Relation of the spectra of symplectic Rarita-Schwinger and Dirac operators on flat symplectic manifolds

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October 5, 2007

Abstract

Consider a flat symplectic manifold (M^{2l}, ω), $l \geq 2$, admitting a metaplectic structure. We prove that the symplectic twistor operator maps the eigenvectors of the symplectic Dirac operator, that are not symplectic Killing spinors, to the eigenvectors of the symplectic Rarita-Schwinger operator. If λ is an eigenvalue of the symplectic Dirac operator such that $-\imath l\lambda$ is not a symplectic Killing number, then $\frac{l-1}{l}\lambda$ is an eigenvalue of the symplectic Rarita-Schwinger operator.

1 Introduction

In the paper, we shall study relations between the spectrum of the symplectic Dirac operator and the spectrum of the symplectic Rarita-Schwinger operator on a flat symplectic manifold with a given metaplectic structure. In the symplectic case, there also exists a non-trivial two-fold covering of the symplectic group $Sp(2l, \mathbb{R})$, called the metaplectic group. We shall denote it by $Mp(2l, \mathbb{R})$. A metaplectic structure on a symplectic manifold (M^{2l}, ω) is a notion parallel to a spin structure on a Riemannian manifold. On a symplectic manifold M with a given metaplectic structure, we shall introduce the symplectic Dirac, the symplectic twistor, and the symplectic Rarita-Schwinger operators. The first operator is acting on sections of the symplectic spinor bundle $\mathcal{S}$, introduced by Bertram Kostant in 1974. The symplectic spinor bundle $\mathcal{S}$ is the vector bundle associated to the metaplectic structure on M by the Segal-Shale-Weil representation of the metaplectic group (see Kostant [10]). The values of

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† I would like to thank to Katharina and Lutz Habermann for an encouragement in this research. The author of this article was supported by the grant GAČR 201/06/P223 of the Grant Agency of Czech Republic for young researchers. The work is a part of the research project MSM 0021620839 financed by MŠMT ČR. Also supported by the SPP 1096 of the DFG.

the twistor operator are sections of the vector bundle $\mathcal{T}$ associated to a more complicated representation of the metaplectic group (described below). The Rarita-Schwinger operator maps the sections of $\mathcal{T}$ to the sections of $\mathcal{T}$.

The Segal-Shale-Weil representation is an infinite dimensional unitary representation of the metaplectic group $Mp(2l, \mathbb{R})$ on the space of all complex valued square integrable functions $\mathbf{L^2}(\mathbb{R}^l)$. It is well known, see, e.g., Kashiwara, Vergne [9], that the underlying Harish-Chandra module of this representation is equivalent to the space $\mathbb{C}[x^1, \ldots, x^l]$ of polynomials in l variables. The Lie algebra $\mathfrak{sp}(2l, \mathbb{C})$ acts on $\mathbf{L^2}(\mathbb{R}^l)$ via the so called Chevalley realization,[1] see Britten, Hooper, Lemire [2]. Thus, the infinitesimal structure of the Segal-Shale-Weil representation could be viewed as the complexified *symmetric* tensor algebra $(\bigoplus_{i=0}^{\infty} \odot^i \mathbb{R}^l) \otimes_{\mathbb{R}} \mathbb{C}$ of the Lagrangian subspace $\mathbb{R}^l$ of the canonical symplectic vector space $\mathbb{R}^{2l} \simeq \mathbb{R}^l \oplus \mathbb{R}^l$. This shows that the situation is completely parallel to the complex orthogonal case where the spinor representation could be realized as the *exterior* algebra of a maximal isotropic subspace. The interested reader is referred to Weil [21], Kashiwara, Vergne [9] and also to Britten, Hooper, Lemire [2] for more details.

In differential geometry, it is quite effective to study geometric and also topological properties of manifolds with a given geometric structure using analytic properties of the corresponding invariant differential operators. This could be illustrated by many examples. For instance, it is known that a Riemannian manifold admitting a spin structure and a nonzero Killing spinor is necessarily Einstein. Therefore the idea of K. Habermann, to introduce a symplectic analogue of the Dirac operator known from Riemannian geometry seems to be very natural in the process of an investigation of symplectic manifolds. The symplectic Dirac operator was introduced with the help of the so called symplectic Clifford multiplication, see Habermann [5]. However, it is not straightforward to generalize this approach to differential operators acting between bundles associated to higher symplectic spinors, i.e., operators which would be similar to the Rarita-Schwinger and the twistor operators studied in the Riemannian case.

In Riemannian geometry, it is well known that the mentioned higher spin differential operators can be realized as certain invariant differential operators acting in the de Rham complex tensored (twisted) by the sections of the orthogonal spinor bundle. In particular, all mentioned operators can be constructed as projections of the associated exterior spinor derivative restricted to invariant parts of the twisted sequence. So it seems to be quite natural to look for the symplectic analogues of the Dirac, Rarita-Schwinger and twistor operators in the twisted de Rham sequence as well. Following the Riemannian case, one should decompose this symplectic spinor valued de Rham sequence into $Mp(2l, \mathbb{R})$-invariant subsequences at the first step. The decomposition is very similar to the Riemannian case and was done in Krýsl [12] at the infinitesimal level using some results of Britten, Hooper, Lemire [2]. In this paper, we present a globalized version of the decomposition. It means that we shall decompose the tensor

[1] The Chevalley homomorphism realizes the complex symplectic Lie algebra as a subalgebra of the algebra of polynomial coefficients differential operators acting on $\mathbb{C}[x^1, \ldots, x^l]$.

product of representations of the metaplectic group, instead of the corresponding Lie algebra. To my knowledge, the symplectic Rarita-Schwinger and the symplectic twistor operators have not yet been defined in the literature. However, there are definitions of contact projective Rarita-Schwinger and twistor operators (at least at an infinitesimal level) in contact projective geometry, see Kadlčáková [7] and Krýsl [13].

Now, let us say a few words about the main result. Let M be a flat symplectic manifold with a chosen metaplectic structure. We shall prove that the eigenvectors of the symplectic Dirac operator different from symplectic Killing spinors[2] embed linearly into the space of eigenvectors of the symplectic Rarita-Schwinger operator. We shall also give a formula relating the corresponding eigenvalues. Notice that in the Riemannian geometry, a similar result is known, see Severa [17], and also some of the methods we shall be using are similar to the ones in the Riemannian case, see again Severa [17]. In the proof of the embedding theorem, there are some statements of an independent interest, which could be also used in a study of operators different from the symplectic Rarita-Schwinger operator.

From a point of view of physics, the embedding theorem is not surprising and is related to the fact that in general, spectra of objects with less symmetries are richer than the spectra of those having more symmetries. The reader interested in applications of this theory in physics is referred to Green, Hull [3], where the symplectic spinors are used in the context of 10 dimensional super string theory. In Reuter [14], where the author found his motivation for this study, symplectic spinors are used in the theory of the so called Dirac-Kähler fields.

In the second section, some basic facts on symplectic vector spaces are recalled. In the subsection 2.1., the Segal-Shale-Weil and the notion of a metaplectic representation together with the symplectic Clifford multiplication are briefly introduced. In the subsection 2.2, we present the decomposition of a part of the symplectic spinor twisted de Rham sequence (Theorem 1). The geometric part of the article starts in the subsection 2.3., where the metaplectic structure, symplectic connection, symplectic Dirac, Rarita-Schwinger and twistor operators are defined. The third section is devoted to a formulation and a proof of the embedding theorem (Theorem 2).

