Twistor operators in symplectic geometry

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Abstract. On a symplectic manifold equipped with a symplectic connection and a metaplectic structure, we define two families of sequences of differential operators, called the symplectic twistor operators. We prove that if the connection is torsion-free and Weyl-flat, the sequences in these families form complexes.

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1. Introduction

Twistor operators on Riemannian spin manifolds are often used in mathematical General relativity and Differential geometry (see [3, 5, 18]). They are usually introduced using local orthogonal frames on the manifold and the Clifford multiplication, or using a tensor product decomposition of appropriate spin-modules into irreducible submodules.

Weil, who searched for symmetries of theta functions (see [26]), and Shale, who searched for symmetries of quantized Klein–Gordon fields (see [20]), discovered a unitary representation of the metaplectic group, a Lie group double cover of the symplectic group. This started a development of the symplectic spin geometry. In the seventies of the last century, Kostant [13] defined a metaplectic structure over a symplectic manifold and enabled a research of symplectic spinor fields, which are sections of bundles that are associated to the representation found by Shale and Weil. Sommen in [21] studied these structures on Euclidean spaces from the point of view of supersymmetry and Clifford algebras. In global analysis, the metaplectic structures were investigated with the help of symplectic Dirac operators, defined in the work of Habermann [6]. See also [1, 8, 17].

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For a Fedosov connection (see Gelfand, Retakh, Shubin [4] and Tondeur [23]), we consider exterior covariant derivatives acting on symplectic spinor fields. We use a decomposition of a tensor product into irreducible submodules over the metaplectic group (Theorem 2.2) to define two families of sequences of differential operators, which we call *symplectic twistor operators*. From the point of view of representation theory, these operators are similar to the Riemannian or Lorentzian twistor operators. They are compositions of appropriate exterior covariant derivatives with projections onto sections of bundles induced by irreducible modules with specific highest weights.

Note that Dolbeault operators on an almost complex manifold form complexes if the almost complex structure is integrable. We prove that symplectic twistor operators form complexes if the Fedosov connection is Weyl-flat. This is already known for two sequences of symplectic twistor operators (see Krýsl [15]). In Theorem 3.4 this result is generalized. We prove that all of the introduced sequences in the two families are complexes under the Weyl-flatness condition. We use the tensor product decomposition of the appropriate Lie group representations into irreducible subrepresentations.

2. Symplectic Spinors

Let (V, ω) be a finite dimensional symplectic vector space over the real numbers. Let us recall that the *symplectic group* $Sp(V)$ of (V, ω) is the Lie subgroup of the general linear group of V consisting of maps preserving ω. Let J be a complex structure on V (linear map satisfying $J^2 = -\mathrm{Id}_V$) such that $g(v, w) = \omega(Jv, w)$, $v, w \in V$, is positive definite. The complex structure determines the unitary group $U(V)$ associated to the triple (V, J, g). From the structure theory of Lie groups, it is known that $U(V)$ is a maximal compact subgroup of $Sp(V)$ (see Knapp [11]). It is well known that the homotopy type of $U(V)$ is that of the circle S^1, i.e., the fundamental group of $Sp(V)$ is isomorphic to $\mathbb{Z}$. By the theory of covering spaces, there is a connected two-fold covering of $Sp(V)$. Moreover, for a fixed covering and a choice of a point in the preimage of the neutral element in $Sp(V)$, there is a unique Lie group structure on the covering space so that the covering map is a Lie group homomorphism and such that the chosen point is the neutral element of the Lie group. The covering space is called the *metaplectic group* and it is denoted by $Mp(V)$. We denote the covering map by λ. It is known that $Mp(V)$ is a non-matrix Lie group, i.e., there is no topological embedding of this group which is a group homomorphism into the general linear group of a finite dimensional vector space (see [17]).

