Segal-Shale-Weil complex

Svatopluk Krýsl

Faculty of Mathematics and Physics, Charles University in Prague

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Srní
$K(H)$ and the Hilbert module

- $H$ be a separable Hilbert space $(,)_H$
- $K(H)$ be the vector space of compact operators on $H = \text{completion of finite rank operators in the operator norm}$
- $K(H) \subseteq$ of bounded operators
- For any $f \in H, a \in K(H)$

$$f \cdot a = a^*(f)$$

defines a right $K(H)$-module
Hilbert modules
Examples of Hilbert modules and $C^*$-algebras
Elliptic complexes of differential operators
Segal-Shale-Weil complex

Algebra of compact operators

- $K(H)$ is a $C^*$-algebra with respect to the adjoint of maps and the norm $|a|_{K(H)} = \sup_{v \in H, |v| = 1} |a(v)|_H$, $a \in K(H)$
- $(,): H \times H \to K(H), (f, g) = f \otimes g^*$, $(f \otimes g^*)(h) = (g, h)_H f$, $f, g, h \in H$
- It maps into rank 1 operator
- **Lemma:** $H$ is a $K(H)$-Hilbert module.
- **Proof.** Check $(f \cdot a, g) = (f, g)a^*$, $(f, g \cdot b) = (f, g)b$
  $(f, f) \geq 0$ and $(f, f) = 0$ implies $f = 0$. $H$ is complete wrt. $|f| = \sqrt{|(f, f)_H|_{K(H)}}$, $f \in H$. □
General definitions

- $K(H)$ is a $C^*$-algebra
- $K(H)$ is associative
- $*: K(H) \to K(H)$ and $*^2 = \text{Id}_{K(H)}$
- $\|: K(H) \to [0, \infty)$ is a norm and $|TT^*| = |T|^2$ ($C^*$-identity)
- Addition, multiplication and scalar multiplication are continuous (consequence of triangle + $C^*$-identity)
- $K(H)$ is complete with respect to $\|$(it is so defined)
Definition of Hilbert and pre-Hilbert $A$-modules

**Definition**

Let $A$ be a $C^*$-algebra and $H$ be a vector space over the complex numbers. We call $(H, (, ))$ a **pre-Hilbert $A$-module** if

- $H$ is a right $A$-module, $\cdot : H \times A \rightarrow H$
- $(, ) : H \times H \rightarrow A$ is a $\mathbb{C}$-bilinear mapping
- $(f \cdot T + g, h) = (f, h) T^* + (g, h)$, $f, g, h \in H$, $T \in K(H)$
- $(f, g) = (g, f)^*$
- $(f, f) \geq 0$ and $(f, f) = 0$ implies $f = 0$

We say $T \in A$ is non-negative ($T \geq 0$) if $T = T^*$ and $\text{Spec}(T) \subseteq [0, \infty)$.

$\text{Spec}(T) = \{ \lambda \in \mathbb{C}; T - \lambda \overline{1} \text{ is not invertible in } A^0 \}$, where $\overline{1} = (0, 1)$ is the unit in $A^0 = A \oplus \mathbb{C}$ (augmentation)
Definition of Hilbert and pre-Hilbert $A$-modules

**Definition**

If $(H, (, ))$ is a pre-Hilbert $A$-module we call it **Hilbert $A$-module** if it is complete with respect to the norm $| | : H \rightarrow [0, \infty)$ defined by $f \ni H \mapsto |f| = \sqrt{|(f, f)|_A}$ where $| |_A$ is the norm in $A$.

