Content: 1. Definition of manifolds and examples
2. Immersion and embedding. Impl. ftion thm.
3. (Tangent bundle). Cov. derivative. Induced cov. der. a geodesics
4. Riemannian connection and Riemannian curvature
5. Constant sectional curvature and Einstein spaces

Literature: Kowalski, O., Zdál. Riemannovy geom.
Do Carmo, P., Riemannian geometry
Kobaiyashi, S., Nomizu, K., Foundations of diff. geom
Spivak, A compresensive intro to Riem. geom I–III
Helgason, S., Diff. geom., Lie groups and symmetric spaces
Curtis, Miller, Diff. Manifolds and Theor. Physics

History: 5^{th} postulate: Saccheri [sakeri], Bolyai [boljaj], Lobačevskij /
Riemann, Klein, Weyl
Whitney

suggestion of a def. (1854)

modern def.

def. of mafld

Facts from topology

Thm. (invariance of domains): $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ inj. and continuous. Then $V := f(U)$ is open in $\mathbb{R}^n$ and f is a homeomorphism.

Proof: ϕ (Brouwer, see blogg of T. Tao). $\square$

Thm. (invariance of dimensions): Let $f : \underset{\text{open}}{U} \twoheadrightarrow \underset{\text{open}}{V \subseteq \mathbb{R}^n}$ be a homeomorphism. Then $m = n$. $\mathbb{R}^n$

Proof: By contradiction

1) $m < n$, $i_{nm}: \mathbb{R}^m \to \mathbb{R}^n$ $\quad i_{nm}(x^1,\dots,x^m) = (x^1,\dots,x^m,\underbrace{0,\dots,0}_{\text{esp. } n-m}) \in \mathbb{R}^n$

i_{nm} is injective and cont. $\Longrightarrow$ $i_{nm} \circ f$ is inj. and continuous. Inv. of domains $\Rightarrow$ $(i_{nm} \circ f)(U)$ is open in $\mathbb{R}^n$. Thus $\forall x_0 \in (i_{nm} \circ f)(U)$ $\exists \varepsilon > 0$ s.t. $U_\varepsilon(x_0) := (x^1-\varepsilon, x^1+\varepsilon) \times \dots \times$

$(x^n-\varepsilon, x^n+\varepsilon) \subseteq (i_{nm} \circ f)(U) \subseteq \mathbb{R}^m \times \underbrace{\{0\} \times \dots \times \{0\}}_{(n-m)\text{-times}}$.

Especially $(x^n-\varepsilon, x^n+\varepsilon) \subseteq \{0\}$, which is impossible due to $\varepsilon > 0$. (last coor.)

2) $n < m$: $f^{-1}: V \subseteq \mathbb{R}^m \longrightarrow U \subseteq \mathbb{R}^n$ is homeo (onto open)
$[\, i_{mn}: \mathbb{R}^n \to \mathbb{R}^m,\ i_{mn} \circ f^{-1}: V \subseteq \mathbb{R}^m \longrightarrow U \subseteq \mathbb{R}^m \,]$ and we proceed as in 1). $\square$

Remark: 1) IoDo cannot drop inj.
Peano curve: $(0,1) \longrightarrow (0,1) \times (0,1)$ continuous but not a homeo (not inj., but surjective).

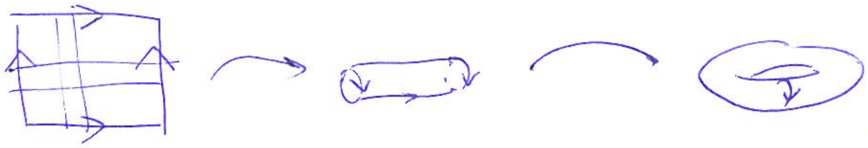
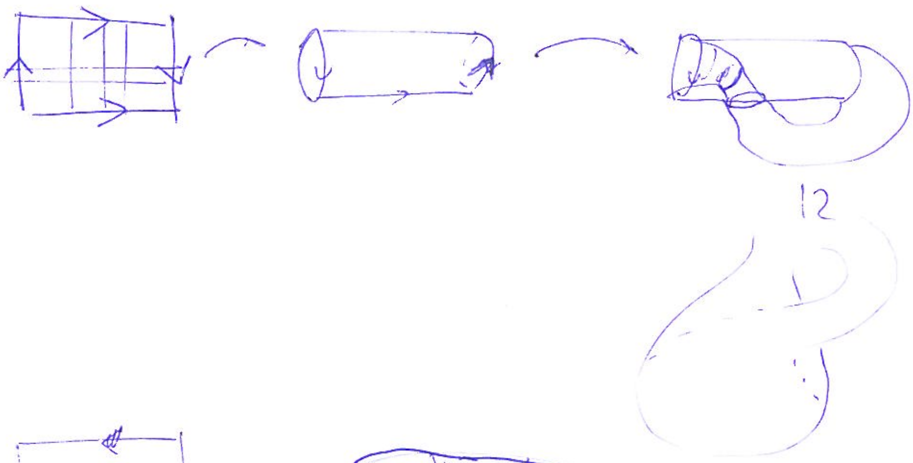
2) IoDo, $I: L^\infty \to L^\infty$, $I(a^1, a^2, \dots) := (0, a^1, a^2, \dots)$ inj. + cont. Image is not open!

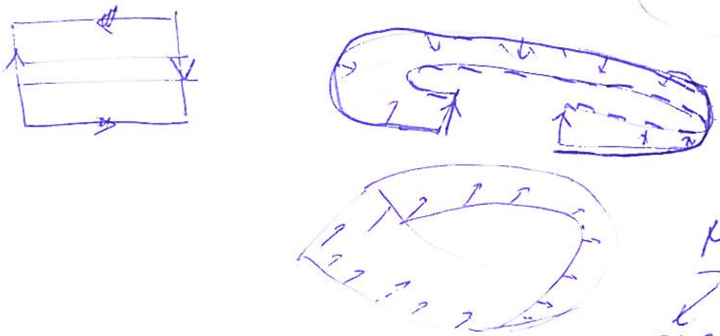
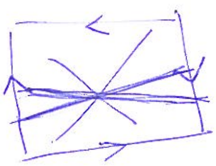
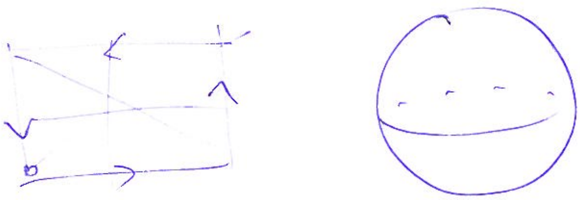
Remark: IoDi (we come to it later as well)
$f: (0,1) \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ $\quad f(x) = (x, 1)$
is homeo onto the image (as a top. space; ~~more~~ formally as a top. space with induced topology)
But not homeo onto open set! (Also #2)

. So homeo onto open is important. Sufficient.

1. Definition: Topological manifold (of dimension n) is a Hausdorff space M with a countable basis (of open neighborhoods), which is locally homeomorphic to (an open set of) $\mathbb{R}^n$, i.e. $\forall m \in M \; \exists U \ni m$ a neighborhood and a homeomorphism $\varphi_m : U \to \mathbb{R}$

Pictures:

- 
  torus
- 

  Klein bottle
- 
  ← „flipping"
  Möbius band
  one-side surface
- 
  Projective space
  (Cohn-Vossen, Hilbert → pictures)
- 
  Sphere

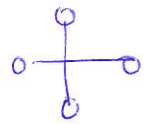
<u>Quotient topology</u>: X topol. space, $\cong \subseteq X \times X$ equiv. relation

$X/\cong$ as a set with topology given by <u>final topol</u> for $\pi : X \to X/\cong$ projection, i.e., <u>finest</u> on $X/\simeq$
canonical
such that π is cont.

<u>Induced top. by inclusion</u>: $i : X \hookrightarrow Y$ (inclusion ($\subseteq$ as a map)) and $X \subseteq Y$ be a subset of the topological space Y. $U \subseteq X$ open in the induced topology iff $\exists V$ open in Y s.t. $U = X \cap V$.
[Eq.: initial for $i : X \hookrightarrow Y$, i.e., coarsest s.t. i is cont.]

$\boxed{\text{open cross}}$

<u>Not a topol. manfd</u>:



$\subseteq \mathbb{R}^2$ ~~$\{(t,t)\}$~~

$X := (-1,1) \times \{0\} \cup \{0\} \times (-1,1)$ with topol induced by $i : X \to Y$ $\quad (i(x,y) = (x,y))$
$(x,y) \in X$

Problem at $(0,0)$. Remove $(0,0) \to$ 4 components

<u>homeo</u> to open. int. (Exercises)

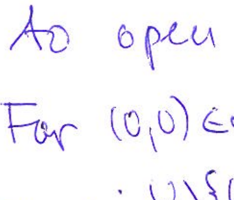
Sketch: For $(0,0) \in$ [cross], there exist (U, φ): $(0,0) \in U$, $\varphi : U \xrightarrow{\text{homeo}} V \subseteq \mathbb{R}$.

Suppose U connected. $\varphi|U \setminus \{(0,0)\} : U \setminus \{(0,0)\} \to V \setminus \{\varphi(0,0)\}$

otherwise take connected component of $(0,0)$

U open in [cross] $\Rightarrow \exists U' \underset{\text{open}}{\subseteq} \mathbb{R}^2$ ~~[cross]~~ $U = [\text{cross}] \cap U' \Rightarrow$ characterization of open sets in metric spaces

$U \subseteq (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$



$\xrightarrow{\text{homeo}} V \setminus \{\varphi(0,0)\}$

But V is a homeo-image of a connected set $U \Rightarrow$
V is open int (since $V \subseteq \mathbb{R}$): <u>4 comp</u> $\xrightarrow{\text{homeo}}$ <u>2 comps</u>
impossible ("easy")

<u>Definition</u>: <u>C^k-atlas on</u> a topol mnfld X of dim n is a set $\mathcal{A}$ consisting of (U, φ), where $U \subseteq X$ open and φ a homeo onto (an open set of) $\mathbb{R}^n$ satisfying:

- $\bigcup pr_1(a) = X$ (X is covered by $\mathcal{A}$), where
$$pr_1(U, \varphi) = U \quad (pr_1 : \mathcal{A} \to 2^X)$$
- $(U_1, \varphi_1), (U_2, \varphi_2) \in \mathcal{A}$ and $U_1 \cap U_2 \neq \emptyset$, the map: $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$ is of class C^k (as a map $\mathbb{R}^n \to \mathbb{R}^n$).

<u>Example</u>: $\mathbb{R}^n$ with std. topol., $\mathcal{A} = \{(\mathbb{R}^n, Id_{\mathbb{R}^n})\}$

$Id_{\mathbb{R}^n} \circ Id_{\mathbb{R}^n}^{-1} = Id_{\mathbb{R}^n}$ is C^k for any $k = 0, \dots, \infty$

<u>Remark</u>: Elements of atlases are called coordinate systems. Compositions $\varphi_1 \circ \varphi_2^{-1}$ are called transition maps

<u>Definition</u>: Let X be a topol. mnfld and $\mathcal{A}_1$, $\mathcal{A}_2$ be C^k-atlases on X. $\mathcal{A}_1$ and $\mathcal{A}_2$ are called compatible iff $\mathcal{A}_1 \cup \mathcal{A}_2$ is a C^k-atlas (on X). We write $\mathcal{A}_1 \cong \mathcal{A}_2$

<u>Thm.</u>: $\simeq$ is an equivalence.

Proof:
- $\mathcal{A} \simeq \mathcal{A}$ $\quad \mathcal{A} \cup \mathcal{A} = \mathcal{A}$ is C^k atlas
- $\mathcal{A} \simeq \mathcal{B} \Rightarrow \mathcal{A} \cup \mathcal{B}$ is C^k atlas. $\mathcal{B} \cup \mathcal{A} = \mathcal{A} \cup \mathcal{B}$ is C^k atlas as well. Thus $\mathcal{B} \simeq \mathcal{A}$.
- $\mathcal{A} \simeq \mathcal{B}$, $\mathcal{B} \simeq \mathcal{C}$. We want $\mathcal{A} \simeq \mathcal{C}$.
$(U, \alpha) \in \mathcal{A}$, $(W, \gamma) \in \mathcal{C}$, $U \cap W \neq \emptyset$.
$\forall p \in U \cap W \underset{\mathcal{B} \text{ atlas}}{\Longrightarrow} \exists (V, \beta), p \in V$, a coord system (around p)

Consider

~~Write~~ $\alpha \circ \gamma^{-1}|_{\gamma(U \cap V \cap W)} : \gamma(U \cap V \cap W) \to \alpha(U \cap V \cap W)$

we may write $\alpha\circ\gamma^{-1}|_{\gamma(U\cap V\cap W)} = (\alpha\circ\beta^{-1})\circ(\beta\circ\gamma^{-1})|_{\gamma(U\cap V\cap W)}$.
But $(\alpha\circ\beta^{-1})|_{\gamma(U\cap V\cap W)}$ is C^k and $(\beta\circ\gamma^{-1})|_{\gamma(U\cap V\cap W)}$ is C^k as well. The comp. is C^k on $\gamma(U\cap V\cap W)$. Esp. it is C^k in p.
($\alpha\circ\gamma^{-1}$ is C^k in any $\gamma(p)\in\gamma(U\cap W)$, it is C^k on $\gamma(U\cap W)$).
Thus $\mathcal{A}\simeq\mathcal{C}$. $\square$

Example: $\{(\mathbb{R}, Id_{\mathbb{R}})\} = \mathcal{A}$, C^∞-atlas on $\mathbb{R}$
$\{(\mathbb{R}, t\mapsto t^3)\} = \mathcal{B}$, —"—
But $\mathcal{A}\cup\mathcal{B} = \{(\mathbb{R}, Id_{\mathbb{R}}), (\mathbb{R}, t\mapsto t^3)\}$ and
$Id_{\mathbb{R}}\circ t^{1/3} = t^{1/3}$ is not C^1 (in 0 it has no derivative). $\mathcal{A}$ and $\mathcal{B}$ are not cptbl.

Definition: Let X be a topol. mfld. Each element in $\mathbb{A}/\simeq$ is called a C^k-differential structure.
Here $\mathbb{A} = \{\mathcal{A} \mid \mathcal{A}$ a C^k-atlas on $X\}$.

Thm.: Each C^k-diff. structure on X contains a maximal C^k-atlas wrt inclusion (of atlases).

Proof. $\mathbf{B}$ a C^k-diff. structure on X, $\mathcal{A}\in B$ a C^k-atlas
$\mathcal{C} = \{(U,\varphi)\in\mathcal{A}' \mid \mathcal{A}'\in B$ s.t. $\{(U,\varphi)\}\cup\mathcal{A}$ is a C^k-atlas$\}$.
- $\mathcal{C}$ is a C^k-atlas: a) $\mathcal{C}\neq\emptyset$, since $\mathcal{A}\subseteq\mathcal{C}$
b) $(U,\alpha), (W,\gamma)\in\mathcal{C} \Rightarrow$
$\Rightarrow \{(U,\alpha)\}\cup\mathcal{A}$ is a C^k-atlas
$\{(W,\gamma)\}\cup\mathcal{A}$ is a C^k-atlas
Is $\alpha\circ\gamma^{-1}: \gamma(U\cap W)\to\alpha(U\cap W)$ C^k-map ($\mathbb{R}^n\to\mathbb{R}^n$)?

