De Rham complex twisted by the oscillator bundle

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**Definition**

A is called a $C^*$-algebra if

- $A$ is an associative algebra, i.e.,
  - $A$ vector space over $\mathbb{C}$
  - multiplication map $A \times A \to A$ associative: $(ab)c = a(bc)$,
    distributes the additive and scalar structure:
    
    $$(a + b)c = ac + bc, \quad a(b + c) = ab + ac,$$
    $$(ka)b = k(ab) = a(kb)$$

- $*: A \to A$ an anti-involution, $(xy)^* = y^*x^*$ and
  
  $** = (*^2) = Id_A$

- $\nu: A \to [0, \infty)$ norm
  
  - $\nu(x + y) \leq \nu(x) + \nu(y)$, $\nu(\lambda x) = |\lambda|\nu(x)$
  - $\nu(x) \geq 0$ and $\nu(x) = 0$ implies $x = 0$
Examples:

**Matrices**
- $V$ vector space of finite dimension $n$ (over complex numbers)
- $A = \{ L : V \rightarrow V | L \text{ is a linear map} \} = \text{End}(A) = M_n(C)$
- addition of linear maps, multiplication is composition of maps (multiplication of matrices)
- $*A = A^\dagger$
- $\nu(A) = \sup\{|Av|; v \in V, |v| = 1\} = \max\{|Av|; v \in V, |v| = 1\}$

**Compact operators**
- $H$ a separable Hilbert space, $(,)_H : H \times H \rightarrow \mathbb{C}, \| \cdot \| = \sqrt{(,)_H}$
- $K(H) = \{ T : H \rightarrow H, \dim \text{Im} T < \infty \} - \text{algebra of compact operators}$
- $|T| = \sup\{|T_ix|_H; 0 \neq x \in H\}$
- $*T = T^*$ - operator adjoint (separability)
The difficulty of axioms for endomorphisms

$K(H)$ is a $C^*$-algebra.

- $K(H)$ is associative (composition of maps is assoc.)
- $*: K(H) \to K(H)$ and $*^2 = \text{Id}_{K(H)}$
- $\|: K(H) \to [0, \infty)$ is a norm because $\|$ on $H$ is a norm
- $|TT^*| = |T|^2$ (quite difficult, spectras) $|T^* T| \leq |T||T^*|$
  (easy)
- $K(H)$ is complete with respect to $\|$ (it is so defined)
Further examples

**Continuous functions**

- $X$ locally compact topological vector space $\implies X$ has a one-point compactification (infinity)
- $A = C_o(X)$ vector space of continuous complex valued functions vanishing in infinity
- $(fg)(x) = f(x)g(x)$, $x \in X$
- $f, g \in A$ implies $fg \in A$
- $|f| = \sup\{|f(x)|; x \in X\}$
- $f(x) = f(x)$, $x \in X$
- $C^*$-identity: easy consequence of the properties of sup: $\sup(|fg|) \leq \sup(|f|)\sup(|g|)$, but $|ff^*| = \sup|ff| = \sup|f|^2 = (\sup|f|)^2 = |f|^2$

**Convolution algebra on a locally compact group** is in general not a $C^*$-algebra.
Topography of the symplectic group

- $Sp(2n, \mathbb{R})$ non-compact, retractible onto $K = Sp(2n, \mathbb{R}) \cap SO(n) \subseteq Sp(2m, \mathbb{R})$, $K$ is isomorphic to $U(n)$
- $U(n)$ is of homotopy type of $S^1 = \{e^{i\phi}; \phi \in [0, 2\pi]\} \subseteq U(n)$.
- $\pi_1(S^1) \simeq \mathbb{Z}$, i.e., $S^1$ can be entangled only by a spiral with $\mathbb{Z}$ leaves.
- Consequently, $Sp(2n, \mathbb{R})$ is also of this type.
- 2-folded covering (unbranched) is called the metaplectic group $Mp(2n, \mathbb{R})$.
- $\lambda : Mp(2n, \mathbb{R}) \rightarrow Sp(2n, \mathbb{R})$ the two-fold covering.
- A (very nice almost irreducible) faithful unitary representation of $Mp(2n, \mathbb{R})$ exists $\sigma : Mp(2n, \mathbb{R}) \rightarrow \mathcal{U}(L^2(\mathbb{R}^n))$. 