Let L be a Lagrangian subspace of the symplectic space (V, ω). On L, we consider the norm induced by g. Let us denote the Hilbert space of square Lebesgue integrable complex valued functions on L modulo equal almost everywhere (ae.) by E, and the unitary group of E by $U(E)$. There is a unitary representation (see, e.g., Weil [26], Shale [20] or Wallach [25]) $\rho : Mp(V) \to U(E)$ of $Mp(V)$ on E called the oscillator (Shale, Shale–Weil, Segal–Shale–Weil, metaplectic or symplectic spinor) representation. We shall call it the *oscillator representation* following Howe [9]. The representation (ρ, E) is the orthogonal direct sum of two irreducible representations, that we denote by E_+ and E_-, where $E_\pm$ are the spaces of even and odd elements in E (considered modulo ae.), respectively.

Remark 2.1. Let I be the two-sided ideal generated by elements $v \otimes w - w \otimes v - \omega(v, w)1$, $v, w \in V$, as a two-sided ideal in the tensor algebra $T(V)$ of V. The quotient $T(V)/I$ is called the *symplectic Clifford algebra* of (V, ω) and we denote it by $Cl_s(V)$. Any associative algebra has a Lie algebra structure defined by the commutator. As in the orthogonal case, we have a Lie algebra monomorphism of the Lie algebra $\mathfrak{sp}(V)$ of $Sp(V)$ into $Cl_s(V)$. This monomorphism makes us able to consider $Cl_s(V)$ as a left $\mathfrak{sp}(V)$-submodule of $Cl_s(V)$. It can be proved that the so-called Harish-Chandra $(\mathfrak{g}, K)$-module of (ρ, E) is a left ideal in $Cl_s(V)$. See Habermann, Habermann [7] and Kirillov [10]. In the considered case, $\mathfrak{g}$ is the Lie algebra of $Mp(V)$ and $K = \lambda^{-1}(U(V))$.

Let λ^* be the representation on V^* which is dual to the representation λ. Representations λ and λ^* are equivalent as follows by considering the equivariant map $T : V \to V^*$ defined by $T(v)(w) = \omega(v, w)$, where $v, w \in V$. We denote the dual representation by λ as well. Let us consider V equipped with the norm induced by the scalar product g, and $\bigwedge^i V$ with the norm induced by a Hodge scalar product. The exterior powers of λ are denoted by λ^i, $\lambda^i : Mp(V) \to GL(\bigwedge^i V)$, where $GL(\bigwedge^i V)$ denotes the set of all linear automorphisms of the appropriate wedge power. Further, let us consider the vector spaces $E^i_\pm = \bigwedge^i V \otimes E_\pm$, $i \in \mathbb{N}_0$, with the Hilbert tensor product topology. No completion is necessary since the exterior powers are finite dimensional. Let $GL(E^i_\pm)$ denote the set of all linear homeomorphisms of $E^i_\pm$. The tensor product representations $\rho^i_\pm : Mp(V) \to GL(E^i_\pm)$ are defined by $\rho^i_\pm(g)(\alpha \otimes w) = \lambda^i(g)\alpha \otimes \rho(g)w$ for $g \in Mp(V)$, $\alpha \in \bigwedge^i V$ and $w \in E_\pm$, and by the linear extension to all elements in the tensor product.

Let $2n$ be the dimension of V and let $\mathfrak{g}^{\mathbb{C}}$ denote the complexification of the Lie algebra of the metaplectic group $Mp(V)$. For a choice of a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}^{\mathbb{C}}$ and a choice of positive roots Φ^+, the set of fundamental weights $\{\varpi_i\}_{i=1}^n$ is uniquely determined. For $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $\lambda = \sum_{i=1}^n \lambda_i \varpi_i \in \mathfrak{h}^*$, we denote the irreducible highest weight $\mathfrak{g}^{\mathbb{C}}$-module $L(\lambda)$ by $L(\lambda_1, \ldots, \lambda_n)$.