Let $p \in U \cap W \Rightarrow \exists (V,\beta) \in \mathcal{A}$ s.t. $p \in V$. Then

$$\alpha \circ \gamma^{-1} = (\alpha \circ \beta^{-1}) \circ (\beta \circ \gamma^{-1})\big|_{\gamma(U \cap V \cap W)} : \gamma(U \cap V \cap W) \to \alpha(U \cap V \cap W)$$

is C^k in p. Consequently, $\alpha \circ \gamma^{-1}$ is C^k in any $\gamma(U \cap W)$.
Thus $\mathcal{C}$ is a C^k-atlas.

not done on lecture but easy: [c) $\mathcal{C}$ covers X: ↯ if ~~$\exists p \in X$~~ not covered is in contradiction that $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{A}$ covers]

- Maximality: ↯ $\mathcal{D} \supsetneq \mathcal{C}$ $\quad (U,\psi) \in \mathcal{D}$ & $(U,\psi) \notin \mathcal{C}$

  $\mathcal{D} \in B \Rightarrow \mathcal{D} \cup \mathcal{A}$ comp. since $\mathcal{A} \in B \Rightarrow$

  $\{(U,\psi)\} \cup \mathcal{A}$ comp. and $(U,\psi) \in \mathcal{C}$ ↯ $\quad \square$

<u>Remark</u>: <u>X</u> topol mfld of dim <u>m</u>, <u>k</u> fixed non-neg. integer

$\mathbb{A} = \{\mathcal{A} \mid \mathcal{A}\ C^k\text{-atlas on } X\}$. If $\mathbb{A} \neq \emptyset \Rightarrow$

$F : \mathbb{A}/\simeq \longrightarrow \mathbb{A}$, $B \mapsto$ maximal atlas contained in B ($\mathcal{C}$ in the proof of prev. thm.),

is a selector (without AC).

If $\mathbb{A} = \emptyset \Rightarrow F = \emptyset$. Thus, the existence is „non-trivial". [Actually, there are top. mflds with no C^k atlases, $k \geq 1$!]

<u>Remark</u>: Further constr. of max. atlas: $B \overset{C^k\text{-}}{\sim}$ diff. structure

$$\mathfrak{B} = \bigcup \{B' \mid B' \in B\}$$

<u>Remark</u>: • Maximal atlas is biggest. For $\tilde{\mathcal{A}} \not\subseteq \mathcal{C}$, take ~~.......~~ $(U,\psi) \in \tilde{\mathcal{A}}$ and not in $\mathcal{C}$.

$\{(U,\psi)\} \cup \mathcal{A}$ comp since $\mathcal{A}, \tilde{\mathcal{A}} \in B$. ~~which~~ (thus they are comp.)
Cons. $(U,\psi) \in \mathcal{C}$. ✓

- Further, maximal atlas is unique („for a diff. str.") ← we don't say it usually

<u>Theorem</u>: Let X be a topological mnfld of dim m and let X with the same topology be a topological mnfld of dimension n. Then $m=n$.

<u>Proof</u>: $p \in X$ $\quad (U,\varphi)$ be a coord. system around p (i.e. $p \in U$).
In particular $\varphi : U \to \mathbb{R}^n$ homeo
(W,ψ) be a coord. system around p (i.e. $p \in W$).
In particular $\psi : W \to \mathbb{R}^m$ homeo

Form $\varphi \circ \psi^{-1} : \underbrace{\psi(U \cap W)}_{\neq \emptyset \text{ since } U \cap W \ni p} \longrightarrow \varphi(U \cap W)$ is a homeo
$\psi(U \cap W) \subseteq \mathbb{R}^m$, $\varphi(U \cap W) \subseteq \mathbb{R}^n$

<u>IoDi</u> $\Longrightarrow m = n$ $\square$

Definition: A C^k-manifold is a pair $(X, \mathcal{A})$, where X is a topological mnfld and $\mathcal{A}$ is a maximal C^k-atlas, C^k-diff. structure. If $k = \infty$, we call the pair a smooth manifold.

Remark: Sometimes, we use maximal C^k-atlas, sometimes C^k-diff. structure.

Definition: Let $(X^m, \mathcal{A})$ and $(Y^n, \mathcal{B})$ be C^k-manifold.
We say that $f: X^m \longrightarrow Y^n$ is a C^k-map, if
$\forall x \in X \ \forall (U, \varphi) \in \mathcal{A} \ \forall (V, \psi) \in \mathcal{B}$ s.t. $x \in U$
$f(x) \in V$, $\psi \circ f \circ \varphi^{-1}: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is C^k.

did not mention at the lecture. We say that it is diffeomorphism*), if f and f^{-1} are C^∞.

! **Remark**: The inverse f^{-1} should exist on whole Y. Particularly, f is onto

Assertion: $f: X \to Y$ is C^k, iff $\exists \tilde{\mathcal{A}}$ in the C^k-differential structure of X, $\exists \tilde{\mathcal{B}}$ in the C^k-differential of Y, s.t. $\forall x \in X$ $\forall (\tilde{U}, \tilde{\varphi}) \in \tilde{\mathcal{A}}$ $\forall (\tilde{V}, \tilde{\psi}) \in \tilde{\mathcal{B}}$ s.t. $x \in \tilde{U}$ and $f(x) \in \tilde{V}$, we have
$\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1}: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is C^k.

Proof: $\Leftarrow$ $(U, \varphi) \in \mathcal{A}$ (maximal atlas on X), $x \in U$
$(V, \psi) \in \mathcal{B}$ (maximal atlas on Y), $f(x) \in V$

Is $\psi \circ f \circ \varphi^{-1}$ C^k?

$$\psi \circ f \circ \varphi^{-1} = \underbrace{(\psi \circ \tilde{\psi}^{-1})}_{C^k} \circ \underbrace{(\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1})}_{C^k \text{ by assumpt.}} \circ \underbrace{(\tilde{\varphi} \circ \varphi^{-1})}_{C^k}$$

compatibility of atlases in max. atlas

compatibility

So it is C^k.

$\Rightarrow$ trivial $\tilde{\mathcal{A}} := \mathcal{A} \wedge \tilde{\mathcal{B}} := \mathcal{B}$. $\square$

*) "category theory approach"

Definition: Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be C^k-mnflds. We say that $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ define the same C^k-manifold structure if there exists a diffeo $f : X \to Y$ (onto Y).

Example: 1 $\mathbb{R} = \langle \{(\mathbb{R}, \mathrm{Id}_{\mathbb{R}})\} \rangle$ maximal $\mathcal{C}^\infty$-atlas cont $\{(\mathbb{R}, \mathrm{Id})\}$

2 $\mathbb{R} = \langle \{(\mathbb{R}, t \mapsto t^3)\} \rangle$ maximal $\mathcal{C}^\infty$-atlas containing $\{(\mathbb{R}, t \mapsto t^3)\}$.

$f : {}^{(1)}\mathbb{R} \to {}^{(2)}\mathbb{R} \quad f(t) = t^{1/3}$ inverse of map on ${}^{(1)}\mathbb{R}$.

$(t \mapsto t^3) \circ (t \mapsto t^{1/3}) \circ \mathrm{Id}^{-1} = \mathrm{Id} \circ \mathrm{Id}^{-1} = \mathrm{Id} : \mathbb{R} \to \mathbb{R}$ $\mathcal{C}^\infty$-map (we use the Assertion).

$g : {}^{(2)}\mathbb{R} \to {}^{(1)}\mathbb{R} \quad g(t) = t^3$

$\mathrm{Id} \circ (t \mapsto t^3) \circ \underbrace{(t \to t^{1/3})}_{\text{inverse of map}} = \mathrm{Id} \circ \mathrm{Id} = \mathrm{Id} : \mathbb{R} \to \mathbb{R}$ $\mathcal{C}^\infty$-map

Thus ${}^1\mathbb{R}$ a ${}^2\mathbb{R}$ defines the same $\mathcal{C}^\infty$ mfold structure.

Results: 1) There are topol. mnflds (C^0-mnflds) not admitting any C^1-atlas and (equivalently) any C^1-diff. structure (Kervaire ?)

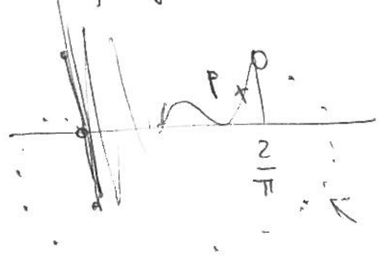
2) If $(X, \mathcal{A})$ is a C^1-manifold. Then there exists for any k a C^k-atlas on X C^1-compatible with $\mathcal{A}$. ($k = \infty$ allowed).

$\mathcal{C}^\infty$-manifold structure on $\mathbb{R}^n$

Examples: Recall examples and "non-examples":
$\mathbb{R}^n$, open sets in $\mathbb{R}^n$, S^n, T^n
cross (even not topological mnfld)

Example:

1. $\{(x,|x|), x\in(-1,1)\} \subseteq \mathbb{R}^2$, $\varphi(x,|x|) =: x$ homeo with inverse $\varphi^{-1}(x) = (x,|x|)$. $\mathcal{A} = \langle\{(\mathrm{Im}\, f, \varphi)\}\rangle$ defines a $\mathcal{C}^\infty$-structure.
2. Topologists sine
$S = \{(x, \sin\frac{1}{x}), x\in(0,\frac{2}{\pi}]\} \cup \{(0,y) \mid y\in(-1,1)\}$. Suppose it is a topolog. mnfld.
By uniquess of dimension, if it is a topol. manifold, the dimesion must be 1.



For $(0,0)\in S$, we must find a homeo. $\varphi(0,0) \in \mathbb{R}$ it lies in an open set $U \subseteq \mathbb{R}$. Take the connected comp. of U. It is an interval $I \ni \varphi(0,0)$.
Any pre-image (φ pre-image) of I contains $(0,0)$ and is an open subset of $S \Rightarrow \exists\, V \subseteq \mathbb{R}^2$ open s.t. the pre-image is ~~all~~ the intersection of V and S. But $V \ni (0,0)$. Since any intersection of V and S is known to be not pathconnected,

we have a homeo $\varphi|_I^{-1} : \underset{\substack{\uparrow \\ \text{path-conn.}}}{I} \longrightarrow \underbrace{V \cap S}_{\text{not path-connected}}$

We get a contradiction.
(Can be made more efficient by an analysis of S.)

2. Embeddings, immersions and implicit function thm.

V: Let $G^1(u^1,\dots,u^n,v^1,\dots,v^d),\dots,G^{n}(u^1,\dots,u^n,v^1,\dots,v^d)$ be def. in a neighb of $(a^1,\dots,a^n,b^1,\dots,b^d)\subseteq \mathbb{R}^n\times\mathbb{R}^d$ and G^j have cont. of k-th part. der. $k\geq 1$.

Let

(i) $G^j(a^1,\dots,a^n;b^1,\dots,b^d)=0$

(ii) $\det\left(\frac{\partial G^j}{\partial u^i}(a^1,\dots,a^n;b^1,\dots,b^d)\right)^{j=1,\dots,n}_{i=1,\dots,n}\neq 0$.

Then $\exists U$ at. a $\exists V$ at b $\forall v\in V\,\exists!\,u\in U$, že $G^j(u^1,\dots,u^n;v^1,\dots,v^d)=0$. Namely $u^1,\dots,u^n$ are fctions of $v^1,\dots,v^d$, i.e. $\phi^i: V\to\mathbb{R}$ defined $\phi^i(v)=r \iff r = pr^i\circ u$, where u is the unique u which exists for v.

Někdy namísto $\mathbb{R}^n[u^1,\dots,u^n]$ píšeme $\mathbb{R}^n[v^1,\dots,v^d]$ +)

Refor: $W\subseteq\mathbb{R}^n[u^1,\dots,u^n]\times\mathbb{R}^d[v^1,\dots,v^d]$ be neighb of (a,b), $a\in\mathbb{R}^n$, $b\in\mathbb{R}^d$, $G: W\to\mathbb{R}^n[t^1,\dots,t^n]$ be C^k map ($1\leq k\leq\infty$) and let

(i) $G(a,b)=0$ (ii) $\det\left(\frac{\partial G}{\partial u}(a,b)\right)\neq 0$.

Then $\exists U\underset{op.}{\ni} a$ $\exists V\ni b$ $\exists!\,\phi: V\to U$ s.t. $\forall u\in U\,\forall v\in V$

$G(u,v)=0 \iff \phi(v)=u$.

Moreover, ϕ is C^k-map.

+) se stejným významem, tj. $v^i(x^1,\dots,x^n)=x^i$ ($v^i=pr^i$).

Inverse function thm. $F: U \subseteq \mathbb{R}^n[u^1,\dots,u^n] \longrightarrow \mathbb{R}^n[v^1,\dots,v^n]$

be C^k a $\left\| \dfrac{\partial(v^j\circ F)}{\partial u^i}\right\| \neq 0$ v a. Pak $\exists V \ni b = F(a)$

and $\exists F: V \to \mathbb{R}^n$. this F^{-1} is C^k, esp. F is diffeo onto a nmgb. of $F(a)$.

Proof: $G(u^1,\dots,u^n,v^1,\dots,v^n) := F(u) - v$.

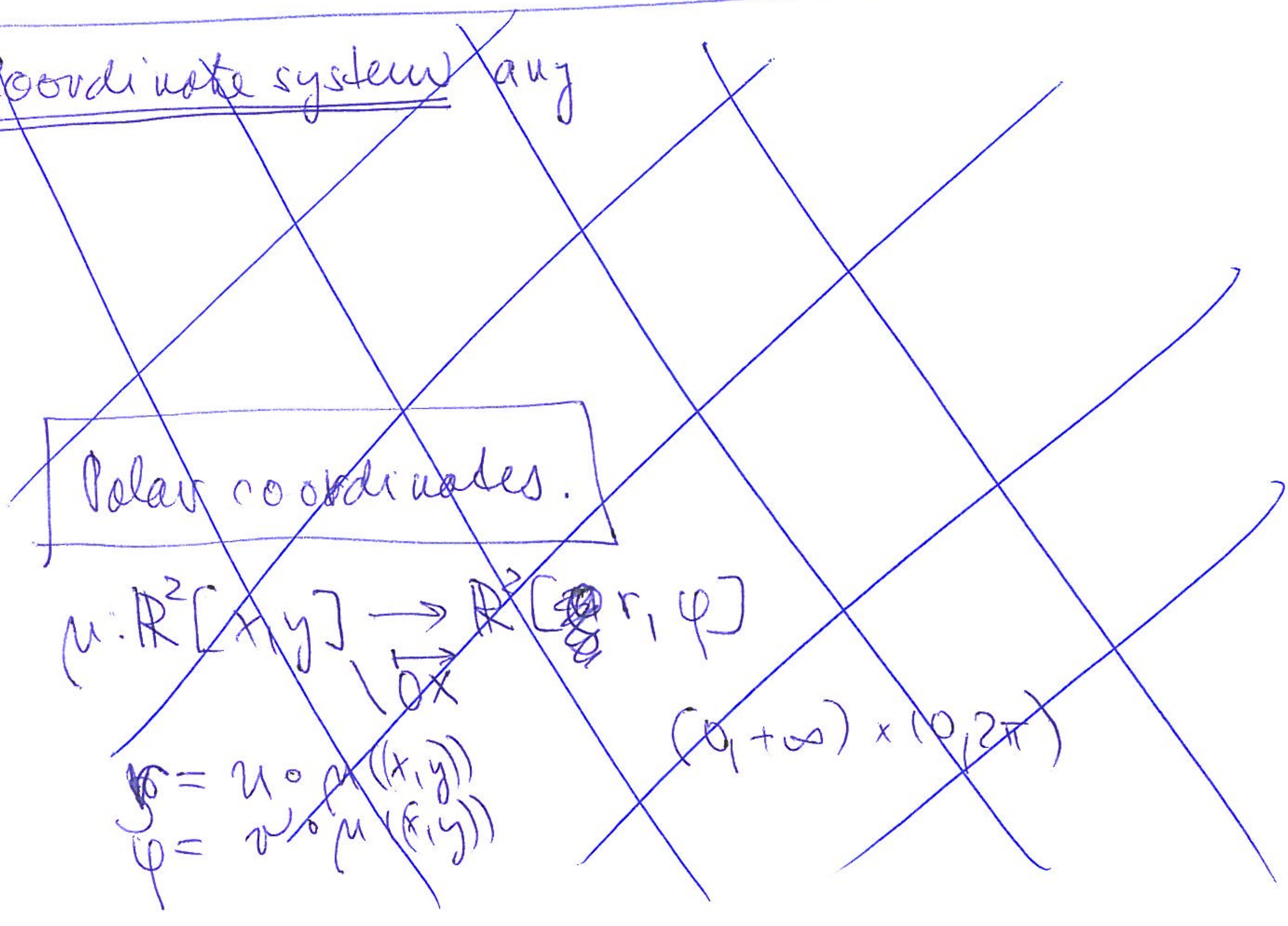
$G(a,b) = 0$. Then $G^j = v^j \circ G$:

$$\frac{\partial G^j}{\partial u^i} = \frac{\partial(v^j\circ F)}{\partial u^i} \neq 0 \Rightarrow \exists \phi: V \to U$$

$$u = \phi(v) \iff G(u,v) = 0 \iff F(u) = v$$

ϕ je inverze k F.

