, Shubin [4]).

We use ρ^{ij} for defining the associated vector bundles $\mathcal{E}^{ij}_{\pm} = \mathcal{P} \times_{\rho^{ij}} E^{ij}_{\pm} = (\mathcal{P} \times E^{ij}_{\pm})/\simeq$, where $(q, f) \simeq (q', f')$ if and only if there exists an element $g \in Mp(V)$ such that $q' = qg$ and $f' = \rho^{ij}_{\pm}(g^{-1})f$. We set $\mathcal{E}^{ij} = \mathcal{E}^{ij}_{+} \oplus \mathcal{E}^{ij}_{-}$. Bundles $\mathcal{E}^{i}_{\pm}$ and $\mathcal{E}^{i}$ are defined by the appropriate representations. We set also $\mathcal{E}_{\pm} = \mathcal{E}^{00}_{\pm}$ and $\mathcal{E} = \mathcal{E}_{+} \oplus \mathcal{E}_{-}$. Associated bundles are considered with the quotient topology. Elements of $\Gamma(\oplus_{i=0}^{2n} \mathcal{E}^{i})$ are called *symplectic spinors* fields.

Any symplectic connection ∇ defines a principal bundle connection on the principal $Sp(V)-$bundle $\mathcal{Q}$. Let us assume that (M, ω) admits a metaplectic structure $(p_P : \mathcal{P} \to M, \Lambda)$. The principal bundle connection lifts to the principal $Mp(V)$-bundle $\mathcal{P}$ (Habermann, Habermann [8]), and induces a covariant derivative on the associated bundle $\mathcal{E}$. We denote the exterior covariant derivatives, that maps $\Gamma(\mathcal{E}^{i})$ to $\Gamma(\mathcal{E}^{i+1})$, by ∇^{i}. The restriction of ∇^{i} to $\Gamma(\mathcal{E}^{ij})$ is denoted by ∇^{ij}. The operator $R^{i} = \nabla^{i+1}\nabla^{i}$ is the so called $(i$th$)$ *symplectic spinor curvature* and $R = \sum_{i=0}^{2n} R^{i}$ is the total symplectic spinor curvature. It factorizes to a map of $\mathcal{E}^{i}$ into $\mathcal{E}^{i+2}$. We denote its restriction to $\mathcal{E}^{ij}$ by R^{ij}.

The equivariant projections $p^{ij}_{\pm} : E^{i}_{\pm} \to E^{ij}_{\pm}$ induce projections $\mathcal{E}^{i}_{\pm} \to \mathcal{E}^{ij}_{\pm}$ that are bundle morphisms, which further induce appropriate projections $\Gamma(\mathcal{E}^{i}_{\pm}) \to \Gamma(\mathcal{E}^{ij}_{\pm})$ of the section spaces. We denote them by $p^{ij}_{\pm}$ and set $p^{ij} = p^{ij}_{+} + p^{ij}_{-}$ for all instances of the meaning of the symbol $p^{ij}_{\pm}$. The meaning of the symbols for the projections depends on the objects on which they are used. (We hope that this will not cause a confusion.)

Definition: The (i, j)-th *symplectic twistor operators* are are the maps

$$T^{ij}_{+} = p^{i+1,j+1}\nabla^{ij}$$

and

$$T^{ij}_{-} = p^{i+1,j-1}\nabla^{ij}.$$

In Vaisman [24], symplectic Ricci and Weyl curvatures of a Fedosov connections are defined and investigated. If the symplectic Weyl tensor is null, we call the connection *Weyl-flat*. Let (M, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Weyl-flat Fedosov connection. The following formula (see [17]) was derived in [16]

$$R = \frac{1}{n+1}(E^{+}\Theta^{\sigma} + 2F^{+}\Sigma^{\sigma}),$$

where the maps $E^{+}, \Theta^{\sigma}, F^{+}$, and Σ^{σ} are defined in [17]. For restrictions of these maps to $\mathcal{E}^{ij}$ the following is proved in Krýsl [16]: $\Theta^{\sigma} : \mathcal{E}^{ij} \to \mathcal{E}^{i,j-1} \oplus \mathcal{E}^{i,j} \oplus \mathcal{E}^{i,j+1}$,