Pre-Hilbert $A$-module is a normed space. Hilbert $A$-module is a Banach space.
Examples of $C^*$-algebras

- $X$ locally compact topological vector space, $A = C_0(X)$ (continuous complex valued functions vanishing at infinity), $(*f)(x) = \overline{f(x)}$, $x \in X$, $|f| = \sup\{|f(x)|; x \in X\}$

- $H$ Hilbert space, $A = B(H)$ bounded on $H$, $*T = T^*$, $|T|$ the supremum norm
Examples of Hilbert $A$-modules

- For $A$ a $C^*$-algebra, $M = A$, $a \cdot b = ab$ and $(a, b) = a^* b$.
  Form $(M, (, ))$ - it is a Hilbert $A$-module

- For $A = K(H)$, the $C^*$-algebra of compact operators on a separable Hilbert space $H$, $M = H$ is a Hilbert $A$-module with respect to $(,) : H \times H \to K(H)$ given by $(f, g) = f \otimes g^*$ and the right action given by the evaluation $f \cdot T = T^*(f)$.

- If $M$ is a Hilbert $A$-module, then $M^n = M \oplus \ldots \oplus M$ is a Hilbert $A$-module with respect to $(m_1, \ldots, m_n) \cdot a = (m_1 \cdot a, \ldots, m_n \cdot a)$ and the product given by $(m_1, \ldots, m_n) \cdot (m'_1, \ldots, m'_n) = \sum_{i=1}^n (m_i, m'_i)$

- Further generalizes to $\ell^2(M)$ controlled by the convergence in $A$. Special case $\ell^2(A)$ ($M = A$)
Definition: Fomenko, Mishchenko [FM]
Attempt: generalize the Atiyah-Singer index theorem

- An $A$-Hilbert bundle is a Banach bundle the fibers of which are homeomorphic to a fixed Hilbert $A$-module $M$ and the transition functions are into $\text{Aut}_A(M)$
- If $\mathcal{F} \to M$ is a Hilbert bundle over a compact $M$, then $\Gamma(\mathcal{E})$ is a pre-Hilbert $A$-module; canonically $(s \cdot a)(m) = s(m) \cdot a$, $m \in M$; $s \in \Gamma(\mathcal{F})$ and $a \in A$.
- Sobolev type completion of $\Gamma(\mathcal{E})$ exists (over compacts)
- These completions form Hilbert $A$-modules
(finite order) differential operators in finite rank vector bundles over a manifold → generalizes

(finite order) differential operators in $A$-Hilbert bundles

symbols of differential operators (as in classical PDE-theory),

$$\sigma : \triangle \mapsto (\sigma(\triangle) : f \mapsto |x|^2 f)$$

(Differential operators) $D \rightarrow \sigma(D)$ (Morphisms in the category of $A$-Hilbert bundles)

Definition

A complex $D = (\Gamma(E^k), D_k)_k$ of differential operators in $A$-Hilbert bundles $E^k$ is called elliptic if its symbol sequence is exact in the category of $A$-Hilbert bundles.
Theorem (Krýsl): Let $M$ be a compact manifold, $A$ a $C^*$-algebra, $(\mathcal{F}^k)_{k \in \mathbb{N}_0}$ a sequence of finitely generated projective $A$-Hilbert bundles over $M$ and $D_k : \Gamma(\mathcal{F}^k) \to \Gamma(\mathcal{F}^{k+1})$, $k \in \mathbb{Z}$, a complex $D$ of differential operators. Suppose that the Laplace operators $\triangle_k = D_{k-1}D^*_{k-1} + D^*_kD_k$ of $D$ have closed image in the norm topology of $\Gamma(\mathcal{F}^k)$. If $D$ is elliptic, then the cohomology of $D$ is finitely generated and projective $A$-module, especially a Banach space and $\Gamma(\mathcal{F}^k) = \text{Ker} \triangle_k \oplus \text{Im} D_{k-1} \oplus \text{Im} D^*_{k+1}$ and $H^k(M, D) \simeq \text{Ker} \triangle_k$. Moreover, the cohomology groups of $D$ are finitely generated and projective Hilbert $A$-modules.