2 Metaplectic structure and higher symplectic spinor representations

Let us consider a real symplectic vector space $(\mathbb{V}, \omega_0)$ of dimension $2l$, i.e., $\mathbb{V}$ is a $2l$ dimensional real vector space and ω_0 is a non-degenerate antisymmetric bilinear form on $\mathbb{V}$. Let us choose two Lagrangian subspaces $\mathbb{L}, \mathbb{L}' \subseteq \mathbb{V}$ such that $\mathbb{L} \oplus \mathbb{L}' = \mathbb{V}$. Throughout this article, we shall use a symplectic basis $\{e_i\}_{i=1}^{2l}$ of $\mathbb{V}$

[2] Symplectic Killing spinors are defined below. To my knowledge, the symplectic Killing spinor equation has been not introduced in the literature till now. In physics, there is a notion of the so called symplectic Killing spinor different of the one we are using.

chosen in such a way that $\{e_i\}_{i=1}^{l}$ and $\{e_i\}_{i=l+1}^{2l}$ are respective bases of $\mathbb{L}$ and $\mathbb{L}'$. Because the notion of symplectic basis is not unique, let us fix one which shall be used in this text. A basis $\{e_i\}_{i=1}^{2l}$ of $\mathbb{V}$ is called symplectic basis of $(\mathbb{V}, \omega_0)$ if $\omega_{ij} := \omega_0(e_i, e_j)$ satisfies $\omega_{ij} = 1$ if and only if $i \leq l$ and $j = i + l$; $\omega_{ij} = -1$ if and only if $i > l$ and $j = i - l$ and finally, $\omega_{ij} = 0$ in other cases. Let $\{\epsilon^i\}_{i=1}^{2l}$ be the basis of $\mathbb{V}^*$ dual to the basis $\{e_i\}_{i=1}^{2l}$. Notice that $\omega_{ij} = -\omega_{ji}$, $i, j = 1, \ldots, 2l$. For $i, j = 1, \ldots, 2l$, we define ω^{ij} by $\sum_{k=1}^{2l} \omega_{ik}\omega^{kj} = \delta_i^j$, $i, j = 1, \ldots, 2l$. Notice that not only $\omega_{ij} = -\omega_{ji}$, but also $\omega^{ij} = -\omega^{ji}$, $i, j = 1, \ldots, 2l$. Let us denote the ω_0-dual basis to the basis $\{e_i\}_{i=1}^{2l}$ by $\{\check{e}_i\}_{i=1}^{2l}$, i.e., $\epsilon^j(\check{e}_i) = \iota_{\check{e}_i}\epsilon^j = \omega_{ij}$ for $i, j = 1, \ldots, 2l$.

Let us denote the symplectic group of $(\mathbb{V}, \omega_0)$ by G, i.e., $G := Sp(\mathbb{V}, \omega_0) \simeq Sp(2l, \mathbb{R})$. Because the maximal compact subgroup K of G is isomorphic to the unitary group $K \simeq U(l)$ which is of homotopy type $\mathbb{Z}$, there exists a nontrivial two-fold covering $\tilde{G}$ of G. This two-fold covering is called metaplectic group of $(\mathbb{V}, \omega_0)$ and it is denoted by $Mp(\mathbb{V}, \omega_0)$. In the considered case, we have $\tilde{G} \simeq Mp(2l, \mathbb{R})$. For a later use, let us reserve the symbol λ for the mentioned covering. Thus $\lambda : \tilde{G} \to G$ is a fixed member of the isomorphism class of all nontrivial $2:1$ coverings of G. Because $\lambda : \tilde{G} \to G$ is a homomorphism of Lie groups and G is a subgroup of the general linear group $GL(\mathbb{V})$ of $\mathbb{V}$, mapping λ is also a representation of the metaplectic group $\tilde{G}$ on the vector space $\mathbb{V}$. Further, let us denote the λ-preimage, $\lambda^{-1}(K)$, of K in $\tilde{G}$ by $\tilde{K}$. It is well known that $\tilde{K} \simeq \widetilde{U(l)} := \{(u, z) \in U(l) \times \mathbb{C}^\times | \det u = z^2\}$, and that $\widetilde{U(l)}$ is a connected Lie group, see Tirao, Vogan, Wolf [19]. For details on the metaplectic group, see, e.g., Habermann, Habermann [6].

From now on, we shall restrict ourselves to the case $l \geq 2$ without mentioning it explicitly. The case $l = 1$ should be handled separately because the shape of the root system of $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R})$ is different from that one of of the root system of $\mathfrak{sp}(2l, \mathbb{R})$ for $l > 1$. Let us denote the Lie algebra of G by $\mathfrak{g}$. Thus $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}, \omega_0) \simeq \mathfrak{sp}(2l, \mathbb{R})$. As usual, we shall denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}^{\mathbb{C}}$. Obviously, $\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sp}(2l, \mathbb{C})$. Fixing a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$ and choosing a system Φ^+ of positive roots with respect to the choice of $\mathfrak{h}^{\mathbb{C}}$, the basis of fundamental weights $\{\varpi_i\}_{i=1}^{l}$ is uniquely determined. For $\mu \in (\mathfrak{h}^{\mathbb{C}})^*$, let us denote the irreducible $\mathfrak{g}^{\mathbb{C}}$-module with highest weight μ by $L(\mu)$.

Now, let us say a few remarks of technical character about a notation we shall be using throughout this article. For a general reductive Lie group G, let HC denote the forgetful Harish-Chandra functor going from the category of admissible finite length representations of G on locally convex Hausdorff vector spaces to the category of Harish-Chandra modules. If $\sigma : G \to \mathrm{Aut}(\mathbf{W})$ is a representation of G from the mentioned category, we denote its underlying Harish-Chandra module $HC(\mathbf{W})$ by W. Considering only the Lie algebra structure of the Harish-Chandra module W, we shall denote it by $\mathbb{W}$. Of course, these conventions are superfluous in the case of finite dimensional representations, because the underlying vector spaces stay the same. It is well known that there exists an adjoint functor to the forgetful Harish-Chandra functor, which is called minimal globalization and which we shall denote by mg. See Kashiwara, Schmid [8] for details. Let us remark, that in our case the reductive group will always be the metaplectic group $Mp(\mathbb{V}, \omega_0)$, and the representations we shall be using belong to the mentioned category, see, e.g., Kashiwara, Vergne [9]. Further, if $(\mathcal{G}, p, M, G)$ is a principal G-bundle, we shall denote the vector bundle associated to this principal bundle via the representation $\sigma : G \to \mathrm{Aut}(\mathbf{W})$ by $\mathcal{W}$, i.e., $\mathcal{W} = \mathcal{G} \times_\sigma \mathbf{W}$.

2.1 Metaplectic representation and symplectic spinors

There exists a distinguished infinite dimensional representation of the metaplectic group $\tilde{G}$ which does not descend to a representation of the symplectic group G. This representation, called Segal-Shale-Weil,[3] plays a fundamental role in geometric quantization of Hamiltonian mechanics, see, e.g., Woodhouse [22] and in the theory of modular forms and theta correspondence, see, e.g., Howe [4]. We shall not give a definition of this representation here, and refer the interested reader to Weil [21] or Habermann, Habermann [6]. We shall only mention that the Segal-Shale-Weil representation, which we shall denote by U here, is an infinite dimensional unitary representation of $\tilde{G}$ on the space of complex valued square Lebesgue integrable functions defined on the Lagrangian subspace $\mathbb{L}$, i.e.,

$$U : \tilde{G} \to \mathcal{U}(\mathbf{L^2}(\mathbb{L})),$$

where $\mathcal{U}(\mathbf{W})$ denotes the group of unitary operators on a Hilbert space $\mathbf{W}$. In order to be precise, let us refer to the space $\mathbf{L^2}(\mathbb{L})$ as to the Segal-Shale-Weil module. It is easy to see that this representation is not irreducible and splits into two irreducible modules $\mathbf{L^2}(\mathbb{L}) \simeq \mathbf{L^2}(\mathbb{L})_+ \oplus \mathbf{L^2}(\mathbb{L})_-$. The first module consists of even and the second one of odd complex valued square Lebesgue integrable functions on the Lagrangian subspace $\mathbb{L}$. Let us remark that a typical construction of the Segal-Shale-Weil representation is based on the so called Schrödinger representation of the Heisenberg group of $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega_0)$ and a use of the Stone-von Neumann theorem.