Coordinate system any



$\boxed{\text{Polar coordinates.}}$

$\mu: \mathbb{R}^2[x,y] \longrightarrow \mathbb{R}^2[r,\varphi]$

$\setminus \overrightarrow{0x}$

$(0,+\infty)\times(0,2\pi)$

$r = u\circ\mu((x,y))$

$\varphi = v\circ\mu((x,y))$

Thm.: Let $U \subseteq \mathbb{R}^n[u^1, u^2, \dots, u^n]$, $\varphi: U \to \mathbb{R}^n[v^1, v^2, \dots, v^n]$ be a C^k-map and $m \in U$.
Then TFAE:

(1) $\left\| \left(\frac{\partial \varphi^j}{\partial u^i} \right)_{ij} (m) \right\| \neq 0$

(2) $\exists V \subseteq U$ ~~open~~ $\varphi_{|V}$ such that $(V, \varphi_{|V})$ is a coord. system.

Proof: $(1) \Rightarrow (2)$: $\exists V \subseteq U$ such that $\varphi_{|V}$ is invertible with inverse of the same order of differentiability
In particular, $\varphi_{|V}$ is homeo.
$\varphi_{|V}$ is coord. $\{(V, \varphi_{|V})\} \in B_{k,n}$.
Suff. for one in $B_{k,n}$: $\varphi_V^{-1} \circ Id$ and $Id \circ \varphi_V^{-1}$ are C^k.

$(2) \Rightarrow (1)$: $\psi := \varphi_{|V}$, ψ coo $\Rightarrow \psi$ homeo
$\psi^{-1} \circ \psi = Id_{|V}$, ψ, ψ^{-1} are C^1 at least.

$jac(\psi^{-1} \circ \psi)(m) = jac(Id_{|V})(m) = 1$

$jac(\psi^{-1})(\psi(m))\, jac(\psi)(m) = 1 \Rightarrow jac(\psi)(m) = \left\| \left(\frac{\partial \psi^j}{\partial u^i} \right) \right\|_{ij} \neq 0.$

Recall
Let $jac(\varphi)$ be the determinant of the p.d. matrix $\frac{\partial \varphi^j}{\partial u^i}$

When we say a C^k-map, $k \geq 1$. (convention) ②

$\operatorname{jac}(\varphi) = \left(\frac{\partial \varphi^j}{\partial u^i}\right)_{ij}$ def.

$\|A_{ij}\| = \det(A_{ij})$ označení

Thm.: A C^k-map $\varphi : U \subseteq \mathbb{R}^n[u^1, \dots, u^n] \to \mathbb{R}^n[v^1, \dots, v^n]$ is a coordinate system iff φ is injective and $\operatorname{jac}(\varphi)$ ~~≠ 0~~ is everywhere non-zero.

Proof: $\Rightarrow$: φ coord $\Rightarrow \varphi$ inj. $\wedge\, C^k, k \geq 1$, prev. thm. $\|\operatorname{jac}(\varphi)\| \neq 0$

$\Leftarrow$ Locally, φ is inv. by a C^k-map, i.e., $\forall m \in M\ \exists U_m$

$\psi_m : \varphi(U_m) \to \mathbb{R}^n$ s.t. $\psi \circ \varphi_{|U_m} = \mathrm{Id}_{|U_m}$

φ inj. $\Rightarrow \varphi$ has glob. inv. Uniq of inv.

$\varphi^{-1}_{|\varphi(U_m)} = \psi_m$ which is C^k. (Everywhere C^k, (always) C^k!)

Remark 1: We spoke about jacobians and coordinate systems.

Remark 2: In some point nonzero jac in its neighb. as well.

~~Definition: M^m, N^d ($m \leq d$) be C^k-manifolds of dimension m and d, respectively. $F: M \to N$ a C^k-map is called an immersion iff $\forall m \in M\ \exists U \subseteq M\ \exists \psi = (y^1, \dots, y^d)$ a coo. system on $V \underset{\text{open}}{\subseteq} N$ s.t. $F(m) \in V$ and $x^i = y^i \circ F$, $i = 1, \dots, m$ is a coord. system on M.~~

~~Embedding is an injective immersion.~~

Definition: M^m, N^d be C^k-manifolds of dimensions m and d. Let $F: M \to N$ be a C^k-map. We call F an <u>immersion</u> iff $\forall m \in M \; \exists U \ni m \; \exists \psi = (y^1, \dots, y^d)$ coordinate system ~~on~~ around $F(m)$ s.t. $x^i = y^i \circ F$, $i = 1, \dots, n$ builds a coordinate system around m. We call F an <u>embedding</u> if it is an injective immersion.

Thm.: Let $U \subseteq \mathbb{R}^m[u^1, \dots, u^m]$ open, $F: U \to \mathbb{R}^d[v^1, \dots, v^d]$ a C^k-map. Then F is an immersion iff $\left(\frac{\partial F^j}{\partial u^i}\right)_{i=1,\dots,n}^{j=1,\dots,d}$ has constant rank n.

Proof: $\boxed{\Leftarrow}$ We comb the proof from the lecture.
Recall $F^j = v^j \circ F$, $j = 1, \dots, d$ (definition & notation). Let $m \in U$
$\underset{\text{assum.}}{\Rightarrow} \exists \pi : \{1, \dots, n\} \to \{1, \dots, d\}$ injective s.t.
(depends on m, possibly)

$$\det\left(\frac{\partial F^{\pi(j)}}{\partial u^i}(m)\right)_{i,j=1,\dots,n} \neq 0.$$ Setting $G^j = F^{\pi(j)} = v^{\pi(j)} \circ F$, we get a coordinate system around m. Thus $\psi = (v^{\pi(1)}, \dots, v^{\pi(n)})$, $\operatorname{Dom} \psi = U$ = open set determined by the thm on coo-syst. $\leftrightarrow$ jacobians! $\{x^i = v^{\pi(i)} \circ F\}$ is coo syst. around m $\Rightarrow$ F is immersion according to the definition

$\boxed{\Rightarrow}$ ~~We want~~ F immersion $\Rightarrow \forall m \in M\ \exists U \ni m\ \exists \psi = (y^1, \dots, y^d)$ coo syst. around $F(m)$ s.t. $x^i = y^i \circ F$ is a coo syst around m (2)

↯ $\exists m_0$ rank $\left(\frac{\partial F^\ell}{\partial u^k}(m_0) \right) < m$
$k = 1, \dots, m$
$\ell = 1, \dots, d$

Chainrule $$\frac{\partial (y^j \circ F)}{\partial u^k}(m_0) = \sum_{\ell=1}^{d} \frac{\partial y^j}{\partial v^\ell}(F(m_0)) \frac{\partial F^\ell}{\partial u^k}(m_0)$$

$(m, y^?)$ lin. alg.

$$\Box = \Box\ \Box \Rightarrow$$

$\Rightarrow$ rank$(\Box) \le$ rank $\Box < m$ ↯ with assum.

$\Box$

rank is m.

Remark: • $F: M^m \to N^d$ immersion $\Rightarrow m \le d$
(↯ $m > d$ ... $(y^1, \dots, y^d)$ cannot be coo syst on M^m)

Examples: • $\gamma: (a,b) \to \mathbb{R}^m$ reg. curve ($\overset{\text{def}}{=}$ C^1 and $\sum_{i=1}^{m} \gamma_i'^2(t) \neq 0$ $\forall t \in (a,b)$). possible classic. Attenzione
$(a,b) = M$, $\mathbb{R}^n = N$, $k = 1$ (C^k-map).

rank $(\gamma_1'(t), \dots, \gamma_m'(t)) = 1$ $\forall t$ $\Leftrightarrow \sum_{i=1}^{m} \gamma_i'^2(t) \neq 0$ $\forall t$

(rank $\Leftrightarrow$ "Lin. indep." of one vect $\Leftrightarrow$ vector is non-zero $\Leftrightarrow$ all comp. are nonzero $\Leftrightarrow \sum \gamma_i'^2 \neq 0$)

Thus reg. curve is an immersion (and vice versa) possible classical def.

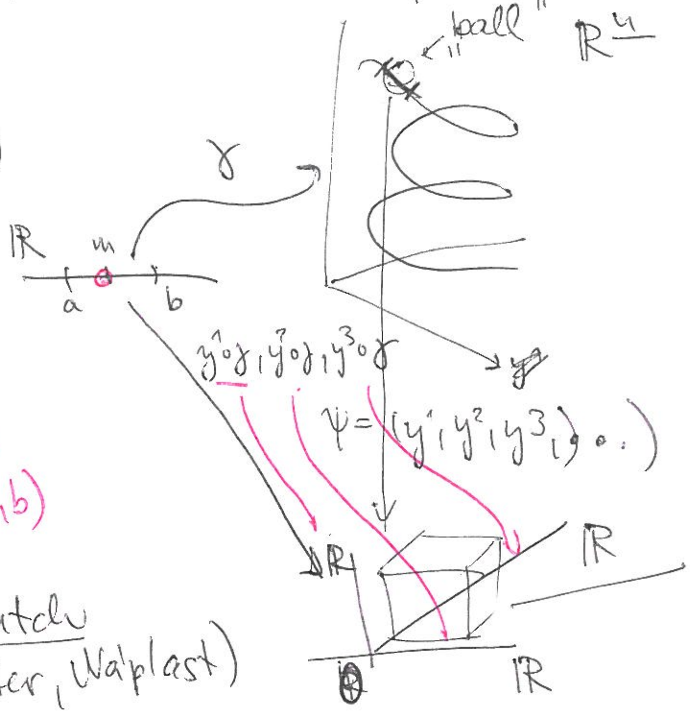
• $S: (a_1, a_2) \times (b_1, b_2) \to \mathbb{R}^3$ reg. surf ($\overset{\text{def}}{=}$ C^1 and
$$\left(\frac{\partial S^1}{\partial u}(p), \frac{\partial S^2}{\partial u}(p), \frac{\partial S^3}{\partial u}(p) \right) \times \left(\frac{\partial S^1}{\partial v}(p), \frac{\partial S^2}{\partial v}(p), \frac{\partial S^3}{\partial v}(p) \right) \neq \vec{0}$$
$\forall p$ ($(a_1, a_2) \subseteq \mathbb{R}[u]$, $(b_1, b_2) \subseteq \mathbb{R}[v]$ ~~notat~~ for convenience))

Vector prod non-zero iff the set of corr. vectors is l. ind. (3)

$$\Longleftrightarrow \operatorname{rank}\begin{pmatrix} \frac{\partial S^1}{\partial u} & \frac{\partial S^2}{\partial u} & \frac{\partial S^3}{\partial u} \\ \frac{\partial S^1}{\partial v} & \frac{\partial S^2}{\partial v} & \frac{\partial S^3}{\partial v} \end{pmatrix}(p) \neq 0 \overset{\text{thm}}{\Longleftrightarrow} S \text{ is an immersion}$$

~~around~~

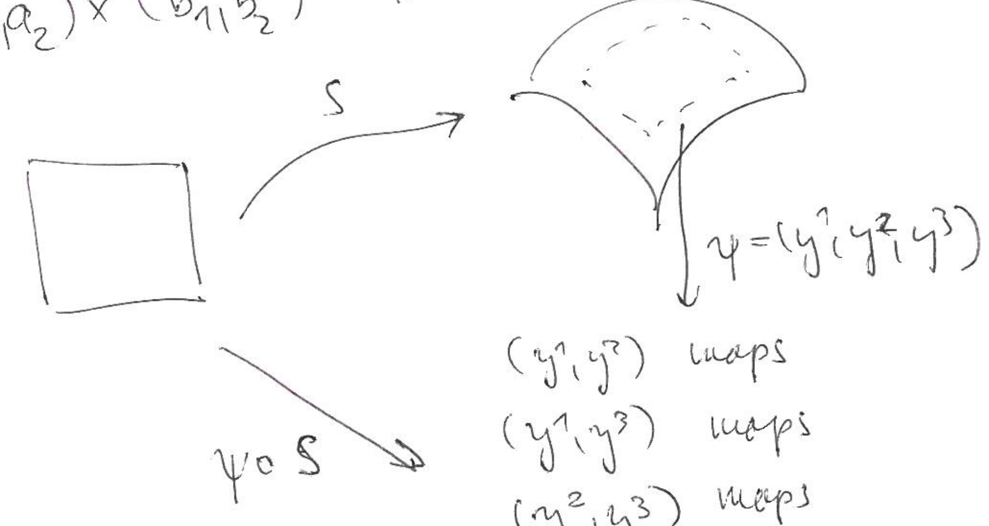
Note: • $\gamma : \mathbb{R}^1 \to \gamma(\mathbb{R}^1)$



The red arrows are coo systems around $m \in (a,b)$ (and maps)

Terminol. remark: γ is called a patch (Pflaster, Wa'plast)

• $S : (a_1, a_2) \times (b_1, b_2) \to \mathbb{R}^3$



S is patch
$(y^i, y^j) \circ S$ might be patches (depends on i,j)

Examples

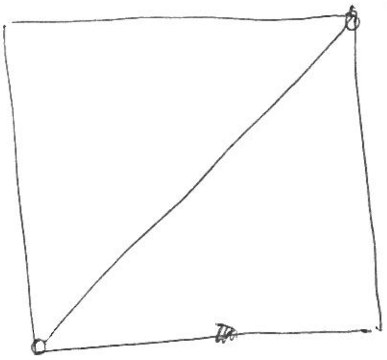
1) torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$. Take $n=2$.

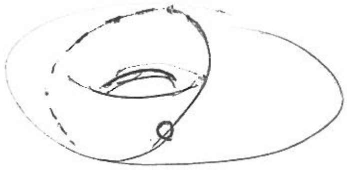
$\gamma_{a,b} : (0,1) \longrightarrow T^2$ $\quad t \in \mathbb{R}$, $(at, bt) \in \mathbb{R}^2$

$\gamma_{a,b}(t) = [(at, bt)]$ $\quad$ [ ] eq. class in $\mathbb{R}^2/\mathbb{Z}^2$.