Definition on elements

Let $\tilde{g} \in Mp(2n, R)$ denotes an element from two-point $\lambda^{-1}(g)$. Let $A \in M_n(R)$ be symmetric ($A^t = A$) and $B \in GL(n, R)$.

$$g_1 = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}, \quad (\sigma(\tilde{g}_1)f)(x) = e^{-i(Ax,x)/2}f(x)$$

$$g_2 = \begin{pmatrix} B & 0 \\ 0 & (B^t)^{-1} \end{pmatrix}, \quad (\sigma(\tilde{g}_2)f)(x) = \sqrt{\det B}f(B^tx)$$

$$g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\sigma(\tilde{g}_3)f)(x) = \pm e^{i\pi n}(\mathcal{F}f)(x),$$

where $\mathcal{F} : L^2(R^n) \rightarrow L^2(R^n)$ denotes the Fourier transform, $f \in L^2(R^n)$ and $x \in R^n$. 
History

- David Shale (when doing Ph.D. on Quantization of Klein-Gordon fields by Segal, Irving Ezra Segal)
- Irving Ezra Segal: constructive quantum theory ($C^*$-algebras, representations of locally compact groups, a definition of the state etc.), use of Stone-von Neumann theorem in QP
- André Weil (French number theorist and geometer, member of the Bourbaki group) - representations of some "discrete" Lie groups arising in number theory
- Berezin - infinitesimal level (angeblich, according to S. Gindikin)
- Bertram Kostant - use in geometric quantization (rediscovering via polarization structures)
Sketch reason for the existence

- **Construction**: Schrödinger representation of the Heisenberg group \((\text{Heisenberg CCR}, H_n)\), \(r : H_n \to \mathcal{U}(L^2(\mathbb{R}^n))\)
- Symplectic twist \(Sp(2n, \mathbb{R}) \times H_n \to H_n\) gives rise to other (twisted) Schrödinger representations \(r_g : H_n \to \mathcal{U}(L^2(\mathbb{R}^n))\)
- Stone-von Neumann: All are equivalent (even if twisted) ;-) 
- The intertwiners \(T_g\) (‘realizing’ the equivalences) compose in the same way as the elements of the symplectic group modulo signs (elements in \(e^{i\phi}\))
- Weil co-cycle computation: it is a true rep of the double-cover of \(Sp(2n, \mathbb{R})\), i.e., of the metaplectic group \(Mp(2n, \mathbb{R})\)
Symplectic manifolds - Phase spaces

$\langle M, \omega \rangle$ a symplectic manifold

1) $\omega \in \Omega^2(M)$ - exterior (anti-symmetric) differential two-form

2) $\omega_m : T_m M \times T_m M \to \mathbb{R}$ non-degenerate for any $m \in M$

3) $d\omega = 0$ (crucial for the Jacobi identity for the Poisson brackets)

1) and 2) imply $\dim T_m M(= \dim M)$ is even

**Basic examples:**

- $(\mathbb{R}^{2n}[q^1, \ldots, q^n, p_1, \ldots, p_n], \sum_{i=1}^{n} dq^i \wedge dp_i)$ canonical symplectic space

- $(T^* M, \omega = d\theta_L)$ cotangent spaces
Examples

- $(S^2(r_0), \omega = r_0^2 \sin \theta d\phi \wedge d\theta)$, i.e., sphere with the volume form
- no other sphere (except perhaps $S^0 = \{-1, 1\}$), Stokes theorem
- even dimensional tori ($T^{2n} = S^1 \times \ldots \times S^1$, $\omega = \sum_{i=1}^{n} d\phi_i \wedge d\theta_i$)
- Kähler manifolds, many of homogeneous spaces (e.g., $G/H$ where $G$, $H$ are complex Lie groups), connection to Einstein manifolds (many Einstein manifolds are Kähler or homogeneous spaces)
Metaplectic structures

- $M$ a symplectic manifold
- $\mathcal{P} = \{ e = (e_1, \ldots, e_{2n}); e$ is a symplectic basis of $(T_m M, \omega_m), m \in M \}$
- bundle of symplectic reperes
- $\mathcal{P}$ is a $Sp(2n, \mathbb{R})$-principal bundle ($Sp$ acts from the right)
- $\mathcal{Q}$ be a two-fold covering of $\mathcal{P}$ metaplectic structure
- $\mathcal{Q} \to M$ defines a bundle over $M$, a principal $Mp(2n, \mathbb{R})$-bundle
- Mild condition on $(M, \omega)$ for the existence of $\mathcal{Q}$
- All cotangent bundles of orientable manifolds
Oscillator bundle and symplectic spinors