For some technical reasons, we shall need the minimal globalization of the underlying $(\mathfrak{g}, \tilde{K})$-module of the introduced Segal-Shale-Weil module, see Kashiwara, Schmid [8] or Vogan [20] for details on globalizations. We shall call this minimal globalization metaplectic representation and denote it by *meta*, i.e.,

$$meta : \tilde{G} \to \mathrm{Aut}(mg(HC(\mathbf{L^2}(\mathbb{L})))),$$

where mg is the minimal globalization functor. For our convenience, let us denote the module $mg(HC(\mathbf{L^2}(\mathbb{L})))$ by $\mathbf{S}$ and similarly we define $\mathbf{S}_+$ and $\mathbf{S}_-$ as minimal globalizations of the underlying Harish-Chandra modules of the modules $\mathbf{L^2}(\mathbb{L})_+$ and $\mathbf{L^2}(\mathbb{L})_-$ introduced above. Accordingly to $\mathbf{L^2}(\mathbb{L}) \simeq \mathbf{L^2}(\mathbb{L})_+ \oplus \mathbf{L^2}(\mathbb{L})_-$, we have $\mathbf{S} \simeq \mathbf{S}_+ \oplus \mathbf{S}_-$. We shall call $\mathbf{S}$ symplectic spinor module and

[3] The names oscillator or metaplectic representation are also used in the literature. We shall use the name Segal-Shale-Weil in this text, and reserve the name metaplectic for certain representation arising from the Segal-Shale-Weil one.

its elements symplectic spinors. For the name "spinor", see Kostant [10] or the Introduction.

One can easily compute the highest weights of the underlying Harish-Chandra modules of $\mathbf{S}_+$ and $\mathbf{S}_-$. The highest weight of $HC(\mathbf{S}_+)$ as a $\mathfrak{g}^{\mathbb{C}}$-module equals $-\frac{1}{2}\varpi_l$ and the highest weight of $HC(\mathbf{S}_-)$ as a $\mathfrak{g}^{\mathbb{C}}$-module is $\varpi_{l-1} - \frac{3}{2}\varpi_l$.

Further notion related to the symplectic vector space $(\mathbb{V}, \omega_0)$ is the so called symplectic Clifford multiplication of elements of the Schwartz space $\mathbf{Sch}(\mathbb{L})$ (of rapidly decreasing smooth functions on $\mathbb{L}$) by vectors from $\mathbb{V}$. For $f \in \mathbf{Sch}(\mathbb{L})$, we define

$$(e_i.f)(x) := \imath x^i f(x),^4$$

$$(e_{i+l}.f)(x) := \frac{\partial f}{\partial x^i}(x), x = \sum_{i=1}^{l} x^i e_i \in \mathbb{L}, i = 1, \ldots, l.$$

Extending this multiplication $\mathbb{R}$-linearly, we get the mentioned symplectic Clifford multiplication of Schwartz functions by vectors from $\mathbb{V}$.

In the following lemma, a kind of $\tilde{G}$-equivariance of the symplectic Clifford multiplication is stated.

Lemma 1: Clifford multiplication is $\tilde{G}$-equivariant, i.e.,

$$meta(g)(v.f) = [\lambda(g)v].meta(g)f$$

for each $v \in \mathbb{V}$, $g \in \tilde{G}$ and $f \in \mathbf{Sch}(\mathbb{L})$.

Proof. See Habermann, Habermann [6], pp. 13. $\square$

In the next lemma, a commutator relation for the symplectic Clifford multiplication is presented.

Lemma 2: If $v, w \in \mathbb{V}$ and $f \in \mathbf{Sch}(\mathbb{L})$, then

$$v.(w.f) - w.(v.f) = -\imath\omega_0(v, w)f.$$

Proof. See again Habermann, Habermann [6], pp. 11. $\square$

Let us notice that the symplectic Clifford multiplication could be restricted to the symplectic spinor module $\mathbf{S}$ because of the interpretation of the minimal globalization as a vector space of analytic vectors, see Schmid [16].

2.2 Higher symplectic spinors

In this subsection, we shall present some results on decomposition of tensor products of the metaplectic representation and the wedge powers of the representation $\tilde{\lambda} : \tilde{G} \to GL(\mathbb{V}^*)$ of $\tilde{G}$ dual to the representation of λ. Because $\lambda \simeq \lambda^*$, we shall often not distinguish between them, and we also identify $\mathbb{V}$ and $\mathbb{V}^*$ as $\tilde{G}$-modules.

Let us reserve the symbol ρ for this mentioned tensor product representation of $\tilde{G}$, i.e.,

$$\rho : \tilde{G} \to \mathrm{Aut}(\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S})$$

[4] The symbol $\imath$ denotes the imaginary unit, $\imath = \sqrt{-1}$.

and

$$\rho(g)(\alpha \otimes s) := \lambda(g)^{*\wedge r}\alpha \otimes meta(g)s$$

for $\alpha \in \bigwedge^r \mathbb{V}^*$, $g \in \tilde{G}$ and $s \in \mathbf{S}$. Sometimes, we shall write $\lambda(g)^*\alpha$ instead of $\lambda(g)^{*\wedge r}\alpha$ briefly. For a definiteness, let us equip the tensor product $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ by the so called Grothendieck tensor product topology, see Vogan [20] for details on this topological structure.

In the next theorem, the modules of exterior 1- and 2-forms with values in the metaplectic representation are decomposed into irreducible summands.

Theorem 1: The following isomorphisms

$$\mathbb{V}^* \otimes \mathbf{S} \simeq \mathbf{S}' \oplus \mathbf{T}\ , \bigwedge^{\mathbf{2}} \mathbb{V}^* \otimes \mathbf{S} \simeq \mathbf{S}'' \oplus \mathbf{T}' \oplus \mathbf{U}$$

hold, where $\mathbf{S}', \mathbf{S}'', \mathbf{T}$, $\mathbf{T}'$ and $\mathbf{U}$ are the minimal globalizations of Harish-Chandra $(\mathfrak{g}, \tilde{K})$-modules $S', S,'' T, T'$ and U which are uniquely determined by the condition that first, they are submodules of the corresponding tensor products and second, their $\mathfrak{g}^{\mathbb{C}}$-module structure fulfill $\mathbb{S}' \simeq \mathbb{S} = HC(\mathbf{S})$, $\mathbb{T} \simeq \mathbb{T}' \simeq L(\omega_1 - \frac{1}{2}\omega_l) \oplus L(\omega_1 + \omega_{l-1} - \frac{3}{2}\omega_l)$ and $\mathbb{U} \simeq L(\omega_2 - \frac{1}{2}\omega_l) \oplus L(\omega_2 + \omega_{l-1} - \frac{3}{2}\omega_l)$.