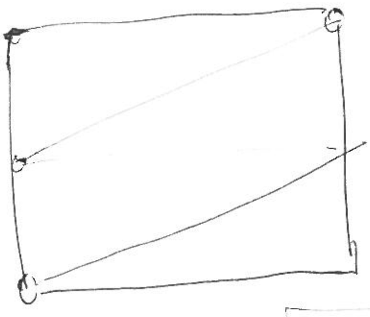
$a, b \in (0,1]$ ← interval

$a=b=1$

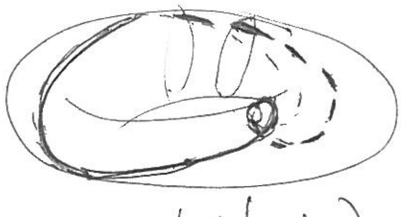




Why?

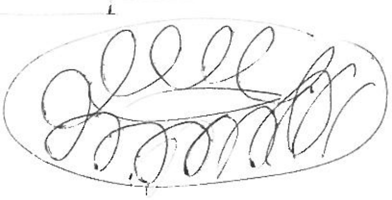


$a=1, b=\frac{1}{2}$

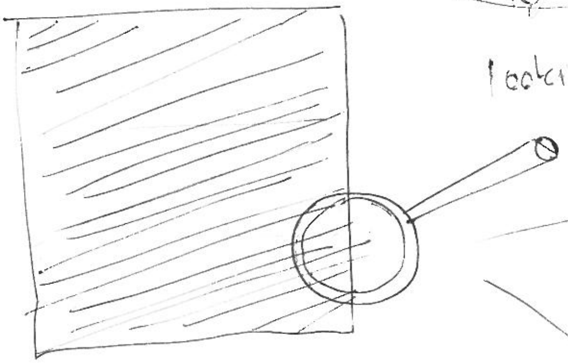


sort of :-)

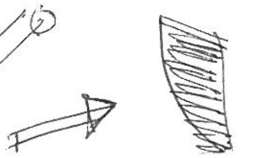
$\frac{a}{b} \notin \mathbb{Q}$



← dense, more dense, more dense, more dense



looking glass



etc.

infinitely many components

Induced topology from inclusion

$\gamma_{a,b}((0,1]) \subseteq T^2$

open set

induced by inclu

open set

Submanifold topology :

But not open in

$\gamma_{a,b}$ is an immersion and it is an embedding (5)

Thus $\gamma_{a,b}([0,1])$ is a submanifold. Tedious to check. Do it.

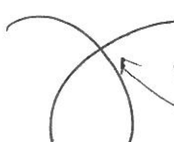
- Another quest.: Is $T^n = \mathbb{R}^n / \mathbb{Z}^n \simeq S^1 \times \cdots \times S^1$

(Demanding :-) : Define atlases and use exponential, one could learn something from $\mathbb{C} \simeq \mathbb{R}^2$.)

---

(2) C^1-curve classical is immersion, but need not be embedding

(to)



in $\mathbb{R}^2$ $a < b \in \mathrm{Dom}(\gamma)^\circ$

$\gamma(a) = \gamma(b)$

---

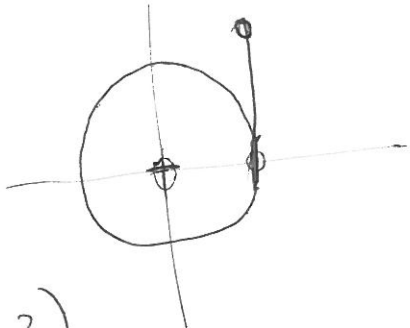
(3) $\varphi(t) = (\cos(t), \sin(t))$, $t \in (0, 2\pi]$

$= (1, t - 2\pi)$, $t \in (2\pi, 7)$

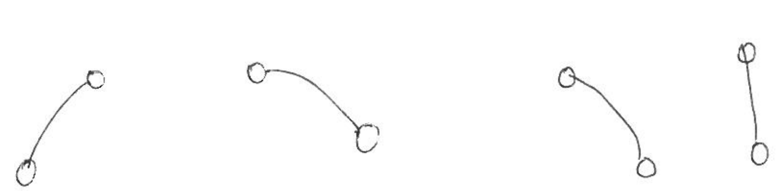
$\varphi'(t) = (-s(t), c(t))$

$= (0, 1)$

it is C^1-map. It is emb. (why?)

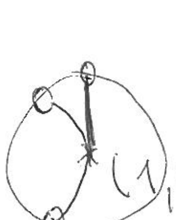


Open sets?



Open sets in ~~induced~~ topology induced by inclusion

open in both



$(1,0)$

but

$\boxed{\text{is not open}}$ because it is not ~~open~~

the intersection of any neighborhood of point $(1,0)$ with S !

⑥ Definition: Let M, N be C^k-manifolds and $X \subseteq N$ a subset of N. X is called a C^k-submanifold of N if $\exists F: M \to N$ embedding of C^k-manifolds s.t.

1) $X = F(M)$
2) topology on X as a C^k-submanifold is given by V open in $X \iff V = F(U)$ $(\cap X)$ for U open in M
   superficial
3) C^k-diff. structure on X is the unique one s.t. F and F^{-1} are C^k-maps

Remark (uniqueness):

- $\mathcal{A}_X', \mathcal{A}_X''$ max atl. $\mathcal{A}_X' \cap \mathcal{A}_X'' = \phi \Rightarrow$ take $\mathcal{A}_X' \cup \mathcal{A}_X''$ compatible? $\frac{\partial(\varphi' \circ \varphi''^{-1})}{\partial u^i} = \frac{\partial}{\partial u^i}(\varphi' \circ F^{-1} \circ F \circ \varphi''^{-1}) =$

$$= \frac{\partial(\varphi' \circ \varphi''^{-1})}{\partial u^i} = \frac{\partial}{\partial u^i}(\varphi' \circ F \circ \psi \circ \psi^{-1} \circ F^{-1} \circ \varphi''^{-1})$$

maps on M

is ok since F & F^{-1} are C^k

Thus they are not maximal (if non-empty, see below)

$\Rightarrow \mathcal{A}_X' \cap \mathcal{A}_X'' \neq \phi$ and the common element proves $\mathcal{A}_X' = \mathcal{A}_X''$ by maximality ("by transitions" as we did

"The Trick:" $(\varphi'' \circ \varphi^{-1}) \circ (\varphi \circ \varphi'^{-1})$

Remark (existence):

F embedding $\Rightarrow$ F immersion : preceeding thm implies $\exists \psi = (y^1, \dots, y^d)$ s.t. $(x^1, \dots, x^n)$ is coo around m $\quad x^i = y^i \circ F$

Take $\mathcal{A}_X = \{(F(U), (x^1, \dots, x^n)) \mid U$ open in $N\}$.

$= \{(F(U), (y^1, \dots, y^n) \circ F \mid U$ open in $N\}$.

~~orhood of af (1,0). Since & pieces of that always occur to unify to ...~~ 6

„Therefore": 7

<u>Definition</u>: We call a C^k-submanifold $X \subseteq N$ proper if the submanifold topology equals the topology given by the inclusion $X \subseteq N$.

<u>Examples</u>: $\frac{a}{b} \notin \mathbb{Q}$ $\quad \gamma_{a,b}([0,1])$ not proper

 not proper

$\frac{a}{b} \in \mathbb{Q}$ $\quad \gamma_{ab}([0,1])$ proper

(γ even not emb.)

<u>Examples</u>: $\Gamma : (-1,1) \to \mathbb{R}^2$ $\quad \Gamma(t) = (t, |t|)$

not even immersion

But $\Gamma(-1,1)$ can be given a submanifold structure!

(„invisible smoothing")

(3). Tangent bundles and connections

Připomeňme, že $k \geq 1$ pro všechny C^k-variety. $(a = m)$ 6

Označení: Nechť $F(M, m)$ je množina všech funkcí definovaných na nějakém okolí $m \in M$.

Definice: $\left(\frac{\partial}{\partial x^i}\right)_a f = \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}(\varphi(a))$, $\varphi : U \ni a \to \mathbb{R}^n[u^1, \dots, u^n]$,

Definice: Zobrazení $S : F(M, m) \to \mathbb{R}$ nazveme tečný vektor v bodě m, pokud

(1) $\forall f, g \;\exists U \ni m \; f_{|U} = g_{|U} \Rightarrow S(f) = S(g)$

(2) $S(fg) = S(f)g(m) + S(g)f(m) \quad \forall f, g \in F(M, m)$

(3) $S(af + bg) = a S(f) + b S(g)$

Množinu tečných vektorů v m označíme $T_m M$.

Pozn.: (1) Formálně $f + g$ def. na $D(f) \cap D(g)$. Obdobně pro fg

(2) $(S + T)(f) = S(f) + T(f) \quad \forall S, T \in T_m M, f, g \in F(M, m)$

$(aS)(f) = a S(f)$

Assertion: $\forall m \in M \quad T_m M$ je vekt. prostor.

Proof: Leibniz $(S + T)(fg) = \dots$

dále už snadné!

Theorem: $\forall m \in M^n \quad \dim T_m M = n (= \dim M)$.

Proof: 1) $M = \mathbb{R}^n[u^1, \dots, u^n]$, $a \in \mathbb{R}^n[u^1, \dots, u^n]$

$$\left(\frac{\partial}{\partial u^i}\right)_a f = \frac{\partial f}{\partial u^i}(a)$$ definují tečné vektory

a) $\left\{\left(\frac{\partial}{\partial u^i}\right)_a\right\}_{i=1}^{n}$ jsou l.n.

$$\sum_{i=1}^{n} \lambda^i \left(\frac{\partial}{\partial x^i}\right)_a = 0 \qquad \underline{f^j(x^1,\dots,x^n) = x^j}$$

$$\sum \lambda^i \left(\frac{\partial f^j}{\partial x^i}\right)(a) = 0 \Rightarrow \sum \lambda^i \delta^j_i = \lambda^j = 0 .$$

b) generují. Nechť $U \ni a$ okolí $a \in V \subseteq U$ hvězdicovité podokolí. $D(f) = V$, $x \in V$

$$f(x) = f(a) + \underbrace{f(x) - f(a)} = f(a) + \int_0^1 \frac{d}{dt} f(a^1 + t(x^1 - a^1), \dots, a^n + t(x^n - a^n))\, dt = f(a) + \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a + t(x-a))(x^i - a^i)\, dt = f(a) + \sum_{i=1}^{n} g_i(a,x)(x^i - a^i),$$ kde

$$g_i(a,x) = \int_0^1 \frac{\partial f}{\partial x^i}(a + t(x-a))\, dt$$

Derivace podle parametru $g_i \in C^\infty(V)$, neb f je C^∞ par. x.ivrámci Lebesgueova int.

$t \in T_a \mathbb{R}^n$

$$tf = t\left(f(a) + \sum_{i=1}^{n} g_i(a,x)(x^i - a^i)\right) = 0 + \sum_{i=1}^{n} t(g_i(a,x))(a^i - a^i) + \sum_{i=1}^{n} g_i(a,a)\, t(x^i) = \sum_{i=1}^{n} g_i(a,a)\, t(x^i) =$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)\, t(x^i), \text{ tj. } \boxed{t = \sum_{i=1}^{n} t(x^i) \left(\frac{\partial}{\partial x^i}\right)_a}$$

$$D(\tilde f) = U \quad f := \tilde f_{|V}$$

$$(1) \Rightarrow t(f) = t(\tilde f_{|V}) \Rightarrow t(f) = \sum_{i=1}^{n} \frac{\partial \tilde f_{|V}}{\partial x^i}(a)\, t(x^i) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)\, t(x^i).$$

c) $\underline{\text{variety}}$

$$t f = \sum t\left[g_i(m, \cdot)\right](m^i - m^i) +$$

$$+ \sum g_i(m, m)\, t(\cancel{x^i} - m^i) =$$

$$= \sum g_i(m, m)\, t(x^i) =$$

$$= \sum \frac{\partial f}{\partial x^i}(m^1, \ldots, m^n) \underbrace{t(x^i)}_{\text{čísla}} =$$

$$= \sum t(x^i) \left(\frac{\partial f}{\partial x^i}\right)(m)$$

$$t = \sum_{i=1}^{n} t(x^i) \left(\frac{\partial}{\partial x^i}\right)_m .$$

generují?

---

c) na varietě:

$t : F(M, m) \to \mathbb{R}$, (U, φ) map around m

$t_\varphi : F(\varphi(U), \varphi(m)) \to \mathbb{R}$

$t_\varphi f := t(f \circ \varphi)$ is t.v. around $\varphi(m)$

t_φ is lin.

is Leibniz type:

$$t_\varphi(fg) = t((fg)\circ\varphi) = t((f\circ\varphi)(g\circ\varphi)) =$$

$$= t(f\circ\varphi)(g\circ\varphi)(m) + t(g\circ\varphi)(f\circ\varphi)(m)$$

$$= t_\varphi(f)\, g(\varphi(m)) + t_\varphi(g)\, f(\varphi(m)).$$

$M_\varphi(t) = t_\varphi$ 1) into $T_m M$ ⑨

2) injective $f: M \to \mathbb{R}$

$M_\varphi(\vec{t}_1)f = M_\varphi(\vec{t}_2)f$

$\cancel{g \circ \varphi^{-1}}$

$t_1(f \circ \varphi) = t_2(f \circ \varphi) \quad \forall f = g \circ \varphi \quad \boxed{g = f \circ \varphi^{-1}}$

$t_1 = t_2$ since any fction can be written as comp.

$M_\varphi^{-1}(\vec{t}_{\mathbb{R}^n}) f = t(f \circ \varphi^{-1})$

$(M_\varphi M_\varphi^{-1})(t_{\mathbb{R}^n}) f = \cancel{M_\varphi(t(f \circ \varphi^{-1}))} =$

$$\boxed{\begin{aligned} &= \left(M_\varphi(M_\varphi^{-1})t_{\mathbb{R}^n}\right)(f) = M_\varphi^{-1}(t_{\mathbb{R}^n})(f \circ \varphi) \\ &= t_{\mathbb{R}^n}(f \circ \varphi \circ \varphi^{-1}) = \\ &= t_{\mathbb{R}^n}(f). \end{aligned}}$$

$[(M_\varphi^{-1} \circ M_\varphi)(t)](f) = f.$

Vector fields

Definition: $TM := \bigcup_{m \in TM} T_m M$ tangent bundle (of M)

Rem.: Sometimes, one writes $\dot{\bigcup}_{m \in M} T_m M$, the disjoint union, e.g.
$\dot{\bigcup}_{m \in M} T_m M = \bigcup_{m \in M} \{m\} \times T_m M$. It is not necessary, since our manifolds are not considered as embedded into $\mathbb{R}^k$.

Definition: Any map $X: M \to TM$ such that $X_m := X(m) \in T_m M \ \forall m \in M$ and such that X is smooth in the foll. sense, is called a vector field.

Remark: ① $\forall m \in M$ $X_m \in T_m M$, $\varphi = (x^1, \dots, x^n)$ be a coord. map on U around m. Then $X_m = \sum_{i=1}^{n} a^i(m) \left(\frac{\partial}{\partial x^i}\right)_m$.
X is smooth if for each m, each (U, φ), a^i is smooth.

② Equivalently X is smooth iff $\forall f \in \mathcal{C}^\infty(M)$ $Xf \in \mathcal{C}^\infty(M)$.
This is "our" definition. Proving $1 \Leftrightarrow 2$ is easy.

③ Alternatively, TM can be given a topology.
$\pi: TM \to M$ $\pi(\xi) = m$ iff $\xi \in T_m M$. TM has the initial top. for π, i.e., the smallest (=: coarsest) one such that π is cont. Further, TM can be given a structure of a $\mathcal{C}^\infty$-manif. (induced by the $\mathcal{C}^\infty$-man. str. on M). Suppose, this is done. $X: M \to TM$, $X(m) \in T_m M$ is smooth iff it is smooth as a manifold map. Note $\dim TM = 2 \dim M$.