Theorem (Krýsl, AGAG15): If $A$ is a $C^*$-subalgebra of the algebra of compact operators $K(H)$, one may drop the closed image assumption on the Laplacians.
Symplectic structures

- $(V, \omega)$ a symplectic space of dimension $2n$ (flat phase space of a system with $n$-degrees of freedom)
- $G = Sp(V, \omega)$ the symplectic group (linear transformation which do not change the form of the Hamilton equations)
- $\pi_1(G) = \mathbb{Z} \implies \exists 2:1$ covering $\lambda : \tilde{G} \rightarrow G$
- $\tilde{G} = Mp(V, \omega)$ the metaplectic group
- non-universal, not compact, non matrix Lie group
Segal-Shale-Weil representation

- $L$ any maximal isotropic (Lagrangian) subspace of $(V, \omega)$
- $J : V \to V$ compatible complex structure, the bilinear form $g(v, w) = \omega(Jv, w)$ is positive definite
- $H = L^2(L)$ Lebesgue square integrable functions on $L$
- $\sigma : \tilde{G} \to U(L^2(L))$ the Segal-Shale-Weil representation
- oscillator, metaplectic, symplectic spinor
- $\sigma(\tilde{\omega}) = \mathcal{F} : L^2(L) \to L^2(L), \tilde{\omega} \in \lambda^{-1}(\omega)$
Properties of SSW

- $\sigma$ is unitary
- decomposes into an orthogonal sum of two irreducible $\tilde{G}$-modules
- they are highest weight modules (and especially in the category $\mathcal{O}$)
- multiplicity bounded reps of $\mathfrak{sp}(V,\omega)$
Symplectic parallels of orthogonal spin geometry

- $(M, \omega)$ symplectic manifold of dimension $2n$ (phase space of a ”curved” system of $n$ freedom degrees)
- $\mathcal{P} = \{ e = (e_1, \ldots, e_{2n}) | e \text{ is a symplectic basis of } T^*_x M, x \in M \}$
- $p : \mathcal{P} \to M$ is a principal $G$-bundle
- $(\Lambda, q)$, where $q : \tilde{\mathcal{P}} \to M$ a $\tilde{G}$-bundle and $\Lambda$ is a bundle homomorphism, is called metaplectic structure if the diagram commutes
Hilbert modules

Examples of Hilbert modules and $C^*$-algebras

Elliptic complexes of differential operators

Segal-Shale-Weil complex

\[ \begin{array}{ccc}
\tilde{\mathcal{P}} \times \tilde{G} & \longrightarrow & \tilde{\mathcal{P}} \\
\Lambda \times \lambda & \downarrow & \Lambda \\
\mathcal{P} \times G & \longrightarrow & \mathcal{P} \\
\end{array} \]

\[ \begin{array}{c}
\Lambda \times \lambda \\
\Lambda \\
p \\
\end{array} \]

\[ \begin{array}{c}
M \\
\end{array} \]
Higher symplectic spins

\[ \sigma^k : \tilde{G} \to \text{Aut}(\bigwedge^k V^* \otimes H), \ k = 0, \ldots, 2n \]

\[ \sigma^k(g)(\alpha \otimes s) = \lambda^* \bigwedge^k(g)(\alpha) \otimes \sigma(g)(s), \ g \in \tilde{G}, \ s \in H \]

\[ E^k = \bigwedge^k V^* \otimes H \] ”Higher symplectic spinors”

Bundle of higher symplectic spinors:
\[ \mathcal{E}^k = \tilde{P} \times_{\sigma^k} (\bigwedge^k V^* \otimes H) \]

Higher symplectic spinor fields \( \Gamma(\mathcal{E}^k) \)

\( K(H) \)-structure on these fields

\[ (\alpha \otimes v) \cdot a = \alpha \otimes a^*(v) \]

\[ (\alpha \otimes v, \beta \otimes w) = g(\alpha, \beta)v \otimes w^*, \ \alpha, \beta \in \bigwedge^k V^*, \ a \in K(H), \ v, w \in H \]
Trivialization - Kuiper theorem