Proof. The proof of this theorem is based on a use of Lemma 2 in Krýsl [12], where the tensor products under consideration are decomposed at the level of Lie algebras. Thus it remains to show how to come from the infinitesimal level to the globalized one. We should do it only for $\mathbb{V}^* \otimes \mathbf{S}$, the remaining case could be handled in the same way. Because the minimal globalization functor is compatible with the Grothendieck tensor product topology, we have $\mathbb{V}^* \otimes \mathbf{S} \simeq \mathbb{V}^* \otimes mg(HC(\mathbf{L^2}(\mathbb{L}))) \simeq mg(\mathbb{V}^* \otimes HC(\mathbf{L^2}(\mathbb{L})))$. Thus we need to decompose the defining representation tensored by the Harish-Chandra $(\mathfrak{g}, \tilde{K})$-module $S = HC(\mathbf{L^2}(\mathbb{L}))$. Because $\tilde{K}$ is connected, it is sufficient to decompose $\mathbb{V}^* \otimes \mathbb{S}$ as a $\mathfrak{g}$-module, see Baldoni [1]. Thus we have reduced the problem to the infinitesimal level. Now, using Lemma 2 from Krýsl [12], we have $\mathbb{V}^* \otimes \mathbb{S} \simeq \mathbb{S} \oplus \mathbb{T}$. Again because $\tilde{K}$ is connected, we have $\mathbb{V}^* \otimes S \simeq S \oplus T$. By applying the minimal globalization functor, the statement follows. $\Box$

Let us notice that in Krýsl [12], all the wedge powers of the defining representation of $\mathfrak{g}^{\mathbb{C}}$ tensored by $\mathbb{S}_+$ are decomposed into irreducible summands. We shall call the irreducible submodules of $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ higher symplectic spinor modules in a parallel with the orthogonal case.

In order to define the symplectic Dirac, Rarita-Schwinger and twistor operators, we introduce the following ($\tilde{G}$-equivariant) projections:

$$p^{10} : \mathbb{V}^* \otimes \mathbf{S} \to \mathbf{S}',\ p^{11} : \mathbb{V}^* \otimes \mathbf{S} \to \mathbf{T},$$

$$p^{21} : \bigwedge^2 \mathbb{V}^* \otimes \mathbf{S} \to \mathbf{T}'.$$

The definitions of these projections are based on the decompositions in Theorem 1.

Due to Theorem 1, we know that there is a $\tilde{G}$-equivariant inclusion $\mathbf{S} \hookrightarrow \mathbb{V}^* \otimes \mathbf{S}$. In the next lemma, this inclusion is described explicitly.

Lemma 3: Module $\mathbf{E} := \{\sum_{i=1}^{2l} \epsilon^i \otimes e_i.s, s \in \mathbf{S}\} \subseteq \mathbb{V}^* \otimes \mathbf{S}$ is isomorphic to the symplectic spinor module $\mathbf{S}$.

Proof. Let us define a mapping $\tau : \mathbf{E} \to \mathbf{S}$ by the prescription

$$\sum_{i=1}^{2l} \epsilon^i \otimes e_i.s \mapsto s.$$

This prescription is correct and the mapping is clearly bijective. What remains to show is the $\tilde{G}$-equivariance of τ. For $g \in \tilde{G}$ and $s \in \mathbf{S}$, we may write

$$\tau(\rho(g)(\sum_{i=1}^{2l} \epsilon^i \otimes e_i.s)) = \tau(\sum_{i=1}^{2l} \lambda(g)^* \epsilon^i \otimes meta(g)(e_i.s)).$$

Using the fact that the dual representation is isomorphic to the contragradient one and the fact that the symplectic Clifford multiplication is $\tilde{G}$-equivariant (see Lemma 1), we get

$$\begin{aligned}
\tau(\rho(g)(\sum_{i=1}^{2l} \epsilon^i \otimes e_i.s)) &= \tau(\sum_{i=1}^{2l} \lambda(g)^* \epsilon^i \otimes \lambda(g) e_i.meta(g)s) \\
&= \tau(\sum_{i,j,k=1}^{2l} \epsilon^k [\lambda(g)^{-1}]_k^{\ i} \otimes [\lambda(g)]_i^{\ j} e_j.meta(g)s) \\
&= \tau(\sum_{i,j,k=1}^{2l} [\lambda(g)^{-1}]_k^{\ i} [\lambda(g)]_i^{\ j} \epsilon^k \otimes e_j.meta(g)s) \\
&= \tau(\sum_{i,k=1}^{2l} \delta_k^i \epsilon^k \otimes e_i.meta(g)s) \\
&= \tau(\sum_{i=1}^{2l} \epsilon^i \otimes e_i.meta(g)s) \\
&= meta(g)s = \rho(g)\tau(\sum_{i=1}^{2l} \epsilon^i \otimes e_i.s).
\end{aligned}$$

Summing up, we have obtained $\tau(\rho(g)(\sum_{i=1}^{2l} \epsilon^i \otimes e_i.s)) = \rho(g)\tau(\sum_{i=1}^{2l} \epsilon^i \otimes e_i.s)$, thus τ is $\tilde{G}$-equivariant and the modules $\mathbf{E}$ and $\mathbf{S}$ are isomorphic. $\Box$

In order to define symplectic the Dirac, Rarita-Schwinger and twistor operators, we should introduce further $\tilde{G}$-invariant operators.

For $r = 0, \ldots, 2l$ and $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, we set

$$X : \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S},\ X(\alpha \otimes s) := \sum_{i=1}^{2l} \epsilon^i \wedge \alpha \otimes e_i.s;$$

$$Y : \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r-1} \mathbb{V}^* \otimes \mathbf{S},\ Y(\alpha \otimes s) := \sum_{i=1}^{2l} \iota_{\check{e}_i} \alpha \otimes e_i.s;$$

$$H : \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^r \mathbb{V}^* \otimes \mathbf{S},\ H := \{X, Y\} = XY + YX.$$

In the following parts of this text, we shall often use the procedure of renumbering of indices and the relations $\omega^{ij} = -\omega^{ji}$, $\omega_{ij} = -\omega_{ji}$ and $\sum_{k=1}^{2l} \omega_{ik}\omega^{kj} = \delta_i^j$ for $i, j = 1, \ldots, 2l$ without mentioning it explicitly.

Lemma 4: The homomorphisms X, Y, H are $\tilde{G}$-equivariant with respect to the representation ρ of $\tilde{G}$.

Proof. Let us prove the equivariance for X, Y and H separately.

(i) For $g \in \tilde{G}$, $r = 0, \ldots, 2l$ and $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, we have

$$\begin{aligned}
\rho(g)X(\alpha \otimes s) &= \rho(g)[\sum_{i=1}^{2l}(\epsilon^i \wedge \alpha \otimes e_i.s)] \\
&= \sum_{i=1}^{2l} \lambda(g)^*\epsilon^i \wedge \lambda(g)^*\alpha \otimes meta(g)(e_i.s) \\
&= \sum_{i,j=1}^{2l} \epsilon^j[\lambda(g)^{-1}]_j{}^i \wedge \lambda(g)^*\alpha \otimes (\lambda(g)e_i).meta(g) \\
&= \sum_{i,j,k=1}^{2l} \epsilon^j[\lambda(g)^{-1}]_j{}^i \wedge \lambda(g)^*\alpha \otimes [\lambda(g)]_i{}^k e_k.(meta(g)s) \\
&= \sum_{i,j,k=1}^{2l} [\lambda(g)^{-1}]_j{}^i[\lambda(g)]_i{}^k \epsilon^j \wedge \lambda(g)^*\alpha \otimes e_k.meta(g)s \\
&= \sum_{j,k=1}^{2l} \epsilon^j \delta_j^k \wedge \lambda(g)^*\alpha \otimes e_k.meta(g)s = X\rho(g)(\alpha \otimes s).
\end{aligned}$$

(ii) Now, we do a similar computation for Y. For $g \in \tilde{G}$, $r = 0, \ldots, 2l$, $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, we obtain

$$\begin{aligned}
\rho(g)Y(\alpha \otimes s) &= \rho(g)[\sum_{i=1}^{2l} \iota_{\check{e}_i}\alpha \otimes e_i.s] \\
&= \sum_{i=1}^{2l} \lambda(g)^*\iota_{\check{e}_i}\alpha \otimes meta(g)(e_i.s).
\end{aligned}$$