Example: X a vector field (U, φ), (V, ψ) maps around m.
$X_m = \sum_{i=1}^{n} a^i(m) \left(\frac{\partial}{\partial x^i}\right)_m$; $X_m = \sum_{j=1}^{n} b^j(m) \left(\frac{\partial}{\partial y^j}\right)_m$, $\varphi = (x^1, \dots, x^n)$, $\psi = (y^1, \dots, y^n)$
$\phi = \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$

Definition (Lie bracket of vector fields): For any $X, Y \in \mathcal{X}(M)$ and $m \in M$, let us set $\forall f \in \mathcal{C}^\infty(M)$ $[X,Y]_m f = X_m(Yf) - Y_m(Xf)$.

Remark: $[X,Y]_m$ is called the Lie bracket of v.f. X, Y in point m.

Theorem: For any $X, Y \in \mathcal{X}(M)$, $m \mapsto [X,Y]_m$ is a vector field ~~on~~ M.

Proof: 1) $[X,Y]_m(f+g) = X_m(Y(f+g)) - Y_m(X(f+g)) =$
$= X_m(Yf+Yg) - Y_m(Xf+Xg) = X_m(Yf) + X_m(Yg) -$
$- Y_m(Xf) - Y_m(Xg) = X_m(Yf) - Y_m(Xf) + X_m(Yg) -$
$- Y_m(Xg) = [X,Y]_m f + [X,Y]_m g$.

2) $\lambda \in \mathbb{R}$: $[X,Y]_m(\lambda g) = X_m(Y(\lambda g)) - Y_m(X(\lambda g)) =$
$= \lambda(X_m(Yg)) - \lambda(Y_m(Xg)) = \lambda [X,Y]_m(g)$.

3) $[X,Y]_m(fg) = X_m(Y(fg)) - Y_m(X(fg)) =$
$= X_m((Yf)g + f(Yg)) - Y_m((Xf)g + f(Xg)) =$
$= [X_m(Yf)]g(m) + \cancel{(Yf)(m)(X_m g)} + \cancel{(X_m f)(Yg)(m)} + f(m)X_m(Yg)$
$- [Y_m(Xf)]g(m) - \cancel{(Xf)(m)Y_m(g)} - \cancel{(Y_m f)(Xg)(m)} - f(m)Y_m(Xg)$
$= ([X,Y]_m f)g(m) + ([X,Y]_m g)f(m)$

Thus $[X,Y]_m \in T_m M$ (if we prove that $[X,Y]_m f = [X,Y]_m g$ whenever $f_{|U} = g_{|U}$ for a neighborhood around $m \Leftrightarrow f_{|U} = 0 \Rightarrow [X,Y]_m f = 0$, But this is easy since for such f $X_{|U} f_{|U} = 0 \wedge Y_{|U} f_{|U} = 0$).

4) We prove the smoothness of $m \to [X,Y]_m$ in the remark below.

$\blacksquare$

Remark: Let (U,φ) be a coordinate system around $m \in M^n$ and $X, Y \in \mathfrak{X}(M)$.
There exist $a^i, b^i \in \mathcal{C}^\infty(M)$, $i=1,\dots,n$ such that

$$X_{|U} = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}, \quad Y_{|U} = \sum_{j=1}^{n} b^j \frac{\partial}{\partial x^j}, \text{ where } \varphi = (x^1,\dots,x^n).$$

For $f \in \mathcal{C}^\infty(U)$, let us compute $m \mapsto$

$$\boxed{X_m(Y_{|U} f) - Y_m(X_{|U} f)} = \sum_i a^i(m)\left(\frac{\partial}{\partial x^i}\right)_m \sum_j b^j \frac{\partial f}{\partial x^j} - \sum_j b^j(m)\left(\frac{\partial}{\partial x^j}\right)_m \sum_i a^i \frac{\partial f}{\partial x^i} =$$

$$= \sum_{i,j} a^i(m)\left[\left(\frac{\partial b^j}{\partial x^i}\right)(m)\frac{\partial f}{\partial x^j}(m) + \cancel{b^j(m)\frac{\partial^2 f}{\partial x^i \partial x^j}(m)}\right]$$

$$- \sum_{i,j} b^j(m)\left[\left(\frac{\partial a^i}{\partial x^j}\right)(m)\frac{\partial f}{\partial x^i}(m) + \cancel{a^i(m)\frac{\partial^2 f}{\partial x^j \partial x^i}(m)}\right] = \text{(after renumbering of indices)}$$

$$= \boxed{\sum_{i,j} \underbrace{\left[a^i(m)\frac{\partial b^j}{\partial x^i}(m) - b^i\frac{\partial a^j}{\partial x^i}(m)\right]}_{m \mapsto \text{ is } \in \mathcal{C}^\infty(U) \text{ since } a^i, b^j \in \mathcal{C}^\infty(U)} \left(\frac{\partial}{\partial x^j}\right)_m f}$$

! Note!: We used:

- $X_{|U} f = (Xf)_{|U}$ (Prove/Why?)
- $\frac{\partial^2 f}{\partial x^i \partial x^j}(m) = 2\,|\, \mathcal{C}^\infty(U) \ni f \longmapsto \frac{\partial f}{\partial x^j} \in \mathcal{C}^\infty(U) \longmapsto \left(\frac{\partial}{\partial x^i}\right)_m\left(\frac{\partial f}{\partial x^j}\right) =: \frac{\partial^2 f}{\partial x^i \partial x^j}(m)$
- We use the def: $X \in \mathcal{C}^\infty$ iff $X_{|U} = \sum a^i \frac{\partial}{\partial x^i}$ for any map $\varphi = (x^1,\dots,x^n)$ on U, for any U is a^i smooth ($\mathcal{C}^\infty$).

Notation: The set of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$.

Theorem: Let $X, Y, Z \in \mathfrak{X}(M)$. Then

1) $[X,Y] = -[Y,X]$
2) $[aX+bY, Z] = a[X,Z] + b[Y,Z] \quad \forall a,b \in \mathbb{R}$
3) $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$

Proof: • 1), 2) easy

3) $f \in \mathcal{C}^\infty(M)$: $[[X,Y],Z]_m f + [[Z,X],Y]_m f =$

$$[X,Y]_m(Zf) - Z_m([X,Y]f) +$$
$$+ [Z,X]_m(Yf) - Y_m([Z,X]f) =$$

$= \underline{X_m(Y(Zf))} - Y_m(\cancel{X}(Zf)) - Z_m(\cancel{X}(Yf)) + Z_m(Y(Xf))$

$+ Z_m(\cancel{X}(Yf)) - X_m(Z(Yf)) - Y_m(Z(Xf)) + Y_m(\cancel{X}(Zf)) =$

$[[Y,Z],X]_m f = [Y,Z]_m(Xf) - X_m([Y,Z]f) =$

$= Y_m(Z(Xf)) - Z_m(Y(Xf)) - X_m(Y(Zf)) + X_m(Z(Yf)) =$

(compare the ~~~~, - - - -, symbols!) $\square$

<u>Remark</u>: Thus $(\mathcal{X}(M), [,])$ is a <u>Lie algebra</u> (over the field $k = \mathbb{R}$)

If $\dim M > 0 \Rightarrow \dim \mathcal{X}(M) = +\infty$.

(If $\dim M = 0 \Rightarrow \dim \mathcal{X}(M) = 0 \ldots$)

"generically", $\mathcal{X}(M)$ is an infinite dimensional Lie algebra.

<u>Remark</u>: $(\mathrm{End}(V), [,])$, V vect. sp., $A, B \in \mathrm{End}(V)$, $[A,B] = A \circ B - B \circ A$ a Lie alg. as well

<u>Definition</u>: We call X a coordinate vector field, if there exists a map φ around each point of m such that
$X_{|U} = \frac{\partial}{\partial x^i}$, where $\varphi : U \to \mathbb{R}^n$, $\varphi = (x^1, \ldots, x^n)$, $m \in U$ and $i = 1, \ldots, n$.

<u>Theorem</u>: If X, Y are coordinate vector fields defined on a neighborhood U for a map $\varphi : U \to \mathbb{R}^n$, then $[X,Y] = 0$.

Proof: $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $\varphi = (x^1, \ldots, x^n)$, $\varphi : U \to \mathbb{R}^n$, $m \in U$

$[X,Y]_m f = X_m(Yf) - Y_m(Xf) = X_m \frac{\partial f}{\partial x^j} - Y_m \frac{\partial f}{\partial x^i} =$

$= X_m\left(\frac{\partial(f\circ\varphi^{-1})}{\partial u^j}\circ\varphi\right) - Y_m\left(\frac{\partial(f\circ\varphi^{-1})}{\partial u^i}\circ\varphi\right)$ $\quad \Big|\ \mathbb{R}^n = \mathbb{R}^n[u^1, \ldots, u^n]$

$= \left[\frac{\partial}{\partial u^i}\left(\frac{\partial(f\circ\varphi^{-1})}{\partial u^j}\circ\cancel{\varphi\circ\varphi^{-1}}\right)\right]\varphi(m) - \left[\frac{\partial}{\partial u^j}\left(\frac{\partial(f\circ\varphi^{-1})}{\partial u^i}\circ\cancel{\varphi\circ\varphi^{-1}}\right)\right]\varphi(m)$

$= \frac{\partial^2(f\circ\varphi^{-1})}{\partial u^i \partial u^j}(\varphi(m)) - \frac{\partial^2(f\circ\varphi^{-1})}{\partial u^j \partial u^i}(\varphi(m)) = 0$ by multivariate calculus. $\square$

Definition: Let $U \subseteq M^n$ be an open set. An n-tuple $X_1, \dots, X_n \in \mathfrak{X}(U)$ is called a (local) frame (on U), if $\forall m \in U$ $\{(X_1)_m, \dots, (X_n)_m\}$ is a basis of $T_m U$.

Remark: 1) A local frame $X_1, \dots, X_n \Rightarrow \{(X_1)_m, \dots, (X_n)_m\}$ a repère.

2) Previous thm can be reversed (e.g., by the Frobenius thm on integrability), i.e., $[X,Y]_{|U} = 0$, U open $\Rightarrow$ $\exists V$ open and a map $\psi : V \to \mathbb{R}^n$ such that

$$X_{|V} = \frac{\partial}{\partial x^i} \quad \text{and} \quad Y_{|V} = \frac{\partial}{\partial x^j} \quad \text{for some } i, j.$$

3) $[X,Y] = 0 \overset{\text{def}}{\Longleftrightarrow} X \,\&\, Y$ commutes.

Example: The example will be added in the Addenda at the end of semester. (It's about 'coordinate vector fields)

Remark: X, Y s.t. $[X,Y] \neq 0 \Rightarrow$ they are not coord. v.f. wrt to a single map $\varphi : U \to \mathbb{R}^n$. ~~(if $U \supseteq \operatorname{supp}([X,Y])$) e.g. a superset of a set where $[X,Y] \neq 0$).~~

May be used to solve a class of first order PDEs.

We probably, omit this example from the time reasons.

Differential forms

Definition: ① $\forall m \in M$, $T_m^*M := (T_m M)^*$ is called the cotangent space (bundle) in m. ($T_m^*M := \{\alpha : T_m M \to \mathbb{R} \mid \alpha$ is linear$\}$.) $T^*M = \bigcup T_m^*M$ is called cotangent bundle

② Any $\alpha : M \to T^*M$ is called a diff. 1-form if $\alpha(m) =: \alpha_m \in T_m^*M$ $\forall m \in M$ and α is smooth in the sense $X \in \mathfrak{X}(M) \Rightarrow \alpha(X) \in \mathcal{C}^\infty(M)$. Here,

(!) $[\alpha(X)](m) = \alpha_m(X_m)$.

Remark: ① T^*M can be given a manif. structure. Smoothness of $\alpha : M \to T^*M$ can be expressed by saying α is smooth iff α is smooth as a manif. map.

② Smoothness of ~~the~~ diff. 1-forms is possible to express by local description as well

! ③ (U,φ) a map around m $\underset{\text{know}}{\Longrightarrow}$ $\left\{ \left(\frac{\partial}{\partial x^1}\right)_m, \dots, \left(\frac{\partial}{\partial x^n}\right)_m \right\}_x$ is a basis of $T_m M$. Let $(\varepsilon^i)_{i=1}^n$ be dual to. Then $\boxed{\alpha_{|U} = \sum_{i=1}^n a_i \varepsilon^i}$ for suitable a_i. If these are $\mathcal{C}^\infty$ $\Rightarrow$ α is smooth (for any U).

More satisfactory is to use the following notion.

Definition: (more compact def. than at the lecture) For any $f \in \mathcal{C}^\infty(M)$ and $m \in M$, we set $(df)_m t = t(f)$ for $t \in T_m M$. The map $m \longmapsto (df)_m$, $m \in M$, is called the (de Rham) differential of f.

Assertion: For any $f \in \mathcal{C}^\infty(M)$, df is a differential 1-form.

Proof: ① $(df) : m \mapsto (df)_m \in T_m^*M$?

- $(df)_m$ is a function on $T_m M$ („not $T_n M$, $n \neq m$")

- $(df)_m(t+s) = (t+s)(f) = tf + sf = (df)_m t + (df)_m s$

linear (additive) def of mult of (tang) vectors

- $(df)(at) = (at)f \overset{\swarrow}{=} a(t(f)) = a(df(t))$

$\mathbb{R}$-linear

② smoothness: $X \in \mathfrak{X}(M)$

$[(df)(X)](m) = (df)_m X_m = X_m f$. Since X is smooth, $m \mapsto X_m f$ is smooth for any $\mathcal{C}^\infty$-function f (around/on M).

Remark: ① $X : M \to TM \rightsquigarrow \tilde{X} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ (we often omit the ~)

$(\tilde{X} f)(m) = X_m f \quad \forall m \in M$

② $\alpha : M \to T^*M \rightsquigarrow \tilde{\alpha} : \mathfrak{X}(M) \to \mathcal{C}^\infty(M)$ (—||—)

$[\tilde{\alpha}(X)](m) = \alpha_m(X_m) \quad \forall m \in M$

③ The "induced" structures (:= used to define the smoothness) are ~~basi~~ equivalent to the original ones in the sense of the next theorem. Before we formulate it, let us set:

④ The space of diff. (1-)forms is denoted by $\Omega^1(M)$. It is equipped with the 'obvious' vect. sp. str.: $(\alpha+\beta)(m) = \alpha_m + \beta_m$ $(= \alpha(m) + \beta(m))$

$(\lambda\alpha)(m) = \lambda\alpha(m)$. It is an $\mathbb{R}$-vect. spc.

We may set $(f\alpha)(m) = f(m)\alpha_m \quad \forall f \in \mathcal{C}^\infty(M)$

as well. Then $\Omega^1(M)$ is made in a module of the ring $\mathcal{C}^\infty(M)$ $\left[\begin{array}{l}(f+g)(m) = f(m) + g(m) \\ (fg)(m) = f(m)g(m)\end{array}\right]$

Assertion / Observation: $T_m^* M = \mathcal{L}(\{(dx^1)_m, \dots, (dx^n)_m\})$. 8

! ((finite) linear combination), where $\varphi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ is a map.

All symbols already defined! Moreover, $\{(dx^1)_m, \dots, (dx^n)_m\}$ is a basis.