We define the map $\mathrm{sgn} : \{+, -\} \to \{0, 1\}$ by $\mathrm{sgn}(+) = 0$ and $\mathrm{sgn}(-) = 1$. For $i = 0, \ldots, 2n$, we set $k_{n,i} = n - |n - i|$ and

$$\mathbb{E}^{ij}_\pm = L(\underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{i}, \underbrace{-\frac{1}{2}, \ldots, -\frac{1}{2}}_{n-i-1}, -1 + \frac{1}{2}(-1)^{\mathrm{sgn}(\pm)+i+j})$$

$$\begin{array}{ccccccc}
E^0 & E^1 & E^2 & E^3 & E^4 & E^5 & E^6 \\
E^{00} & E^{10} & E^{20} & E^{30} & E^{40} & E^{50} & E^{60} \\
 & E^{11} & E^{21} & E^{31} & E^{41} & E^{51} & \\
 & & E^{22} & E^{32} & E^{42} & & \\
 & & & E^{33} & & &
\end{array}$$

FIGURE 1. Decomposition for $n = 3$

for $i = 0, \ldots, n-1$, $j_i = 0, \ldots, k_{n,i}$, and for $i = n$ and $j_i = 0 \ldots, n-1$. For $i = n$ and $j = n$, we set $\mathbb{E}_+^{nn} = L(\frac{1}{2}, \ldots, \frac{1}{2})$, $\mathbb{E}_-^{nn} = L(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{5}{2})$; and $\mathbb{E}_\pm^{n+i,j} = \mathbb{E}_\pm^{n-i,j}$, where $i = 1, \ldots, n$ and $j = 0, \ldots, k_{n,i}$.

For each $i = 0, \ldots, 2n$, a decomposition of the representation $\rho_\pm^i$ on $E_\pm^i$ into irreducible $Mp(V)$-subrepresentation is described in [16].

Theorem 2.2. *For $i = 0, \ldots, 2n$, $j_i = 0, \ldots, k_{n,i}$, there are topological vector spaces $E_\pm^{ij_i}$ and irreducible representations $\rho_\pm^{ij_i}$ of $Mp(V)$ on $E_\pm^{ij_i}$ such that the $\mathfrak{g}^{\mathbb{C}}$-structure of the Harish-Chandra $(\mathfrak{g}^{\mathbb{C}}, K)$-module of $E_\pm^{ij_i}$ is isomorphic to the $\mathfrak{g}^{\mathbb{C}}$-module $\mathbb{E}_\pm^{ij_i}$. The representation $\rho_\pm^i$ on $E_\pm^i$ is equivalent to the direct sum $\bigoplus_{j_i=0}^{k_{n,i}} \rho_\pm^{ij_i}$.*

Let us set $E^{ij} = E_+^{ij} \oplus E_-^{ij}$, $E^i = E_+^i \oplus E_-^i$, $\rho^{ij} = \rho_+^{ij} \oplus \rho_-^{ij}$ and $\rho^i = \rho_+^i \oplus \rho_-^i$ for the representations on E^{ij} and E^i, respectively, where $i = 0, \ldots, 2n$ and $j = 0 \ldots, k_{n,i}$. The decompositions of the representations E^i into irreducible subrepresentations can be seen at Figure 1 for $n = 3$.

Remark 2.3.
1. Spaces E^{ij} are endowed with the topology inherited by the inclusion $E^{ij} \subseteq E^i$.
2. Each (ρ^i, E^i) is multiplicity-free, i.e., when (ρ', E'), (ρ'', E'') are different irreducible subrepresentations of ρ^i, they are not equivalent. (See [16].)
3. Representation $\rho_\pm^{ij}$ is equivalent to $\rho_\mp^{i+1,j}$ for each i and j satisfying $0 \leq i \leq 2n$, $0 \leq i+1 \leq 2n$, $1 \leq j \leq \min\{k_{n,i}, k_{n,i+1}\}$ (see [16]).
4. For $j < 0$ and for $j > k_{n,i}$, we set $\rho_\pm^{ij} = 0$, $\rho^{ij} = 0$, $E_\pm^{ij} = 0$ and $E^{ij} = 0$.