It is easy to see that $\lambda(g)^*(\iota_v\alpha) = \iota_{\lambda(g)v}(\lambda(g)^*\alpha)$ for $v \in \mathbb{V}^*$. Using this relation, we get

$$\sum_{i=1}^{2l} \lambda(g)^*(\iota_{\check{e}_i}\alpha) \otimes meta(g)(e_i.s)$$

$$= \sum_{i,j=1}^{2l} \iota_{\lambda(g)\check{e}_i}(\lambda(g)^*\alpha) \otimes [\lambda(g)]_i{}^j e_j.meta(g)s.$$

Using the formula $\check{e}_i = \sum_{k=1}^{2l} \omega^{ik} e_k$, we can rewrite the preceding equation as

$$\begin{aligned}
& \sum_{i,j=1}^{2l} \iota_{\lambda(g)\sum_{k=1}^{2l}\omega^{ik}e_k}(\lambda(g)^*\alpha) \otimes [\lambda(g)]_i{}^j e_j.meta(g)s \\
= & \sum_{i,j,k,n=1}^{2l} [\lambda(g)]_i{}^j \omega^{ik} \iota_{[\lambda(g)]_k{}^n e_n}(\lambda(g)^*\alpha) \otimes e_j.meta(g)s \\
= & \sum_{i,j,k,n=1}^{2l} [\lambda(g)]_i{}^j \omega^{ik} [\lambda(g)]_k{}^n \iota_{e_n}(\lambda(g)^*\alpha) \otimes e_j.meta(g)s.
\end{aligned}$$

Because $\lambda(g) \in Sp(\mathbb{V}, \omega_0)$, we have $\sum_{i,k=1}^{2l} [\lambda(g)]_i{}^j \omega^{ik} [\lambda(g)]_k{}^n = \sum_{i,k=1}^{2l} [\lambda(g)^{\perp}]^j{}_i \omega^{ik} [\lambda(g)]_k{}^n = \omega^{jn}$ for $j, n = 1, \ldots, 2l$. Substituting this relation into the previous computation, we get

$$\begin{aligned}
& \sum_{i,j,k,n=1}^{2l} [\lambda(g)]_i{}^j \omega^{ik} [\lambda(g)]_k{}^n \iota_{e_n}(\lambda(g)^*\alpha) \otimes e_j.meta(g)s \\
= & \sum_{j,n=1}^{2l} \omega^{jn} \iota_{e_n}(\lambda(g)^*\alpha) \otimes e_j.meta(g)s \\
= & \sum_{j=1}^{2l} \iota_{\sum_{n=1}^{2l}\omega^{jn}e_n}(\lambda(g)^*\alpha) \otimes e_j.meta(g)s \\
= & \sum_{j=1}^{2l} \iota_{\check{e}_j}(\lambda(g)^*\alpha) \otimes e_j.meta(g)s \\
= & Y\rho(g)(\alpha \otimes s).
\end{aligned}$$

(iii) H is $\tilde{G}$-equivariant, because $H = \{X, Y\}$ and X and Y are $\tilde{G}$-equivariant.

$\square$

In the next lemma, the values of H on homogeneous components of $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ are computed.

Lemma 5: Let $(\mathbb{V}, \omega_0)$ be a $2l$ dimensional symplectic vector space. Then for $r = 0, \ldots, 2l$, we have

$$H_{|\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}} = \imath(l - r)\mathrm{Id}_{|\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}}.$$

Proof. We shall compute the value of H in three items.

(i) First, let us prove that for $r = 0, \dots, 2l$ and $\alpha \in \bigwedge^r \mathbb{V}^*$, we have

$$\sum_{i,j=1}^{2l} \omega_{ij} \epsilon^i \wedge \iota_{\check{e}_j} \alpha = r\alpha. \tag{1}$$

It is sufficient to prove the statement for homogeneous basis elements. We proceed by induction.

I. For $r = 1$, we have $\sum_{i,j=1}^{2l} \omega_{ij} \epsilon^i \wedge \iota_{\check{e}_j} \epsilon^k = \sum_{i,j=1}^{2l} \omega_{ij} \epsilon^i \omega^{jk} = \delta_i^k \epsilon^i = \epsilon^k$ for each $k = 1, \dots, 2l$.

II. For $r = 0, \dots, 2l$, $\alpha \in \bigwedge^r \mathbb{V}^*$ and $k = 1, \dots, 2l$, we have $\sum_{i,j=1}^{2l} \omega_{ij} \epsilon^i \wedge \iota_{\check{e}_j} (\epsilon^k \wedge \alpha) = \sum_{i,j=1}^{2l} \omega_{ij} \epsilon^i \wedge \iota_{\check{e}_j} \epsilon^k \wedge \alpha - \sum_{i,j=1}^{2l} \omega_{ij} \epsilon^i \wedge \epsilon^k \wedge \iota_{\check{e}_j} \alpha = \sum_{i,j=1}^{2l} \omega_{ij} \omega^{jk} \epsilon^i \wedge \alpha + \sum_{i,j=1}^{2l} \omega_{ij} \epsilon^k \wedge \epsilon^i \wedge \iota_{\check{e}_j} \alpha = \epsilon^k \wedge \alpha + r\epsilon^k \wedge \alpha = (r+1)\epsilon^k \wedge \alpha$, where we have used the induction hypothesis in the second last equation.

(ii) Now, let us prove that for each $s \in \mathbf{S}$,

$$A := \sum_{i,j=1}^{2l} \omega^{ij} e_i.e_j.s = \imath l s. \tag{2}$$

Using Lemma 2 about the commutator of the symplectic Clifford multiplication, we can write $A = \sum_{i,j=1}^{2l} \omega^{ij} e_i.e_j.s = \sum_{i,j=1}^{2l} \omega^{ij} (e_j.e_i. - \imath \omega_{ij}) s = \sum_{i,j=1}^{2l} \omega^{ji} e_i.e_j.s + \sum_{i=1}^{2l} \imath \delta_i^i s = -\sum_{i,j=1}^{2l} \omega^{ij} e_i.e_j.s + 2\imath l s = -A + 2\imath l s$. Comparing the left hand and right hand side, we get $A = \imath l s$.

(iii) Now, we shall compute the value of H.

For $r = 0, \dots, 2l$ and $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, we have $H(\alpha \otimes s) = XY(\alpha \otimes s) + YX(\alpha \otimes s) = \sum_{i,j=1}^{2l} \epsilon^i \wedge \iota_{\check{e}_j} \alpha \otimes e_i.e_j.s + \sum_{i,j=1}^{2l} \iota_{\check{e}_i} (\epsilon^j \wedge \alpha) \otimes e_i.e_j.s = \sum_{i,j=1}^{2l} \epsilon^i \wedge \iota_{\check{e}_j} \alpha \otimes e_i.e_j.s + \sum_{i,j=1}^{2l} \iota_{\check{e}_i} \epsilon^j \alpha \otimes e_i.e_j.s - \sum_{i,j=1}^{2l} \epsilon^j \wedge \iota_{\check{e}_i} \alpha \otimes e_i.e_j.s = \sum_{i,j=1}^{2l} \epsilon^j \wedge \iota_{\check{e}_i} \alpha \otimes e_j.e_i.s + \sum_{i,j=1}^{2l} \alpha \otimes \omega^{ij} e_i.e_j.s - \sum_{i,j=1}^{2l} \epsilon^j \wedge (\iota_{\check{e}_i} \alpha) \otimes (e_j.e_i.s - \imath \omega_{ij} s) = \sum_{i,j=1}^{2l} \alpha \otimes \omega^{ij} e_i.e_j.s - \sum_{i,j=1}^{2l} \imath \omega_{ji} \epsilon^j \wedge \iota_{\check{e}_i} \alpha \otimes s = \imath l \alpha \otimes s - \imath \alpha \otimes s$, where we have used equations (2) and (1) in the last step. $\square$

2.3 Metaplectic structure and symplectic Dirac, Rarita-Schwinger and twistor operators

After we have finished the algebraic part of this section, let us start describing the geometric structure with help of which the symplectic Dirac, Rarita-Schwinger and twistor operators are defined. This structure, called metaplectic, is a precise symplectic analogue of the notion of a spin structure in the Riemannian geometry.