Proof: linear indep. $\sum_{i=1}^{n} \lambda_i (dx^i)_m = 0$, $\lambda_i \in \mathbb{R}$

$\forall j = 1, \dots, n$ $\left(\frac{\partial}{\partial x^j}\right)_m$ (insert): $\left\{\sum_{i=1}^{n} \lambda_i (dx^i)_m\right\} \left(\frac{\partial}{\partial x^j}\right)_m = 0$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i (dx^i)_m \left(\frac{\partial}{\partial x^j}\right)_m = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x^j}\right)_m x^i = 0$$

$$\Rightarrow \sum \lambda_i \delta^i_j = 0 \Rightarrow \lambda_j = 0 \ \forall j.$$

We used $\left(\frac{\partial}{\partial x^j}\right)_m x^i = \left[\frac{\partial}{\partial u^j}(x^i \circ \varphi^{-1})\right](\varphi(m)) =$ constant Kronecker-δ

$= \frac{\partial}{\partial u^j}\left[pr^i \circ \varphi \circ \varphi^{-1}\right](\varphi(m)) = \left(\frac{\partial}{\partial u^j} pr^i\right)(\varphi(m)) = \delta^i_j(\varphi(m)) =$

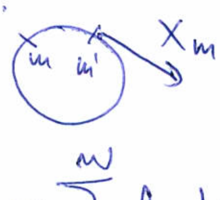
$= 1 \quad i = j$

$= 0 \quad i \neq j$

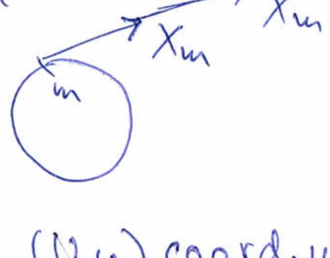
generates: from dimensional reasons (e.g.)

Remark: ① $m \in S^1 \Rightarrow X \in \mathfrak{X}(S^1) \rightsquigarrow X_m \in T_m S^1$

Not:



Rather



Local descript. ② $\alpha_{|U} = \sum_{i=1}^{n} f_i \, dx^i$ $\forall \alpha \in \Omega^1(M)$, (U, φ) coord. map, $\varphi = (x^1, \dots, x^n)$ since $(\alpha_{|U})_m = \sum_{i=1}^{n} \underbrace{a_i(m)}_{\in \mathbb{R}} (dx^i)_m$, set $f_i : m \mapsto a_i(m)$, $m \in M$.

! $\alpha_{|U}\left(\frac{\partial}{\partial x^j}\right)_{(|U)} = \sum_{i=1}^{n} f_i \, dx^i\left(\frac{\partial}{\partial x^j}\right) = f_j$ (defines f_j)

Smooth tensor fields

$$T^r_s M = \bigcup_{m\in M}\Big[\underbrace{T^*_m M\otimes\cdots\otimes T^*_m M}_{s\text{-times}}\otimes\underbrace{(T_m M\otimes\cdots\otimes T_m M)}_{r\text{-times}}\Big]$$

tensor bundle.

Tensor field of type (r,s): $T: M \to T^r_s M$, $m \mapsto T(m)$ $=: T_m \in \underbrace{(T^*_m M\otimes\cdots\otimes T^*_m M)}_{s\text{-times}}\otimes\underbrace{(T_m M\otimes\cdots\otimes T_m M)}_{r\text{-times}}$ s.t.

it is smooth in the sense:

$$(*)\quad \tilde{T}(\underbrace{X_1,\dots,X_s}_{\in\mathfrak{X}(M)},\underbrace{\alpha_1,\dots,\alpha_r}_{\in\Omega^1(M)})(\underbrace{m}_{\in M}) = T((X_1)_m,\dots,(X_s)_m,(\alpha_1)_m,\dots,(\alpha_s)_m)$$

has to be smooth function for any smooth X_i, α_j, $i=1,\dots,s$, $j=1,\dots,r$.

Remark: ① Again $T^r_s M$ can be made a manifold (of dimension n^{r+s}).

② Note $T^1_1 M$ is not $T^*M\otimes TM$ e.g. because TM and T^*M is not a vector space (at least for $M\neq\emptyset$ and $M\neq\{pt\}$). Even $T^1_1 M \neq T^*M\otimes T M$ locally! (Again, what does it mean.) $T^1_1 M = \bigcup_{m\in M}(T^*_m M\otimes T_m M)$

Theorem: If $T: M\to T^r_s M$ is a tensor field, then the induced $\tilde{T}: \underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{s}\times\underbrace{\Omega^1(M)\times\cdots\times\Omega^1(M)}_{r}\to\mathcal{C}^\infty(M)$ is $\mathcal{C}^\infty(M)$-linear and vice-versa, i.e., if $T:\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)\times\Omega^1(M)\times\cdots\times\Omega^1(M)\to\mathcal{C}^\infty(M)$ is $\mathcal{C}^\infty(M)$-linear, it defines a tensor field such that the 'induced' tensor field by formula $(*)$ is the tensor field T.

Proof: 1) The first part is easy, e.g.: $[\tilde{T}(fX_1,\dots,X_s,\alpha_1,\dots,\alpha_r)](m)$

$= T_m((fX_1)_m,\dots,(X_s)_m,(\alpha_1)_m,\dots,(\alpha_r)_m) =$

$= T_m(f(m)(X_1)_m,\dots,(X_s)_m,(\alpha_1)_m,\dots,(\alpha_r)_m) =$

$= f(m)T_m((X_1)_m,\dots,(X_s)_m,(\alpha_1)_m,\dots,(\alpha_r)_m) \Rightarrow$

$\tilde{T}(fX_1,\dots,X_s,\alpha_1,\dots,\alpha_r) = f\,T(X_1,\dots,X_s,\alpha_1,\dots,\alpha_r)$,

where again $(fT)(m) = f(m)T(m)$ (point-wise mult on tensor fields; it makes the set of tensor fields a $C^\infty(M)$-module „bfw").

The opposite direction

2) T is $C^\infty(M)$-linear. (suppose)

a) locallity: $T(X_1,\dots,X_i,\dots,\alpha_1,\dots,\alpha_r)_{|U} = T(X_1,\dots,Y_i,\dots,\alpha_1,\dots,\alpha_r)_{|U}$, whenever $(X_i)_{|U} = (Y_i)_{|U}$. For it:

Let $m \in U$, $f(m) = 0$ and $f = 1$ outside of U

It is sufficient $X_{i|U} = 0 \Rightarrow T(X_1,\dots,X_i,\dots\alpha_1,\dots,\alpha_s)_{|U} = 0$.

We have $\underset{(=0)}{fX_{i|U}} = \underset{(=0)}{X_{i|U}}$ & $fX_i = X_i$ outside of U.

$T(X_1,\dots,X_i,\dots,\alpha_1,\dots,\alpha_r)(m) = T(X_1,\dots,fX_i,\dots,\alpha_1,\dots,\alpha_r)(m)$

$= [f\,T(X_1,\dots,X_i,\dots,\alpha_1,\dots,\alpha_r)](m) = \underset{=0}{f(m)}\,T(\dots\dots)(m) = 0$

b) point-wise behaviour: $(X_i)_m = (Y_i)_m \Rightarrow T(X_1,\dots,X_i,\dots)(m) = 0$.

By linearity, it is sufficient $(X_i)_m = 0 \Rightarrow T(X_1,\dots,X_i\dots)(m) = 0$

For a map $\varphi: U \to \mathbb{R}^n$ around m with $\varphi = (x^1,\dots,x^n)$,

$X_{i|U} = \sum_{j=1}^{n} f^j \frac{\partial}{\partial x^j}$ for suitable $f^j \in C^\infty(U)$. Let Y_j be a smooth extension of $\frac{\partial}{\partial x^j}$ and g^j a smooth extension of f^j. We have $X_{i|U} = (\sum_{j=1}^{n} g^j Y_j)_{|U}$. By proved locallity

$$T(X_1,\dots,X_i,\dots)(m) = T\Big(X_1,\dots,\sum_{j=1}^{n} g^j Y_j,\dots\Big)(m) =$$

$$= \sum_{j=1}^{n} g^j(m)\, T(X_1,\dots,Y_j,\dots)(m) = \sum_{j=1}^{n} 0\; T(\dots)(m) = 0$$

$\mathcal{C}^\infty(M)$-lin and plugging in the point

since $g^j(m) = f^j(m) = 0$.

$\square$

Remark: 1) Existence of f  and Y_j and g^j, $j=1,\dots,n$, follows from the so-called partition of unity. (See Kowalski or Spivak for instance.)

2) We do not dist. smooth tensor fields of type (r,s) and $\mathcal{C}^\infty(M)$-maps $\underbrace{\mathcal{X}(M)\times\dots\times\mathcal{X}(M)}_{s}\times\underbrace{\Omega^1(M)\times\dots\times\Omega^1(M)}_{r} \to \mathcal{C}^\infty(M)$.

3) Diff. 1-forms $\equiv \alpha : \mathcal{X}(M) \to \mathcal{C}^\infty(M)$ $\quad\mathcal{C}^\infty(M)$-lin.

$\equiv$ $(0,1)$-tensor fields

Vector fields $\equiv \tilde{X} : \Omega^1(M) \to \mathcal{C}^\infty(M)$ by

$\tilde{X}(\alpha)(m) = \alpha_m(X_m)$

$(1,0)$-tensor fields

4) The last theorem is one of the fundamental theorems of tensor calculus.

Push-forwards of tangent vectors and pull-backs of 1-forms

Definition: $\phi : M \to N$ be a $\mathcal{C}^\infty$-map and $m \in M$. Then $\phi_{*m} : T_m M \to T_{\phi(m)} N$ is defined by $(\phi_{*m} t) f = t(f \circ \phi)$ $\forall t \in T_m M$ $\forall f \in \mathcal{F}(m, M)$.

Remark: The definition is correct, i.e. $\phi_{*m} t$ is a tangent vector (HW we did it!)

Assertion: For any $m \in M$ 1) $Id_{*m} = Id_{T_m M}$

2) $\forall \phi : M \to N, \psi : N \to L$ $\mathcal{C}^\infty$-maps $\forall m \in M$:

$$(\psi \circ \phi)_{*m} = \psi_{*\phi(m)} \circ \phi_{*m}$$

Proof: ① Easy (HW)

② $[(\psi \circ \phi)_{*m} t)] f = t(f \circ \psi \circ \phi) = (\phi_{*m} t)(f \circ \psi) = [\psi_{*\phi(m)}(\phi_{*m} t)] f$

$\forall f \in \mathcal{F}(\psi(\phi(m)), L)$, $\forall t \in T_m M$.

Consequence: $\phi : M \to N$ diffeomorphism $\Rightarrow \phi_{*m}$ is isomorphism $\forall m \in M$.

Proof: 1) $\phi_{*m} : T_m M \to T_{\phi(m)} N$ is linear $(t_1, t_2 \in T_m M, \alpha, \beta \in \mathbb{R}, \dots)$

2) $\left.\begin{array}{l} \phi^{-1} \circ \phi = Id \stackrel{prev.}{\Rightarrow} (\phi^{-1})_{*\phi(m)} \circ \phi_{*m} = Id_{T_m M} \\ \phi \circ \phi^{-1} = Id \stackrel{prev.}{\Rightarrow} \phi_{*\phi^{-1}(\phi(m))} \circ \phi^{-1}_{*\phi(m)} = Id_{T_{\phi(m)} M} \end{array}\right\} \Rightarrow$

$\phi_{*m} : T_m M \to T_{\phi(m)} M$ is an isomorphism. $\square$

Definition: Let $\phi : M \to N$, $m \in M$. We define the $\mathrm{rank}_m \phi$ to be $\mathrm{rank}\, \phi_{*m}$.

Assertion: $\mathrm{rank}_m(\psi \circ \phi \circ \varphi^{-1})$ does not depend on the maps $\psi : N \to \mathbb{R}^n$ and $\varphi : M \to \mathbb{R}^l$.

Proof: In the assertion, $m \in \mathbb{R}^n$.

$(\psi \circ \phi \circ \varphi^{-1})_{*m} = \psi_{*\phi(\varphi^{-1}(m))} \circ \phi_{*\varphi^{-1}(m)} \circ (\varphi^{-1})_{*m}$ and

ψ_{*x} and φ^{-1}_{*y} are isomorphisms by the above consequence. $\square$

Remark!: ϕ_{*m} is called the push-forward in m.

Definition: Let $\phi: M \to N$ be a diffeomorphism and $X \in \mathfrak{X}(M)$. 13
Then $(\phi_* X)(n) = \phi_{*\phi^{-1}(n)} X_{\phi^{-1}(n)} \quad \forall n \in N$

Remark: 1) Obviously $\phi_* X \in \mathfrak{X}(N)$; We write also $(\phi_* X)_n$ as usual.

2) If ϕ is not a diffeo, the definition of ϕ_* does not work. At least, it needn't produce a vector field on N.

3) $\boxed{*m}$ $\to$ reminds on 'covariant functors.

Definition: If $\phi: M \to N$ is a diffeomorphism, $m \in M$ and $\alpha \in T^*_{\phi(m)} N$, then $(\phi^*_m \alpha) \in T^*_m M$ is defined via

$$(\phi^*_m \alpha)(t) = \alpha(\phi_{*\phi^{-1}(m)} t) \quad \forall t \in T_m M.$$

Remark: $\phi^*_n \alpha$ is called the pull-back of α in n (to $\phi^{-1}(n)$).

Assertion: For $\phi: N \to L$ and $\psi: M \to N$ diffeo's,

$$(\phi \circ \psi)^*_m = \psi^*_m \circ \phi^*_{\phi(m)}$$

Proof: Similarly as for pull-backs. More precisely, evaluate on tang. vector and use the theorem of push-forwards of the composition.

Remark: 1) $\boxed{\overset{*}{m}}$ reminds on contra-variant tensors.

2) One can pull-back not only 1-forms, but also differential 1-forms. Again $\phi: M \to N$ diffeo,

$$(\phi^* \alpha)(m) = \phi^*_{\phi(m)} \alpha_{\phi(m)} \quad \text{for } \alpha \in \Omega^1(N)$$

Some formulas

(1) $\boxed{X(f\circ\phi) = (\phi_* X)(f)\circ\phi}$ $\forall f\in\mathcal{C}^\infty(M)$ $\forall\phi$ diffeo, $\underset{M\to N}{}$ $X\in\mathfrak{X}(M)$

Proof: $[X(f\circ\phi)](m) = X_m(f\circ\phi) = (\phi_{*m}X_m)f$

$[(\phi_* X)(f)\circ\phi](m) = [(\phi_* X)f](\phi(m)) = [\phi_{*m}(X_m)]f.$

Also: $\phi_* X = \phi_*\circ X\circ\phi^{-1}$

(2) $\phi_*([X,Y])(f) \overset{(1)}{=} [[X,Y](f\circ\phi)]\circ\phi^{-1} =$

$[X(\underbrace{Y(f\circ\phi)}) - Y(X(f\circ\phi))]\circ\phi^{-1} =$

$= [X(\underbrace{Y(f\circ\phi)\circ\phi^{-1}}\circ\phi) - Y(X(f\circ\phi)\circ\phi^{-1}\circ\phi]\circ\phi^{-1} =$

$\overset{(1)}{=} [X(\underbrace{(\phi_* Y)(f)}\circ\phi) - Y((\phi_* X)(f)\circ\phi)]\circ\phi^{-1} =$ $\qquad$ def of $[,]$ & $\phi\circ\phi^{-1} = Id$

$\overset{(1)}{=} [\phi_* X((\phi_* Y)(f)\circ\phi) - (\phi_* Y)((\phi_* X)(f)\circ\phi)]\circ\phi^{-1} \Longleftarrow$

$= [\phi_* X, \phi_* Y](f)$. Thus, $\boxed{\phi_*[X,Y] = [\phi_* X, \phi_* Y]}$

(3) a) $\phi_*(fX)_{\phi(m)} = \phi_{*m}(fX)_m = \phi_{*m}[f(m)X_m] = f(m)\,\phi_{*m}(X_m)$

b) $[(f\circ\phi^{-1})(\phi_* X)]_{\phi(m)} = f(\phi^{-1}(\phi(m)))\,\phi_{*m}(X_m) = f(m)\phi_{*m}(X_m)$

Thus, $\boxed{\phi_*(fX) = (f\circ\phi^{-1})\,\phi_* X}$

(4) $(\phi^*_m(dg)_{\phi(m)})(t) = (dg)_{\phi(m)}(\phi_{*m}t) = (\phi_{*m}t)g =$

$= t(g\circ\phi) = [d(g\circ\phi)]_m t \Rightarrow$

$\boxed{\phi^*_m(dg)_{\phi(m)} = d(g\circ\phi)_m}$ $\quad$ Any $g\in\mathcal{C}^\infty(N)$ $(t\in T_m M)$

Curves and their tangent fields

Let $\gamma : I \to M$ be a C^∞-map, $I = (a,b) \subseteq \mathbb{R}$ open. γ is called a curve.