For each $i = 0, \ldots, n$, and $j_i = 0, \ldots, k_{n,i}$, let $p_\pm^{ij_i}$ denote the *unique* $Mp(V)$-equivariant projections of $E_\pm^i$ onto $E_\pm^{ij_i}$. The correctness of the definition of the projections, regarding their uniqueness, follows from an appropriate version of the Schur lemma (Dixmier [2], p. 87), from the fact that E^i are Hilbert space globalizations (see, e.g., the overview article of Schmid [19]), and from the multiplicity freeness of each E^i (item 2 of Remark 2.3 above).

$$\begin{tikzcd}
\mathcal{P} \times Mp(V) \arrow[r] \arrow[dd, "\Lambda\times\lambda"] & \mathcal{P} \arrow[dd, "\Lambda"] \arrow[dr, "\pi_P"] & \\
& & M \\
\mathcal{Q} \times Sp(V) \arrow[r] & \mathcal{Q} \arrow[ur, "\pi_Q"'] &
\end{tikzcd}$$

FIGURE 2. Diagram - metaplectic structure

3. Symplectic twistor operators and complexes

Let (M, ω) be a symplectic manifold of dimension $2n$, $n \in \mathbb{N}$, and (V, ω_0) be a symplectic vector space over the real numbers of the same dimension. We consider the set of symplectic frames $\mathcal{Q} = \{A : V \to T_mM \,|\, \omega(Av, Aw) = \omega_0(v, w), v, w \in V, m \in M\}$ and the map $p_Q : \mathcal{Q} \to M$ defined by setting $p_Q(A) = m$ if and only if $A : V \to T_mM$. The topology on $\mathcal{Q}$ is given by considering the so-called frame topology, which is the final topology for the set of the inverses of canonical charts (see Sternberg [22]). It is well known that the canonical charts define also a smooth bundle atlas. Let us consider the right action of $Sp(V)$ on $\mathcal{Q}$ given by the map composition from the right. Then $p_Q : \mathcal{Q} \to M$ is a principal $Sp(V)$-bundle on M. We call a principal $Mp(V)$-bundle $p_P : \mathcal{P} \to M$ on M and a morphism of principal fibre bundles $\Lambda : \mathcal{P} \to \mathcal{Q}$ a *metaplectic structure* on (M, ω) if and only if $\Lambda(Ag) = \Lambda(A)\lambda(g)$ for any $A \in \mathcal{P}$ and $g \in Mp(V)$. Thus those fibre bundle morphisms Λ are allowed for which the diagram at Figure 2 commutes (the horizontal arrows represent actions of the appropriate groups).

Definition 3.1. An affine connection ∇ is called a *symplectic connection* if $\nabla\omega = 0$. It is called a *Fedosov connection* if it is symplectic and torsion-free.

Remark 3.2. In contrast to Riemannian connections, there are infinitely many Fedosov connections if $n \geq 1$. Moreover, these connections form an infinite dimensional affine space in this case (see Gelfand, Retakh, Shubin [4]).

We use $\rho_\pm^{ij}$ for defining the associated vector bundles $\mathcal{E}_\pm^{ij} = \mathcal{P} \times_{\rho^{ij}} E_\pm^{ij} = (\mathcal{P} \times E_\pm^{ij})/\simeq$, where $(q, f) \simeq (q', f')$ if and only if there exists an element $g \in Mp(V)$ such that $q' = qg$ and $f' = \rho_\pm^{ij}(g^{-1})f$. We set $\mathcal{E}^{ij} = \mathcal{E}_+^{ij} \oplus \mathcal{E}_-^{ij}$. Bundles $\mathcal{E}_\pm^i$ and $\mathcal{E}^i$ are defined by the appropriate representations $\rho_\pm^i$ and ρ^i. We also set $\mathcal{E}_\pm = \mathcal{E}_\pm^{00}$ and $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$. Associated bundles are considered with the quotient topology. Elements of $\Gamma(\oplus_{i=0}^{2n}\mathcal{E}^i)$ are called *symplectic spinors* fields.