Let (M^{2l}, ω) be a symplectic manifold of dimension $2l$ and let us denote the bundle of symplectic reperes in TM by $\mathcal{P}$ and the foot-point projection of $\mathcal{P}$ onto M by p. Thus $(\mathcal{P}, p, M, G)$, where $G \simeq Sp(2l, \mathbb{R})$, is a principal G-bundle over M. As in subsections 2.1. and 2.2., let $\lambda : \tilde{G} \to G$ be a member of the isomorphism class of the non-trivial two-fold coverings of the symplectic group G. In particular, $\tilde{G} \simeq Mp(2l, \mathbb{R})$. Further, let us consider a principal $\tilde{G}$-bundle $(\mathcal{Q}, q, M, \tilde{G})$ over the symplectic manifold (M, ω). We call a pair $(\mathcal{Q}, \Lambda)$ metaplectic structure if $\Lambda : \mathcal{Q} \to \mathcal{P}$ is a surjective bundle homomorphism over the identity on M and if the following diagram

$$\begin{tikzcd}
\mathcal{Q} \times \tilde{G} \arrow[r] \arrow[dd, "\Lambda\times\lambda"] & \mathcal{Q} \arrow[dd, "\Lambda"] \arrow[dr, "q"] & \\
 & & M \\
\mathcal{P} \times G \arrow[r] & \mathcal{P} \arrow[ur, "p"'] &
\end{tikzcd}$$

with the horizontal arrows being respective actions of the displayed groups commutes.

Let us mention that a precise obstruction to the existence of a metaplectic structure over a symplectic manifold (M, ω) is the parity of its first Chern class. A Chern class of a symplectic manifold (M, ω) is defined to be the Chern class of the so called ω-compatible almost complex structure which always exists and on which the Chern class does not depend. See Habermann, Habermann [6] and Kostant [10] for details on this construction and metaplectic structures in general. Let us only remark, that typical examples of symplectic manifolds admitting a metaplectic structure are cotangent bundles of orientable manifolds (also called phase spaces), complex projective spaces $\mathbb{CP}^{2k+1}$, $k \in \mathbb{N}_0$. For more details, see Habermann, Habermann [6].

Let us denote the vector bundle associated to the introduced principal $\tilde{G}$-bundle $(\mathcal{Q}, q, M, \tilde{G})$ via the representation ρ restricted to $\mathbf{S}$ by $\mathcal{S}$ and call this associated vector bundle symplectic spinor bundle. Thus we have $\mathcal{S} = \mathcal{Q} \times_\rho \mathbf{S}$. Further, we shall define the following associated vector bundles. $\mathcal{S}' := \mathcal{Q} \times_\rho \mathbf{S}'$, $\mathcal{T} := \mathcal{Q} \times_\rho \mathbf{T}$ and $\mathcal{T}' := \mathcal{Q} \times_\rho \mathbf{T}'$.

Because the projections p^{10}, p^{11}, p^{21} and the operators X, Y and H are $\tilde{G}$-equivariant, they lift to operators acting on sections of the corresponding associated vector bundles. We shall use the same symbols as for the defined operators as for their lifts to the associated vector bundle structure.

Now, we are in a position to define the differential operators we shall be dealing with. Let ∇ be an affine torsion free symplectic connection on M, i.e., ∇ is torsion free connection on TM and $\nabla\omega = 0$. Let us notice that there is no unique torsion free symplectic connection in contrast to the Riemannian geometry, where the Levi-Civita connection is the only one candidate. Obviously, the chosen connection ∇ induces a principal connection Z on the principal G-bundle $(\mathcal{P}, p, M, G)$. This principal connection lifts to a connection $\hat{Z}$ on the principal $\tilde{G}$-bundle $(\mathcal{Q}, q, M, \tilde{G})$. Let us denote the associated covariant derivative on the symplectic spinor bundle $\mathcal{S}$ by ∇^S, i.e., $\nabla^S : \Gamma(M, \mathcal{S}) \to \Gamma(M, TM^* \otimes \mathcal{S})$, and call it symplectic spinor covariant derivative. Finally, let us denote the associated exterior covariant derivative acting on symplectic spinor valued exterior differential forms by d^{∇^S}, i.e., $d^{\nabla^S}: \Gamma(M, \bigwedge^r TM^* \otimes \mathcal{S}) \to \Gamma(M, \bigwedge^{r+1} TM^* \otimes \mathcal{S})$, $r = 0, \ldots, 2l$.

Let us consider an operator $D_0 : \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S}')$ defined by $D_0 := p^{10}\nabla^S$. Using this operator, we can define the symplectic Dirac operator $\mathfrak{D} : \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S})$ by the equation

$$\mathfrak{D} := -Y D_0.$$

Further, we can define an operator $R_1 : \Gamma(M, \mathcal{T}) \to \Gamma(M, \mathcal{T}')$ by the equation $R_1 = p^{21} d^{\nabla^S}_{|\Gamma(M,\mathcal{T})}$. The symplectic Rarita-Schwinger operator $\mathfrak{R} : \Gamma(M, \mathcal{T}) \to \Gamma(M, \mathcal{T})$ is then defined by

$$\mathfrak{R} := -Y R_1.$$

We can also introduce symplectic analogs of two twistor operators known from the Riemannian geometry. Let us define an operator $T_0 : \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{T})$ by the prescription $T_0 = p^{11}\nabla^S$ and call it symplectic twistor operator. Finally, let us define the second symplectic twistor operator $T_1 : \Gamma(M, \mathcal{S}') \to \Gamma(M, \mathcal{T}')$ by $T_1 = p^{21} d^{\nabla^S}_{|\Gamma(M,\mathcal{S}')}$.

In the next picture, the operators D_0, R_1, T_0 and T_1 are displayed.

$$\begin{tikzcd}
\Gamma(M,\mathcal{S}) \arrow[r, "D_0"] \arrow[dr, "T_0"] & \Gamma(M,\mathcal{S}') \arrow[dr, "T_1"] & \circ \\
& \Gamma(M,\mathcal{T}) \arrow[r, "R_1"] & \Gamma(M,\mathcal{T}') \\
& & \circ
\end{tikzcd}$$

The symbol $\circ$ indicates a possible further summand according to the decomposition of $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}$, see Theorem 1.

Actually, in the rest of this text, we shall be proving that under the assumption of an anti-commutativity of the preceding diagram, the embedding theorem (see Introduction) holds. Before doing so, let us recall the following facts. It is obvious that if ∇ is a flat torsion free symplectic connection, then the Riemann curvature tensor field R^S, defined by $R^S := d^{\nabla^S}\nabla^S$, of ∇^S is zero. For further details, see Proposition 3.2.9 in Habermann, Habermann [6].