Definition: $\frac{d\gamma}{dt}(t_0) := \gamma_{*t_0}\left(\frac{d}{du}\right)_{t_0}$, $\mathbb{R}[u] \supseteq I$, $t_0 \in I$.

This vector is called the tangent vector to γ at t_0 (ocassionally, at $\gamma(t_0)$).

Remark: 1) Let $m \in \operatorname{Im}\gamma$, $m = \gamma(t_0)$ and $\varphi = (x^1, \ldots, x^n)$ be a map on $U \subseteq M$ around m. Denote by $\gamma^i : I \subseteq \mathbb{R}[u] \to \mathbb{R}^n$ the composition $x^i \circ \gamma$. Then

$$\frac{d\gamma}{dt}(t_0) = \sum_{i=1}^{n} \frac{d\gamma^i}{du}(t_0)\left(\frac{\partial}{\partial x^i}\right)_{\gamma(t_0)}.$$

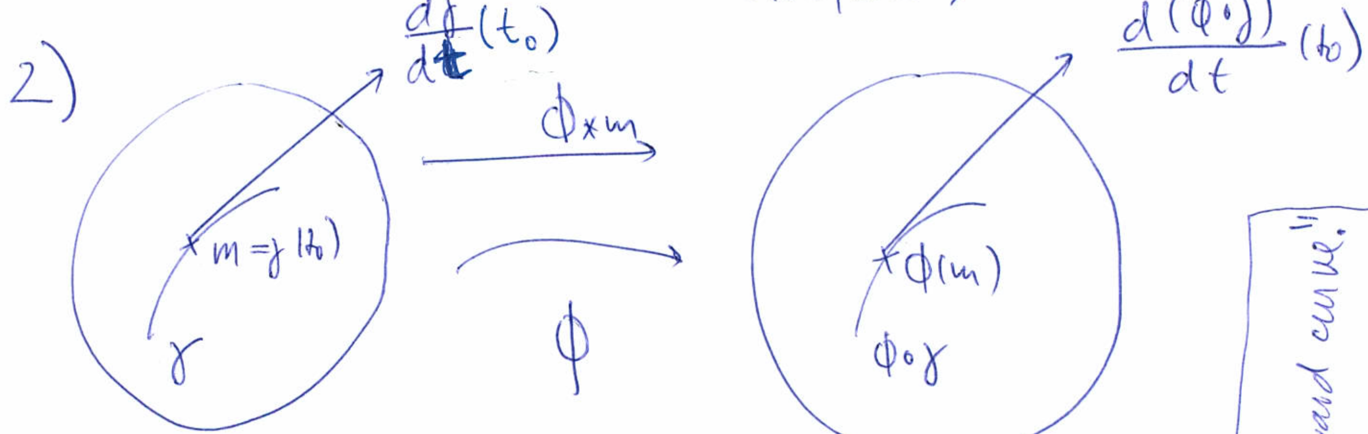
Proof:

LHS $\frac{d\gamma}{dt}(t_0) f = \left[\gamma_{*t_0}\left(\frac{d}{du}\right)_{t_0}\right] f = \left(\frac{d}{du}\right)_{t_0}(f \circ \gamma) =$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\gamma(t_0)) \frac{d\gamma^i}{du}(t_0)$$

by the manifold chain rule, which readily implies (is equal) the RHS.

2)



$$\phi_{*m}\left(\frac{d\gamma}{dt}(t_0)\right) \overset{\text{def}}{=} \phi_{*\gamma(t_0)} \circ \gamma_{*t_0}\left(\frac{d}{du}\right)_{t_0} \overset{\text{thm}}{=} (\phi \circ \gamma)_{*t_0}\left(\frac{d}{du}\right)_{t_0} \overset{\text{def}}{=} \frac{d(\phi \circ \gamma)}{dt}(t_0)$$

: "Push-forward of tangent vector is the ~~tangent~~ tangent vector of the pushed-forward curve."

4. Connections and covariant derivatives

<u>Definition</u>: Let M be a C^∞-manifold. We call a map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ an <u>affine connection</u> (on M) if

1) ∇ is $\mathbb{R}$-linear in both arguments
2) $\nabla_{fX} Y = f \nabla_X Y \quad \forall X, Y \in \mathcal{X}(M) \; \forall f \in C^\infty(M)$ and
3) $\nabla_X (fY) = (Xf) Y + f \nabla_X Y \quad \forall X, Y \in \mathcal{X}(M) \; \forall f \in C^\infty(M)$

Here, $\nabla_X Y := \nabla(X, Y)$.

<u>Remark</u>: Smoothness of ∇ "assured" by $\xrightarrow{\nabla} \mathcal{X}(M)$ (into C^∞-vector fields). We mean smoothness in the sense
$X, Y \in \mathcal{X}(M) \Rightarrow \nabla_X Y \in \mathcal{X}(M)$.

<u>Proposition</u>: Let ∇ be an affine connection on M. Then for any $X, Y \in \mathcal{X}(M)$ and $m \in M$, $(\nabla_X Y)(m)$ depends on X_m and $Y_{|U}$ in an arbitrary neighborhood U of m.

Proof: 1) The 'point-wise' property in X follows from 1) & 2) by the fundam. thm. of tensor calculus ("above")

2) Locality in Y. Let $Y_{|U} = 0$ for some $U \ni m$.
Choose (partition of unity): $f \equiv 1$ on $M \setminus U$, $f(m) = 0$

Then $fY = Y$. Compute

$$(\nabla_X Y)(m) = (\nabla_X fY)(m) = (Xf)(m) \underset{\overset{\shortparallel}{0}}{Y_m} + \underset{\overset{\shortparallel}{0}}{f(m)} (\nabla_X Y)(m)$$
$$= 0 \qquad \square$$

Remark: If $\dim M > 0$, 3) says that ∇ is not a $((1,2))$-tensor field.

Definition: Let $\gamma : I \to M$ be a $\mathcal{C}^\infty$-map (I open int. in $\mathbb{R}$), i.e., a curve. Then $V : I \to TM$ is called a vector field along γ, if $\exists \tilde{V} \in \mathcal{X}(M)$ s.t. $V(t) = \tilde{V}_{\gamma(t)}$ $\forall t \in I$.

Definition: Let ∇ be an affine connection on M^n and $e = (e_1, \dots, e_n)$ be a local frame on $U \subseteq M$. Then the functions $\Gamma_{ij}^k : U \to \mathbb{R}$, $i, j, k = 1, \dots, n$, uniquelly defined by

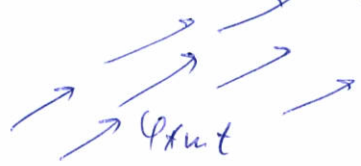
$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \Gamma_{ij}^k e_k$$

are called the Christoffel functions (symbols / coefficients) w.r. to e on U.

Definition: For any $t \in T_m M$, $m \in M$ and $X \in \mathcal{X}(M)$, we set $\nabla_t X = (\nabla_Y X)(m)$, where $Y \in \mathcal{X}(M)$ s.t. $Y_m = t$. The map $\nabla : (t, X) \mapsto \nabla_t X$ is called the covariant derivative (of X)

Proposition: $\nabla_t X$ is well defined.

Proof: 1) Does Y exist? (U, φ), $m \in U$, $\varphi_{*m} t \in \mathbb{R}^n$, $n = \dim M$. Do the „parallel" transport of $\varphi_{*m} t$ in the whole $\mathbb{R}^n$



. Then push it forward by φ^{-1} to $U \subseteq M$. ~~again~~

2) Does $\nabla_t X$ depends on Y? No!, by the previous proposition item 1, $\nabla_Y X$ depends on Y_m only. $\square$ 18

in m

<u>Definition</u>: Let V be a vector field along $\gamma : I \to M$.

We set $\frac{DV}{du}(t_0) = \nabla_{\frac{d\gamma}{du}(t_0)} \tilde{V}$, where $\tilde{V}$ is any "extension" of V in the sense $\tilde{V}_{\gamma(t)} = V_t$ $\forall t \in I$ and $\tilde{V} \in \mathfrak{X}(M)$. We call it the cov. derivative of V along γ.

<u>Remark</u>: 1) $\nabla_{\frac{d\gamma}{du}(t_0)} \tilde{V}$ is well defined (when $\tilde{V}$ is given) according to the previous proposition.

2) Let us set $\dot{\gamma}^i(t_0) = \frac{d\gamma^i}{du}(t_0)$, where $\gamma^i = x^i \circ \gamma$ for a chosen coordinate system (U, φ), $\varphi = (x^1, \ldots, x^n)$

3) $(DV/dt)(t_0)$ instead of $(DV/du)(t_0)$ is also „OK" (symbol)

<u>Proposition</u> (correctness of covar. deriv. along a curve):

For any M^n, ∇, γ and $t_0 \in I$ as above, $\frac{DV}{du}(t_0)$ is well defined.

Proof: $\tilde{V}$ an extension of V, i.e., $\tilde{V}_{\gamma(t)} = V_t$, $m \in U$, $\varphi : U \to \mathbb{R}^n$, $\Gamma_{ij}^k : U \to \mathbb{R}$ Christoffel of ∇ wrt $e = \left(\frac{\partial}{\partial x^i}\right)_{i=1}^n$, $\varphi = (x^1, \ldots, x^n)$,

$\tilde{V}_{|U} = \sum_{i=1}^n \tilde{V}^i \left(\frac{\partial}{\partial x^i}\right)_m$. We omit the existence of $\tilde{V}$!

$$\nabla_{\frac{d\gamma}{du}(t_0)} \tilde V = \sum_{i=1}^n \nabla_{\frac{d\gamma}{du}(t_0)} \tilde V^i \left(\frac{\partial}{\partial x^i}\right) = \sum_{i=1}^n \nabla_{\sum_{j=1}^n \dot\gamma^j(t_0)\left(\frac{\partial}{\partial x^j}\right)_{\gamma(t_0)}} \tilde V^i \frac{\partial}{\partial x^i} =$$

locality of ∇ in upper arg. (prop. above) + linearity of ∇

Notation for $\dot\gamma^j(t_0)$ and formula in the 'curve'-part of the lecture

linearity o ∇

$$= \sum_{i,j=1}^n \dot\gamma^j(t_0)\, \nabla_{\left(\frac{\partial}{\partial x^j}\right)_{\gamma(t_0)}} \tilde V^i \frac{\partial}{\partial x^i}$$

correctness of ∇X

$$= \sum_{i,j=1}^n \dot\gamma^j(t_0)\left(\nabla_{\frac{\partial}{\partial x^j}} \tilde V^i \frac{\partial}{\partial x^i}\right)(\gamma(t_0))$$

Leibniz, i.e., 3) of ∇

$$= \sum_{i,j=1}^n \dot\gamma^j(t_0)\left[\left(\frac{\partial}{\partial x^j}\right)_{\gamma(t_0)} \tilde V^i \left(\frac{\partial}{\partial x^i}\right)_{\gamma(t_0)} + \tilde V^i(\gamma(t_0))\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}\right)(\gamma(t_0))\right]$$

def of Γ^k_{ij}

$$= \sum_{i,j=1}^n \dot\gamma^j(t_0)\left[\left(\frac{\partial}{\partial x^j}\right)_{\gamma(t_0)} \tilde V^i \left(\frac{\partial}{\partial x^i}\right)_{\gamma(t_0)} + \overset{V^i}{\overset{\parallel}{\tilde V^i(\gamma(t_0))}}\, \Gamma^k_{ji}(\gamma(t_0)) \left(\frac{\partial}{\partial x^k}\right)_{\gamma(t_0)}\right]$$

$$= \sum_{i=1}^n \left[\frac{d(V\circ\gamma)^i}{dt}(t_0)\left(\frac{\partial}{\partial x^i}\right)_{\gamma(t_0)} + \sum_{j=1}^n \dot\gamma^j(t_0)\, V^i(t_0) \sum_{k=1}^n \Gamma^k_{ji}(\gamma(t_0)) \left(\frac{\partial}{\partial x^k}\right)_{\gamma(t_0)}\right]$$

chain rule on manif. or push-forwards (via curve e.g.)

Thus $\frac{DV}{du}(t_0)$ does not depend on the extension $\tilde V$ of V. $\square$

<u>Remark:</u> (∇) $\frac{D\gamma}{du}(t_0) = \sum_{i=1}^n \left(\frac{d(V\circ\gamma)^i}{dt}(t_0)\left(\frac{\partial}{\partial x^i}\right)_{\gamma(t_0)}\right) + \sum_{j,k=1}^n \dot\gamma^j(t_0)\, V^i(t_0)\, \Gamma^k_{ji}(\gamma(t_0)) \frac{\partial}{\partial x^k}$

<u>Definition:</u> 1) $X \in \mathfrak{X}(M)$ is called parallelly transported along $\gamma: I \to M$ if $\frac{D\hat{X}}{dt}(t_0) = 0 \; \forall t_0 \in I$, where $\hat{X} = X \circ \gamma$.

2) γ is called a geodesics if the vector field of its tangent vectors is parallelly transported along γ.

<u>Remark:</u> 1) A curve is called a geodesics if is {somehow} "parallel along itself".

2) More precisely, we mean $\frac{D\frac{d\gamma}{du}}{du}(t_0) = 0$ (as the def. of geodesics). In order "not to speak about extensions and and restrictions".

<u>Theorem</u>: Let (U, φ), $\varphi = (x^1, \dots, x^n)$, be a coordinate system around a point $m \in M$. For $\gamma: I \to U$ to be a geodesics it is necessary and sufficient that for all $t \in I$,

$$\ddot{\gamma}^i(t) + \sum_{j,k=1}^{n} \Gamma^i_{jk}(\gamma(t))\, \dot{\gamma}^j(t)\, \dot{\gamma}^k(t) = 0$$

for each $i = 1, \dots, n$.

Proof: Follows from the abreviation $\dot{\gamma}^j(t)$ and the formula ∇. $\square$

Torsion & curvature fields of an affine connection

<u>Definition</u>: M $\mathcal{C}^\infty$-manifold, ∇ affine connection on M.

$$T^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$R^\nabla(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

<u>Theorem</u>: T^∇ and R^∇ are tensor fields of type $(1,2)$ and $(1,3)$, respectively.