Any symplectic connection ∇ defines a principal bundle connection on the principal $Sp(V)$-bundle $\mathcal{Q}$. Let us assume that (M, ω) admits a metaplectic structure $(p_P : \mathcal{P} \to M, \Lambda)$. The principal bundle connection lifts to the principal $Mp(V)$-bundle $\mathcal{P}$ (Habermann, Habermann [7]), and induces a covariant derivative on the associated bundle $\mathcal{E}$. We denote the exterior covariant derivatives (see, e.g., [12]) by ∇^i. They map $\Gamma(\mathcal{E}^i)$ to $\Gamma(\mathcal{E}^{i+1})$. The restriction of ∇^i to $\Gamma(\mathcal{E}^{ij})$ is denoted by ∇^{ij}. The operator $R^i = \nabla^{i+1}\nabla^i$

is the so called (ith) *symplectic spinor curvature*, and $R = \sum_{i=0}^{2n-2} R^i$ is the total symplectic spinor curvature. It factorizes to a map of $\mathcal{E}^i$ into $\mathcal{E}^{i+2}$. We denote its restriction to $\mathcal{E}^{ij}$ by R^{ij}.

The $Mp(V)$-equivariant projections $p^{ij}_{\pm} : E^i_{\pm} \to E^{ij}_{\pm}$ induce projections $\mathcal{E}^i_{\pm} \to \mathcal{E}^{ij}_{\pm}$ that are bundle morphisms, which further induce appropriate projections $\Gamma(\mathcal{E}^i_{\pm}) \to \Gamma(\mathcal{E}^{ij}_{\pm})$ of the section spaces. We denote them by $p^{ij}_{\pm}$ and set $p^{ij} = p^{ij}_{+} + p^{ij}_{-}$ for all meanings of the symbols $p^{ij}_{\pm}$. The meaning of the symbol for a projection (on the modules, bundles and section spaces) depends on the objects on which they are used. We hope that this causes no confusion.

Definition 3.3. The (i,j)-th symplectic twistor operators are the maps

$$T^{ij}_{+} = p^{i+1,j+1}\nabla^{ij} \text{ and } T^{ij}_{-} = p^{i+1,j-1}\nabla^{ij},$$

where $i, j \in \mathbb{Z}$.

Symplectic Ricci and Weyl curvatures of a Fedosov connection are defined in Vaisman [24]. If the symplectic Weyl tensor is null, we call the connection *Weyl-flat.*

Let (M, ω) be a symplectic manifold admitting a metaplectic structure and let ∇ be a Weyl-flat Fedosov connection. In [17], p. 19, linear maps $E^{+}, \Theta^{\sigma}, F^{+}$, and Σ^{σ} are defined. The following formula

$$R = \frac{1}{n+1}(E^{+}\Theta^{\sigma} + 2F^{+}\Sigma^{\sigma}) \tag{3.1}$$

is derived in [15]. The operators F^{+}, E^{+} used here and in [17] differ from that ones in [15] by a multiplicative constant only.

In Krýsl [15], the next properties are proved for the restrictions of the mentioned linear maps to $\mathcal{E}^{ij}$ in the case of a Weyl-flat connection

$$\begin{aligned}
&\Theta^{\sigma} : \mathcal{E}^{ij} \to \mathcal{E}^{i,j-1} \oplus \mathcal{E}^{i,j} \oplus \mathcal{E}^{i,j+1},\\
&\Sigma^{\sigma} : \mathcal{E}^{i+1,j} \to \mathcal{E}^{i+1,j-1} \oplus \mathcal{E}^{i+1,j} \oplus \mathcal{E}^{i+1,j+1},\\
&F^{+} : \mathcal{E}^{i+1,j} \to \mathcal{E}^{i+2,j} \text{ and } E^{+} : \mathcal{E}^{ij} \to \mathcal{E}^{i+2,j}.
\end{aligned}$$

Using these properties and Eq. 3.1, we have for the curvature of a Weyl-flat connection

$$R^{ij} : \mathcal{E}^{ij} \to \mathcal{E}^{i+2,j-1} \oplus \mathcal{E}^{i+2,j} \oplus \mathcal{E}^{i+2,j+1}. \tag{3.2}$$