Example 1: This example is taken almost without any change from Habermann, Habermann [6] and should only illustrates how to deal with the definitions introduced above. Let us consider the symplectic vector space $(\mathbb{R}^2, \omega_0)$. Because the bundle of symplectic frames on $\mathbb{R}^2$ is trivial, there exists only one, up to an isomorphism, metaplectic structure, namely the trivial principal bundle $Mp(2, \mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$. This implies that also the symplectic spinor bundle $\mathcal{S}$ is trivial. Thus a section of this bundle is simply a function $\phi : \mathbb{R}^2 \to \mathbf{Sch}(\mathbb{R})$. This mapping could be equivalently considered as any mapping $\psi : \mathbb{R}^3 \to \mathbb{C}$ such that for any $(s, t) \in \mathbb{R}^2$, we have that $x \in \mathbb{R} \mapsto \psi(s, t, x) \in \mathbb{C}$ lies in the Schwartz space $\mathbf{Sch}(\mathbb{R})$. Choosing the flat connection ∇ on $\mathbb{R}^2$, we obtain the following prescription for the symplectic Dirac operator $\mathfrak{D}$. Namely,

$$(\mathfrak{D}\psi)(s,t,x) = \imath x \frac{\partial \psi}{\partial t}(s,t,x) - \frac{\partial^2 \psi}{\partial x \partial s}(s,t,x).$$

Considering the substitution $z = s + \imath t$, we easily obtain that the function $\psi(z,x) := e^{-x^2/2}h(z)$ lies in the kernel of the symplectic Dirac operator $\mathfrak{D}$ on $(\mathbb{R}^2, \omega_0)$, where h is any entire function, $z \in \mathbb{C}$ and $x \in \mathbb{R}$.

In the next text, we shall need the following technical

Lemma 6: For $X, Y \in \Gamma(M, TM)$ and $s \in \Gamma(M, \mathcal{S})$,

$$\nabla^S_X(Y.s) = (\nabla_X Y).s + Y.\nabla^S_X s.$$

Proof. See Habermann, Habermann [6], pp. 41. $\square$

Let us mention that our definition of the symplectic Dirac operator coincides with that one in the book of Habermann, Habermann [6]. In the work of Kadlčáková [7], symplectic Rarita-Schwinger and symplectic twistor operators were introduced but for a slightly different geometries, namely for the projective contact ones. The construction in Kadlčáková [7] is completely parallel to our construction.

3 Relation of the spectra of symplectic Dirac and Rarita-Schwinger operators

We start with a simple lemma which we shall use in the proof of the main theorem (Theorem 2).

Lemma 7: For a symplectic manifold (M, ω^{2l}) admitting a metaplectic structure and equipped by an affine torsion free symplectic connection ∇, the equation

$$X\nabla^S + d^{\nabla^S} X = 0 \tag{3}$$

hold on $\Gamma(M, \mathcal{S})$.

Proof. Take a local Darboux coordinates $(x^1, \ldots, x^{2l})$ in a neighborhood U of a point $m \in M$. For $i = 1, \ldots, 2l$, let us denote the local vector field $\frac{\partial}{\partial x^i}$ by e_i and the exterior differential form dual to e_i by ϵ^i, i.e., $\epsilon^i = dx^i$, for $i = 1, \ldots, 2l$. For a symplectic spinor $\phi \in \Gamma(M, \mathcal{S})$, we have $(X\nabla^S + d^{\nabla^S} X)\phi = \sum_{i=1}^{2l} X(\epsilon^i \otimes \nabla^S_{e_i}\phi) + d^{\nabla^S}(\sum_{i=1}^{2l} \epsilon^i \otimes e_i.\phi) = \sum_{i,j}^{2l} \epsilon^j \wedge \epsilon^i \otimes e_j.\nabla^S_{e_i}\phi + \sum_{i,j=1}^{2l}(d\epsilon^i \otimes e_i.\phi - \epsilon^i \wedge \nabla^S(e_i.\phi)) = \sum_{i,j}^{2l} \epsilon^j \wedge \epsilon^i \otimes e_j.\nabla^S_{e_i}\phi - \sum_{i,j=1}^{2l} \epsilon^i \wedge \epsilon^j \otimes \nabla^S_{e_j}(e_i.\phi)$, where we have used the fact that for $i = 1, \ldots, 2l$, $d\epsilon^i = 0$ which follows from the choice of ϵ^i's as exact differential forms. Further, we have $\sum_{i,j}^{2l} \epsilon^j \wedge \epsilon^i \otimes e_j.\nabla^S_{e_i}\phi - \sum_{i,j=1}^{2l} \epsilon^i \wedge \epsilon^j \otimes \nabla^S_{e_j}(e_i.\phi) = \sum_{i,j}^{2l} \epsilon^j \wedge \epsilon^i \otimes e_j.\nabla^S_{e_i}\phi - \sum_{i,j=1}^{2l} \epsilon^i \wedge \epsilon^j \otimes ((\nabla_{e_j} e_i).\phi - e_i.\nabla^S_{e_j}\phi)$, where we have used Lemma 6. Now, we can write $\sum_{i,j}^{2l} \epsilon^j \wedge \epsilon^i \otimes e_j.\nabla^S_{e_i}\phi - \sum_{i=1}^{2l} \epsilon^i \wedge \epsilon^j \otimes ((\nabla_{e_j} e_i).\phi - e_i.\nabla^S_{e_j}\phi) = \sum_{i,j}^{2l} \epsilon^j \wedge \epsilon^i \otimes e_j.\nabla^S_{e_i}\phi - \sum_{i=1}^{2l} \epsilon^i \wedge \epsilon^j \otimes (\Gamma^k_{ji} e_k).\phi - \sum_{i,j=1}^{2l} \epsilon^i \wedge \epsilon^j \otimes e_i.\nabla^S_{e_j}\phi$, where Γ^k_{ij}, $i, j, k = 1, \ldots, 2l$ are the Christoffel symbols of the symplectic connection ∇. Using the fact that ∇ is a torsion free connection, we have that $\Gamma^k_{ij} = \Gamma^k_{ji}$, and thus the term with Γ vanishes being a contraction of symmetric and antisymmetric tensor fields. Renumbering the indices in the last term, we get precisely the first one. Thus the relation $X\nabla^S + d^{\nabla^S}X = 0$ holds. $\Box$

For some technical reasons, let us introduce the following notion. Let (M,ω) be a symplectic manifold admitting a metaplectic structure $(\mathcal{Q},\Lambda)$ and let ∇ be an affine torsion free symplectic connection. Then $0 \neq s \in \Gamma(M,\mathcal{S})$ is called symplectic Killing spinor if

$$\nabla^S s = \lambda X s$$

for some $\lambda \in \mathbb{C}$. In this case, λ is called symplectic Killing number. Let us denote the set of symplectic Killing spinors by $\mathcal{K}ill$ and the set of symplectic Killing numbers by $Kill$. As a short-hand, we shall use the symbol $\mathrm{Eigen}(A)$ for the eigenfunctions of an operator A and the symbol $\mathrm{Spec}(A)$ for its spectrum.

Example 2: Let us consider the equation for a symplectic Killing spinor field on the symplectic manifold $(\mathbb{R}^2,\omega_0)$. For a discussion of the metaplectic structure on this manifold, see Example 1. Let $\phi:\mathbb{R}^2 \to \mathbf{Sch}(\mathbb{R})$ be a symplectic spinor. Instaed of ϕ, consider the associated $\psi:\mathbb{R}^3 \to \mathbb{C}$ as in the Example 1. The symplectic connection ∇^S associated to the flat connection on $\mathbb{R}^2$ takes the form $\nabla^S\phi = \epsilon^1 \otimes \frac{\partial\phi}{\partial s} + \epsilon^2 \otimes \frac{\partial\phi}{\partial t}$, where (s,t) are the canonical coordinates on $\mathbb{R}^2$. Obviously, we have also $X\phi = \epsilon^1 \otimes e_1.\phi + \epsilon^2 \otimes e_2.\phi = \epsilon^1 \otimes \imath x\phi + \epsilon^2 \otimes \frac{\partial\phi}{\partial x}$. The equation for a symplectic Killing spinor is equivalent to the following system of partial differential equations:

$$\frac{\partial\psi}{\partial t} = \lambda\frac{\partial\psi}{\partial x}$$

$$\frac{\partial\psi}{\partial s} = \lambda\imath x\psi.$$

It easy to see that in this case, $Kill = \{0\}$ and each symplectic Killing spinor ϕ is constant. More precisely, $\phi(s,t) = f$, where $0 \neq f \in \mathbf{Sch}(\mathbb{R})$ and $(s,t) \in \mathbb{R}^2$. Thus, the situation is very similar to the orthogonal case.