$\Sigma^\sigma : \mathcal{E}^{i+1,j} \to \mathcal{E}^{i+1,j-1} \oplus \mathcal{E}^{i+1j} \oplus \mathcal{E}^{i,+1j+1}$. $F^+ : \mathcal{E}^{i+1,j} \to \mathcal{E}^{i+2,j}$ and $E^+ : \mathcal{E}^{ij} \to \mathcal{E}^{i+2,j}$. In particular, we obtain that $R^{ij} : \mathcal{E}^{ij} \to \mathcal{E}^{i+2,j-1} \oplus \mathcal{E}^{i+2,j} \oplus \mathcal{E}^{i+2,j+1}$.

Theorem 2: Let (M,ω) be a symplectic manifold which admits a metaplectic structure and ∇ be a Weyl-flat Fedosov connection. For any integers i,j, the sequences $(\Gamma(\mathcal{E}^{i+k,j\pm k}), T_\pm^{i+k,j\pm k})_{k\in\mathbb{Z}}$ are cochain complexes.

Proof. After a possible renumbering of i,j, it is sufficient to evaluate

$$\begin{aligned} T_\pm^{i+1,j\pm 1}T_\pm^{ij} &= p^{i+2,j\pm 2}\nabla^{i+1,j\pm 1}p^{i+1,j\pm 1}\nabla^{ij} \\ &= p^{i+2,j\pm 2}\nabla^{i}p^{i+1,j\pm 1}\nabla^{ij}. \end{aligned}$$

Since the image of ∇^{ij} is a linear subspace of $\Gamma(\mathcal{E}^{i+1,j-1} \oplus \mathcal{E}^{i+1,j} \oplus \mathcal{E}^{i+1,j+1})$ (Theorem 4, [15]), we may write $\mathrm{Id}_{\mathcal{E}^{i+1}} - p^{i+1,j} - p^{i+1,j\mp 1}$ instead of $p^{i+1,j\pm 1}$ in the above formula obtaining

$$p^{i+2,j\pm 2}\nabla^{i+1}\nabla^{ij} - p^{i+2,j\pm 2}\nabla p^{i+1,j}\nabla^{ij} - p^{i+2,j\pm 2}\nabla^{E}p^{i+1,j\mp 1}\nabla^{ij}.$$

Using the fact about the image of the restrictions of the connections in the case of $\nabla^{i+1,j}$ and in the case $\nabla^{i+1,j\pm 1}$, we get that the last two therms are null, because the outer projections cancel the appropriate images. In the next picture, we see the vanishing of the first term in the $+$-case. (Dotted arrows point towards spaces which are not in the image.)

$$\begin{tikzcd}[row sep=tiny]
E^{i+1,j-2} & E^{i+2,j-2} \\
E^{i+1,j-1} & E^{i+2,j-1} \\
E^{i+1,j} \arrow[ruu, dotted] \arrow[ru] \arrow[r] \arrow[rd] \arrow[rdd, dotted] & E^{i+2,j} \\
E^{i+1,j+1} & E^{i+2,j+1} \\
E^{i+1,j+2} & E^{i+2,j+2}
\end{tikzcd}$$

Pic. 3. Images of exterior covariant derivatives.

Summing-up, $T_\pm^{i+1,j\pm 1}T_\pm^{ij} = p^{i+2,j\pm 2}R^{ij}$. By the paragraph in front of the formulation of this theorem, the symplectic spinor curvature restricted to $\mathcal{E}^{ij}$ is a map into $\mathcal{E}^{i+2,j-1} \oplus \mathcal{E}^{i+2,j} \oplus \mathcal{E}^{i+2,j+1}$. Consequently, the above composition is null and $T_\pm^{i+k,j\pm k}$, $k \in \mathbb{Z}$, form complexes.

$\square$

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