Theorem 3.4. *Let (M, ω) be a symplectic manifold which admits a metaplectic structure and ∇ be a Weyl-flat Fedosov connection. For any integers i, j, the two sequences $(\Gamma(\mathcal{E}^{i+k,j\pm k}), T^{i+k,j\pm k}_{\pm})_{k\in\mathbb{Z}}$ are cochain complexes.*

Proof. After a possible renumbering of i and j, it is sufficient to compute the composition

$$\begin{aligned}
T^{i+1,j\pm 1}_{\pm} T^{ij}_{\pm} &= p^{i+2,j\pm 2}\nabla^{i+1,j\pm 1} p^{i+1,j\pm 1}\nabla^{ij}\\
&= p^{i+2,j\pm 2}\nabla^{i+1} p^{i+1,j\pm 1}\nabla^{ij}.
\end{aligned}$$

$$\begin{tikzcd}
E^{i+1,j-2} & E^{i+2,j-2} \\
E^{i+1,j-1} & E^{i+2,j-1} \\
E^{i+1,j} \arrow[uur, dotted] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr, dotted] & E^{i+2,j} \\
E^{i+1,j+1} & E^{i+2,j+1} \\
E^{i+1,j+2} & E^{i+2,j+2}
\end{tikzcd}$$

FIGURE 3. Image of the restricted exterior covariant derivative

Since the image of ∇^{ij} is a linear subspace of $\Gamma(\mathcal{E}^{i+1,j-1} \oplus \mathcal{E}^{i+1,j} \oplus \mathcal{E}^{i+1,j+1})$ (Theorem 4 in Krýsl [14]), we may write $\mathrm{Id}_{\Gamma(\mathcal{E}^{i+1})} - p^{i+1,j} - p^{i+1,j\mp1}$ instead of $p^{i+1,j\pm1}$ in the above formula, obtaining

$$p^{i+2,j\pm2}\nabla^{i+1}\nabla^{ij} - p^{i+2,j\pm2}\nabla^{i+1}p^{i+1,j}\nabla^{ij} - p^{i+2,j\pm2}\nabla^{i+1}p^{i+1,j\mp1}\nabla^{ij} \quad (3.3)$$

Using the mentioned fact about the image of the exterior covariant derivatives for $\nabla^{i+1,j}$ (see also Figure 3), we have that $\nabla^{i+1,j}$ maps into $\Gamma(\mathcal{E}^{i+2,j-1} \oplus \mathcal{E}^{i+2,j} \oplus \mathcal{E}^{i+2,j+1})$. Consequently, the second term in expression 3.3 is null. Similarly, one proves that the last term ($+$ and $-$ case) is null, too. Summing-up, $T_{\pm}^{i+1,j\pm1}T_{\pm}^{ij} = p^{i+2,j\pm2}\nabla^{i+1}\nabla^{ij} = p^{i+2,j\pm2}R^{ij}$. By 3.2, the ith symplectic spinor curvature restricted to $\Gamma(\mathcal{E}^{ij})$ is a map into the vector space $\Gamma(\mathcal{E}^{i+2,j-1} \oplus \mathcal{E}^{i+2,j} \oplus \mathcal{E}^{i+2,j+1})$ since ∇ is Weyl-flat. Consequently, $T_{\pm}^{i+1,j\pm1}T_{\pm}^{ij} = p^{i+2,j\pm2}R^{ij} = 0$, proving that $(\Gamma(\mathcal{E}^{i+k,j\pm k}), T_{\pm}^{i+k,j\pm k})_{k\in\mathbb{Z}}$ are complexes. $\square$

At Figure 3, the conclusion of Theorem 4 in [14] for $\nabla^{i+1,j}$ is depicted. The full arrows point to bundles in whose sections' spaces the image of $\nabla^{i+1,j}$ is contained and the dotted arrows point to bundles whose sections have null intersection with the image of $\nabla^{i+1,j}$.

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