Theorem 2: Let (M^{2l},ω) be a symplectic manifold of dimension $2l$, $l \geq 2$, admitting a metaplectic structure $(\mathcal{Q},\Lambda)$ and ∇ be an affine torsion free flat symplectic connection.

(1) If $s \in \mathrm{Eigen}(\mathfrak{D})\backslash\mathcal{K}ill$, then $T_0 s \in \mathrm{Eigen}(\mathfrak{R})$.

(2) If $\lambda \in \mathrm{Spec}(\mathfrak{D})\backslash -\imath l(Kill)$, then $\frac{l-1}{l}\lambda \in \mathrm{Spec}(\mathfrak{R})$.

Proof. We prove the statement of the theorem in the following four items.

(i) Because the flatness of ∇ implies the flatness of ∇^S, we have in particular, $0 = p^{21}R^S = p^{21}d^{\nabla^S}\nabla^S = 0$. Thus we have $0 = p^{21}d^{\nabla^S}\nabla^S = p^{21}d^{\nabla^S}(p^{10} + p^{11})\nabla^S$. The last relation clearly implies

$$T_1 D_0 + R_1 T_0 \quad = \quad 0 \tag{4}$$

i.e., the mentioned anti-commutativity.

(ii) Let $s \in \Gamma(M, \mathcal{S})$ be an eigenvector of the symplectic Dirac operator $\mathfrak{D}$, i.e., $s \neq 0$ and there exists a complex number $\lambda \in \mathbb{C}$ such that $\mathfrak{D}s = \lambda s$. Applying the operator X on the last written equation and using the definition of Dirac operator, we get

$$-XYD_0 s = \lambda X s \tag{5}$$

Now, let us choose a local symplectic frame $\{e_i\}_{i=1}^{2l}$ and its dual local coframe $\{\epsilon^i\}_{i=1}^{2l}$. Due to the definition of D_0, we know that $D_0 s \in \Gamma(M, \mathcal{S}')$. Thus using Lemma 3, there exists an element $\phi \in \Gamma(M, \mathcal{S})$ such that $D_0 s = \sum_{i=1}^{2l} \epsilon^i \otimes e_i.\phi$ holds locally. We may write $-XYD_0 s = -XY(\sum_{i=1}^{2l} \epsilon^i \otimes e_i.\phi) = -X(\sum_{i=1}^{2l} \iota_{\check{e}_j} \epsilon^i e_j.e_i.\phi) = -X(\sum_{i=1}^{2l} \omega^{ji} e_j.e_i.\phi)$. Using equation (2), we get $-X(\sum_{i=1}^{2l} \omega^{ji} e_j.e_i.\phi) = -X(\imath l \phi) = -\imath l \sum_{i=1}^{2l} \epsilon^i \otimes e_i.\phi = -\imath l D_0 s$. Thus we have

$$-XYD_0 s = -\imath l D_0 s \tag{6}$$

Combining equations (5) and (6), we obtain

$$D_0 s \quad = \quad \imath \frac{\lambda}{l} X s \tag{7}$$

(iii) In this item, we prove that

$$T_1 X + X T_0 \quad = \quad 0 \tag{8}$$

Due to Lemma 7, we have $0 = X\nabla^S + d^{\nabla^S} X$. Applying the projection p^{21} to this equation, we get $0 = p^{21}(X\nabla^S + d^{\nabla^S} X) = p^{21}X(p^{10} + p^{11})\nabla^S + p^{21}d^{\nabla^S} X = p^{21}Xp^{10}\nabla^S + p^{21}Xp^{11}\nabla^S + p^{21}d^{\nabla^S} X = 0 + XT_0 + T_1 X$ and the relation (8) follows. In the last equation, we have used that the mapping $p^{21}X_{\Gamma(M,\mathcal{S}')} : \Gamma(M, \mathcal{S}') \to \Gamma(M, \mathcal{T}')$ is zero which follows from the fact that $p^{21}X$ is $\tilde{G}$-equivariant, and $\mathbf{S}'$ and $\mathbf{T}'$ have no irreducible submodule in common.

(iv) Now, let us take the eigenvector s of the symplectic Dirac operator chosen above. Using the definition of the symplectic Rarita-Schwinger operator, we have $\mathfrak{R}T_0 s = -YR_1 T_0 s$. Further, we can write $-YR_1T_0 s = YT_1 D_0 s = \imath \frac{\lambda}{l} YT_1 X s$, where we have used equation (4) and (7). Equation (8) implies that $\imath \frac{\lambda}{l} YT_1 X s = -\imath \frac{\lambda}{l} YXT_0 s$. Using the relation $\{X, Y\} = H$, we have $-\imath \frac{\lambda}{l} YXT_0 s = -\imath \frac{\lambda}{l}(-XY + H)T_0 s = -\imath \frac{\lambda}{l}(-XYT_0 s + HT_0 s)$. Due to the definition of T_0, we know that $T_0 s \in \Gamma(M, \mathcal{T})$. Restricting Y to $\Gamma(M, \mathcal{T})$, we get a $\tilde{G}$- invariant operator going from $\Gamma(M, \mathcal{T})$ to $\Gamma(M, \mathcal{S})$, which must be zero because $\mathbf{T}$ and $\mathbf{S}$ have no irreducible submodule in common. Thus in particular, we have $YT_0 s = 0$. Using this fact together with the fact that H acts by the $\imath(l - r)$th multiple of the identity on symplectic spinor valued exterior differential r-forms (see Lemma 5), we have $-\imath\frac{\lambda}{l}(-XYT_0s+HT_0s) = -\imath\frac{\lambda}{l}(0+\imath(l-1)T_0s) = \frac{l-1}{l}\lambda T_0s$. Summing up, we have obtained that

$$\mathfrak{R}(T_0s) = \frac{l-1}{l}\lambda T_0s.$$

The last thing which remains to check, is that T_0s is nonzero. Suppose that $T_0s = 0$ for a contradiction. For the eigenvector s of the symplectic Dirac operator chosen above, we have $D_0s = \imath\frac{\lambda}{l}Xs$, see equation (7). Using the definition of D_0, we get $p^{10}\nabla^S s = D_0s = \imath\frac{\lambda}{l}Xs$. Because we suppose that $T_0s = p^{11}\nabla^S s = 0$, we obtain that $\nabla^S s = (p^{10}+p^{11})\nabla^S s = \imath\frac{\lambda}{l}Xs+0$. Thus s is a symplectic Killing spinor (and $\imath\frac{\lambda}{l}$ is a symplectic Killing number), which we have excluded in the formulation of this theorem. $\Box$

Let us notice that the assumption $R^\nabla = 0$, where R^∇ is the Riemann curvature tensor field of ∇, could be weakened to $p^{21}R^S = 0$ in order Theorem 2 remains true. (Analyzing this condition, one finds out that it is satisfied by certain broader class of symplectic manifolds then the flat ones.) Let us also remark that in Rudnick [15], the case of the sphere S^2 (equipped by the unique metaplectic structure and the Levi-Civita connection of the round metric) is treated. In particular, a few eigenvalues of the corresponding symplectic Dirac operator are computed there. Thus Theorem 2 enables us to compute at least a part of the spectrum of the symplectic Rarita-Schwinger operator in this case.

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