\[ H = L^2(L), \mathcal{E}^0 = \mathcal{H}, \]
\[ \nabla \text{ a flat connection on } \mathcal{E}^0 \]
 Exists because \( \mathcal{E}^0 \) is trivial: trivialization, horizontal distribution, horizontal directions define the connection
\[ \nabla \text{ induces } d_k^{\nabla} : \Gamma(\mathcal{E}^k) \to \Gamma(\mathcal{E}^{k+1}) \text{ by the Leibniz formula} \]
\[ \nabla_X(s \cdot a) = (\nabla_X s) \cdot a, \ s \in \Gamma(\mathcal{H}), \ k = 0, \ldots, 2n \]
\[ \nabla_X(s, t) = (\nabla_X s, t) + (s, \nabla_X t), \ s, t \in \Gamma(\mathcal{H}), \ X \in \mathfrak{X}(M) \]
 It is a hermitian \( A \)-connection
The Segal-Shale-Weil complex

Let \((M, \omega)\) be a symplectic manifold admitting a metaplectic structure and \(\nabla\) be a trivial connection on \(E^0\)

**Definition**

The complex \(0 \to \Gamma(E^0) \xrightarrow{\partial_0} \Gamma(E^1) \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{2n-1}} \Gamma(E^{2n}) \to 0\) is called the Segal-Shale-Weil complex.

This complex is elliptic, i.e., the symbol sequence is exact (equivalent to symbols of Laplacians are isomorphisms) (out of the zero section of the cotangent bundle)

Laplacians \(\triangle_k = d_{k-1}d_{k-1}^* + d_k^*d_k, \ k = 0, \ldots, 2n\)
Azumaya bundle, matrix densities

\( K(H) \) algebra / vector space of compact operators on \( H \)
\begin{equation*}
\rho : \tilde{G} \to \text{Aut}(K(H))
\end{equation*}
\begin{equation*}
\rho(g)a = \sigma(g)a\sigma(g)^{-1}, \ g \in \tilde{G}, \ a \in K(H)
\end{equation*}
\begin{equation*}
\mathcal{A} = \tilde{\mathcal{P}} \times \rho K(H)
\end{equation*}

So called Azumaya bundle, sections form sheaves of Azumaya algebras (Bundle of ”matrix densities”, ”Filtern”, ”measuring devices”)

Kuiper theorem + \( K(H) = H \hat{\otimes} H \longrightarrow \mathcal{A} \) is trivial, represented by 0 in \( H^3(M, \mathbb{Z}) \)
Construction in more detail

$\mathcal{E}^{k'}$ is $\lambda^{-1}(U(n))$-reduction of $\mathcal{E}^k$

$\cdot : \mathcal{E}^k \otimes \mathcal{A} \to \mathcal{E}^k$

$$[(e, v)] \cdot [(e, a)] = [(e, v \cdot a)]$$

where $e \in \tilde{\mathcal{P}}$, $v \in \mathcal{E}^k$, and $a \in K(H)$

$() : \mathcal{E}^{k'} \otimes \mathcal{E}^{k'} \to \mathcal{A}'$

$$([(e, v)], [(e, w)]) = [(e, (v, w))] \in \mathcal{A}'$$

where $e \in \tilde{\mathcal{P}}$, and $v, w \in \mathcal{E}^k$

$\psi : \mathcal{A} \to M \times K(H)$ trivialization

$\mathcal{M} : \mathcal{E}^k \times \mathcal{E}^k \to \mathcal{A}$

$\mathcal{M}(s', s'') = \text{pr}_2(\psi((s', s'')))$, $s', s'' \in \mathcal{E}^k$

$\mathcal{A} : \mathcal{E}^k \times \mathcal{A} \to \mathcal{E}^k$

$\mathcal{A}(s, a) = s \cdot \psi^{-1}(p(s), a)$, $s \in \mathcal{E}^k$, $a \in \mathcal{A}$
Analytic properties of the cohomology of the SSW-complex

**Theorem:** If $M$ is a compact symplectic, admits a metaplectic structure, and $\nabla$ is a flat connection on $\mathcal{H}$, then the cohomology groups are finitely generated projective Hilbert $K(H)$-modules and the Hodge decomposition holds for it.