<u>Proof</u>: We use the thm on tens. fields $\longleftrightarrow$ $\mathcal{C}^\infty(M)$-linearity

1) T^∇ easy (HW)

2) $R^\nabla(X,Y)Z = -R^\nabla(Y,X)Z$ thus sufficient:

a) $R^\nabla(fX,Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z =$

$= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{[fX,Y]} Z =$

$= \underline{f\nabla_X\nabla_Y Z} - (Yf)\nabla_X Z - \underline{f\nabla_Y\nabla_X Z} - \underline{\nabla_{f[X,Y]} Z} + \nabla_{(Yf)X} Z =$

$= fR^\nabla(X,Y)Z - (Yf)\nabla_X Z + (Yf)\nabla_X Z = fR^\nabla(X,Y)Z$

[used $[fX,Y] = f[X,Y] - (Yf)X$]

b) $R^\nabla(X,Y)(fZ) = \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) =$

$= \nabla_X((Yf)Z + f\nabla_Y Z) - \nabla_Y((Xf)Z + f\nabla_X Z)$

$-([X,Y]f)Z - f\nabla_{[X,Y]} Z = (X(Yf))Z + \cancel{(Yf)\nabla_X Z}$

$+ \cancel{(Xf)\nabla_Y Z} + f\nabla_X\nabla_Y Z - (Y(Xf))Z - \cancel{(Xf)\nabla_Y Z} -$

$- \cancel{(Yf)\nabla_X Z} - f\nabla_Y\nabla_X Z - ([X,Y]f)Z - f\nabla_{[X,Y]} Z =$

$= f \nabla_X \nabla_Y Z - f \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z = f R^\nabla(X,Y)Z$. 2

Summing-up (a) & b)), R^∇ is a tensor field. $\square$

Remark: T^∇ is called the torsion field of ∇.

R^∇ is called the curvature field of ∇.

Definition: $D : \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)$

$$D_X Y = \sum_{i=1}^n X(Y^i) \frac{\partial}{\partial u^i}, \text{ where } Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial u^i},$$

is called the Euclidean connection.

Remark: In the above definition, we regard $\mathbb{R}^n = \mathbb{R}^n[u^1, \ldots, u^n]$, where u^i are std. coordinates, i.e., $u^i = pr^i$, $i = 1, \ldots, n$.

Remark: Definition of Euclidean connection is correct.

Proof: 1) $\mathbb{R}$-lin. is obvious

2) $$D_{fX} Y = \sum_{i=1}^n (fX)(Y^i) \frac{\partial}{\partial u^i} = \sum_{i=1}^n f X(Y^i) \frac{\partial}{\partial u^i} =$$

$$= f \sum_{i=1}^n X(Y^i) \frac{\partial}{\partial u^i} = f D_X Y$$

3) $$D_X(fY) = \sum_{i=1}^n X(fY^i) \frac{\partial}{\partial u^i} = (Xf) \sum_{i=1}^n Y^i \frac{\partial}{\partial u^i} + f \sum_{i=1}^n X(Y^i) \frac{\partial}{\partial u^i}$$

$$= (Xf) Y + f D_X Y. \quad \square$$

(4) Riemannian connections and curvature tensors

<u>Definition</u>: Any (0,2)-tensor field g such that $\forall m \in M$ $g_m \in T_m^* M \otimes T_m M$ is symmetric and positive-definite ~~####~~ is called a <u>Riemannian metric.</u>

<u>Remark</u>: Riemannian metrics exist on any mflds by the decomposition of unity (e.g.).

<u>Remark</u>: If we demand g_m to be of type (p,q), $p+q=n$, g is called pseudo-Riemann. (If $p=1$, g_m is called Lorentzian; esp. $n=4$ & $p=1$ — socalled space-times)

<u>Definition</u>: An affine connection ∇ on a manifold M equipped with a Riemannian metric g is called <u>Riemannian</u> if $T^\nabla = 0$ and $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for all $X, Y, Z \in \mathcal{X}(M)$.

<u>Remark</u>: In some books, such ∇ is called Levi-Civita. In the next thm, we shall see that such a connection is necessarily unique.

∇ **Thm** (Fundamental theorem of Riemannian geometry):

Let M be a manifold equipped by a Riemannian metrics g. Then there exists one and only one Riemannian connection ∇ for it.

Proof: Let ∇ exists. Then

$$Xg(Y,Z)=g(\nabla_X Y,Z)+g(Y,\nabla_X Z) \quad (1)$$
$$Yg(Z,X)=g(\nabla_Y Z,X)+g(Z,\nabla_Y X) \quad (2)$$
$$Zg(X,Y)=g(\nabla_Z X,Y)+g(X,\nabla_Z Y) \quad (3)$$

(1)+(2)−(3) gives: $Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)=$

e.g. (nicer than at the lecture)

$$= g(Y,\nabla_X Z-\nabla_Z X)+g(X,\nabla_Y Z-\nabla_Z Y)+$$
$$+g(Z,\nabla_X Y+\nabla_Y X) = g(Y,T^\nabla(X,Z)+[X,Z])+$$
$$+g(X,T^\nabla(Y,Z)+[Y,Z])+g(Z,T^\nabla(X,Y)+$$
$$+2\nabla_Y X+[X,Y]) \overset{T=0}{=} g(Y,[X,Z])+$$
$$+g(X,[Y,Z])+2g(Z,\nabla_Y X)+g(Z,[X,Y]),$$

i.e.,

(Fmla)
$$g(Z,\nabla_Y X)=\frac{1}{2}\Big[Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-$$
$$-g([X,Z],Y)-g([Y,Z],X)-g([X,Y],Z)\Big]$$

Since (in each point) g is non-degenerate, it determines $\nabla_Y X$ (in the point). Thus $\nabla_Y X$ is unique.

Now, set (define ∇)

$$g(Z,\nabla_Y X)=\frac{1}{2}\Big[\quad \dots \quad\Big]$$ the fmla above.

We must verify that $\nabla_Y X$ is an affine conn.

Linearity obvious ✓

Leibniz in X. ?

Linear in Y. ??

$T^\nabla = 0$: $2g(Z, \nabla_Y X - \nabla_X Y) = \cancel{X g(Y,Z)} + \cancel{Y g(Z,X)} - \cancel{Z g(X,Y)}$
$- g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)$
$- \cancel{Y g(X,Z)} - \cancel{X g(Z,Y)} + \cancel{Z g(Y,X)} +$
$+ g([Y,Z],X) + g([X,Z],Y) - g([Y,X],Z)$

$= 0$

$X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$?

(omitted) $\square$

Remark / Example: Let $(e_i)_{i=1}^n$ be a local frame on M. Moreover let $e_i = \frac{\partial}{\partial x^i}$, $i = 1, \dots, n$ for some coordinate frame $\varphi = (x^1, \dots, x^n)$ of M. Then $[e_i, e_j] = 0$ (thm/assertion) and thus the fmla for Riemannian connection from the proof above simplifies

$$g(e_i, \nabla_{e_j} e_k) = \frac{1}{2}\left[e_k(g_{ji}) + e_j(g_{ik}) - e_i(g_{kj})\right]$$

where $g_{ij} = g(e_i, e_j)$. Denoting $e_k(g_{ab}) =: g_{ab,k}$ and using the def of Christoffel symbols, we get (summ.-conv.)

$$g(e_i, \Gamma^l_{jk} e_l) = \frac{1}{2}\left(g_{ji,k} + g_{ik,j} - g_{kj,i}\right) =$$

$$g_{il}\Gamma^{l}_{jk} = \frac{1}{2}\left(g_{ji,k} + g_{ik,j} - g_{kj,i}\right) \quad / \; g^{mi}\cdot$$

$$\delta^{m}_{l}\Gamma^{l}_{jk} = \frac{1}{2} g^{mi}\left(g_{ji,k} + g_{ik,j} - g_{kj,i}\right)$$

$$\Gamma^{m}_{jk} = \frac{1}{2} g^{mi}\left(g_{ji,k} + g_{ik,j} - g_{kj,i}\right) \text{ or}$$

more std.:

$$\boxed{\Gamma^{k}_{ij} = \frac{1}{2} g^{km}\left(g_{jm,i} + g_{mi,j} - g_{ij,m}\right)}$$

(also sum-conv., at right-hand side).

__Definition__: M mnfld, g Riem. metric, ∇ Riem. connection

$$R(X,Y,Z,U) = g(R^\nabla(X,Y)Z,U).$$

__Remark__: R is a $(0,4)$-tensor field.

__Theorem__: For any affine connection ∇, we have

$$R^\nabla(X,Y)Z + R^\nabla(Z,X)Y + R^\nabla(Y,Z)X = 0$$

(1st Bianchi)*, if ∇ is torsion-free, i.e., $T^\nabla = 0$

__Proof__: $\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y -$

$-\nabla_{[Z,X]}Y + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X =$

$= \nabla_X(\underbrace{T^\nabla(Y,Z)}_{=0} + [Y,Z]) +$

$+ \nabla_Y(\underbrace{T^\nabla(Z,X)}_{=0} + [Z,X]) +$

$+ \nabla_Z(\underbrace{T^\nabla(X,Y)}_{=0} + [X,Y]) - \nabla_{[X,Y]}Z - \nabla_{[Z,X]}Y$

$-\nabla_{[Y,Z]}X = \nabla_X([Y,Z]) + \nabla_Y[Z,X]$

$+ \nabla_Z[X,Y] - \nabla_{[X,Y]}Z - \nabla_{[Z,X]}Y - \nabla_{[Y,Z]}X =$

$= \underbrace{T^\nabla(X,[Y,Z])}_{=0} + [X,[Y,Z]]$

$+ \underbrace{T^\nabla(Y,[Z,X])}_{=0} + [Y,[Z,X]]$

$+ \underbrace{T^\nabla(Z,[X,Y])}_{=0} + [Z,[X,Y]] =$

$= 0 +$ Jacobi id. (thm when we proved that $(\mathcal{X}(M),[\,,\,])$ is a Lie alg. $\square$

* identity

Theorem: M manifold, g Riem. metric and $\nabla = \nabla^g$ Riemannian connection. Then

1) $R(X,Y,Z,U) = -R(Y,X,Z,U)$
2) $R(X,Y,Z,U) = -R(X,Y,U,Z)$
3) $R(X,Y,Z,U) = R(Z,U,X,Y) \quad \forall X,Y,Z,U \in \mathfrak{X}(M)$

Proof: 1) $R(X,Y,Z,U) = g(R^\nabla(X,Y)Z,U) \underset{\uparrow}{=} -g(R^\nabla(Y,X)Z,U)$

easy! (... was)

$= -R(Y,X,Z,U)$

2) It's sufficient $R(X,Y,Z,Z)=0$ since then

$0 = R(X,Y,Z+U,Z+U) = R(X,Y,Z,Z) + R(X,Y,Z,U)$
$+ R(X,Y,U,Z) + R(X,Y,U,U) = 0 +$
$+ R(X,Y,Z,U) + R(X,Y,U,Z) + 0$ ~~implying~~
that implies $R(X,Y,Z,U) = -R(X,Y,U,Z)$.

Thus $R(X,Y,Z,Z) = 0$? Why:

$R(X,Y,Z,Z) = g(R^\nabla(X,Y)Z,Z) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, Z)$. We know

$Xg(\nabla_Y Z, Z) = g(\nabla_X\nabla_Y Z, Z) + g(\nabla_Y Z, \nabla_X Z)$
$Yg(\nabla_X Z, Z) = g(\nabla_Y\nabla_X Z, Z) + g(\nabla_X Z, \nabla_Y Z)$ this implies:

- $g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z, Z) = Xg(\nabla_Y Z, Z) - Yg(\nabla_X Z, Z)$

since $g(\nabla_Y Z, \nabla_X Z)$ & $g(\nabla_X Z, \nabla_Y Z)$ "cancel" each other.

The term $\nabla_{[X,Y]} Z$ is determined by the defining formula from the proof of the FTRG (fund...)

$$\bullet\bullet\; g(\underset{Z}{Z}, \nabla_{\underset{Y}{[X,Y]}} \underset{X}{Z}) = \frac{1}{2}\Big(Z g(\cancel{[X,Y], Z}) + [X,Y] g(Z,Z) - Z g(Z, \cancel{[X,Y]}) - g(\underbrace{[Z,Z]}_{0}, [X,Y]) - g(\cancel{[[X,Y], Z], Z}) - g([Z, \cancel{[X,Y]]}, Z)\Big) =$$

$$= \frac{1}{2}[X,Y](g(Z,Z)).$$

Using $\bullet$ & $\bullet\bullet$, we get $g(\nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, Z) =$

$$= X g(\nabla_Y Z, Z) - Y g(\nabla_X Z, Z) - \frac{1}{2}[X,Y] g(Z,Z) =$$

$$= X\big(Y g(Z,Z) - g(Z, \nabla_Y Z)\big)$$

$$- Y\big(X g(Z,Z) - g(Z, \nabla_X Z)\big) - \frac{1}{2}[X,Y] g(Z,Z) =$$

[Comparing the underlined terms (and using the symm. of g), we get:
① ⇔ ①
② ⇔ ②]

$$= \frac{1}{2} X(Y g(Z,Z)) - \frac{1}{2} Y(X g(Z,Z)) - \frac{1}{2}[X,Y] g(Z,Z)$$

$$= 0$$

✓

3) Bianchi implies: $R(X,Y,Z,U) + R(Z,X,Y,U) + R(Y,Z,X,U) = 0$. Thus, writing $(X,Y,Z,U) := R(X,Y,Z,U)$, we get:

$$(X,Y,Z,U)+(Y,Z,X,U)+(Z,X,Y,U)=0$$
$$(Y,Z,U,X)+(Z,U,Y,X)+(U,Y,Z,X)=0$$
$$(Z,U,X,Y)+(U,X,Z,Y)+(X,Z,U,Y)=0$$
$$(U,X,Y,Z)+(X,Y,U,Z)+(Y,U,X,Z)=0$$

Adding the terms together, using #2), we get:

$$\underline{(Z,X,Y,U)}+(U,Y,Z,X)+\underline{(X,Z,U,Y)}+(Y,U,X,Z)=0$$

which according to 1) & 2) simplifies into

$$2(Z,X,Y,U)+2(U,Y,Z,X)=0 \Rightarrow$$

$$\Rightarrow (Z,X,Y,U)=$$
$$=(Y,U,Z,X),$$ what was to prove. $\square$

<u>Definition</u>: $Ric(X,Y):=Tr(Z\mapsto R(Z,X)Y)$ is called the Ricci form. It defines by $g(Ric^g(X),Y)=Ric(X,Y)$ the Ricci endomorphism Ric^g.

$K:=Tr(X\mapsto Ric^g(X))$ is called scalar curvature.

<u>Remark</u>: g pseudo-Riem $\rightsquigarrow \nabla^g$ by the same formulas. Let $\lambda\in\mathbb{R}$ and T be a symm. (0,2)-tensor field on M. Then the PDE $Ric-\lambda g=T$ for $\underline{g}$ is called the <u>Einstein eq.</u> :-)

'BOUT application in physics

$T = 0$ ... vacuum Einstein equation.

~~λ ... famous cosmo no!~~

$Ric = Tr\ R$ " $R = \nabla\nabla - \nabla\nabla - \nabla$ "

$\nabla \Rightarrow \Gamma = g_{ij,k}$ "

$\nabla(\Gamma) = g_{ij,kl} \rightsquigarrow R$ is obtainable by the 2^{nd} der of g. The formula goes smth like

$$R^i_{jkl} = (\partial_m \Gamma^{\cancel{j}}_{\cancel{j}kl} - \partial\Gamma - \Gamma\Gamma - \Gamma\Gamma) g^{im}$$

$$= (\partial_m g_{kl,l} \cdots$$

$$\rightsquigarrow g_{kl,\underline{\underline{lm}}}$$

- Thus Einstein eq are 2 order.
  ~~If g is positive def, they are elliptic~~
  actually, ~~I~~ do not know!

- They are non-linear.
- Special solutions known:
  - spherically symmetric with $T=0 \to$ Schwarzschild ($\simeq$ 1915 ~~1918~~)
  - Reissner-Nordström, Newman-Kerr (rotating magnetic spheres etc.)
  ....