- Set $H = L^2(\mathbb{R}^n)$
- $\mathcal{H} = Q \times \sigma H$ associated bundle, induced bundle, fiber change
- $\mathcal{H} = Q \times H/ \simeq$
- $(e, f) \simeq (eg, \sigma(g^{-1})f)$
- "From observers to observable quantities"
- metaplectic, symplectic spinor, Kostant's spinor, Segal-Shale-Weil, Weil, oscillator bundle
- An analogue of the spinor bundle (at the algebraic and geometric level)
- one can construct Dirac-type operators on $\Gamma(\mathcal{H})$ (K. Habermann)
Definition of Hilbert and pre-Hilbert $A$-modules

**Definition**

Let $A$ be a $C^*$-algebra and $H$ be a vector space over the complex numbers. We call $(H, (,))$ a **pre-Hilbert $A$-module** if

- $H$ is a right $A$-module – operation $\cdot : H \times A \to H$
- $(,): H \times H \to A$ is a $\mathbb{C}$-bilinear mapping
- $(f \cdot T + g, h) = T^*(f, h) + (g, h)$
- $(f, g) = (g, f)^*$
- $(f, f) \geq 0$ and $(f, f) = 0$ implies $f = 0$

We say $T \in A$ is non-negative ($T \geq 0$) if $T = T^*$ and $\text{Spec}(T) \subseteq [0, \infty)$.

$\text{Spec}(T) = \{ \lambda \in \mathbb{C}; T - \lambda \overline{1} \text{ is not invertible in } A^0 \}$, where $\overline{1} = (0, 1)$ is the unit in $A^0 = A \oplus \mathbb{C}$ (augmentation)
Definition of Hilbert and pre-Hilbert $A$-modules

**Definition**

If $(H, (,) )$ is a pre-Hilbert $A$-module we call it **Hilbert $A$-module** if it is complete with respect to the norm $\| : H \to [0, \infty)$ defined by $f \ni A \mapsto |f| = \sqrt{||f, f||_A}$ where $||A$ is the norm in $A$.

**Trivial example:** $A$ a $C^*$-algebra
Define $\cdot : A \times A \to A$ by $a \cdot b = ab$ and $(a, b) = a^*b$.
Right: $a \cdot (b \cdot c) = a \cdot (bc) = a(bc) = (ab)c = (ab) \cdot c$
Further: $(a \cdot b, c) = (ab, c) = (ab)^*c = (b^*a^*)c = b^*(a, c)$.
$(a, b)^* = (a^*b)^* = b^*a = (b, a)$
Examples of Hilbert $A$-modules

- For $A = K(H)$, the $C^*$-algebra of compact operators on a separable Hilbert space $(H, (,)_H)$, $M = H$ is a Hilbert $A$-module with respect to
  - $f, g, h \in H$ and $T \in K(H)$
  - $f \cdot T := T^*(f) \in H$
  - $(f, g) = f \otimes g^* \in K(H)$ where $(f \otimes g^*)(h) = (g, h)_H f$

- Proof.
Examples of Hilbert $A$-modules

- $A^n = A \oplus \ldots \oplus A$ is a Hilbert $A$-module with respect to $a \cdot (a_1, \ldots, a_n) = (aa_1, \ldots, aa_n)$ and the product given by $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = \sum_{i=1}^n a_i^* b_i$.

- If $M$ is a Hilbert $A$-module, then $M^n = M \oplus \ldots \oplus M$ is a Hilbert $A$-module with respect to $a \cdot (m_1, \ldots, m_n) = (a \cdot m_1, \ldots, a \cdot m_n)$ and the product given by $(m_1, \ldots, m_n) \cdot (m'_1, \ldots, m'_n) = \sum_{i=1}^n (m_i, m'_i)$.

- Further generalizes to $\ell^2(M)$ controlled by the convergence in $A$. Special case $\ell^2(A)$ ($M = A$).
Distinguished features of the $K(H)$-module $H$

$H = L^2(R^n)$

- $H$ is a $Mp(2n, R)$-module and it is a $K(H)$-module
- The $K(H)$-structure make us able to measure the quantities in $H$
- They do not commute or anti-commute.
- The metaplectic structure makes us able to place $H$ on our manifold (in accordance with the dynamics/geometry)
- On a manifold - on the oscillator bundle - the $K(H)$ and $Mp(2n, R)$-structures are compatible
Banach bundles

- $p : G \to M$ be a Banach bundle, bundle of Banach spaces with transitions into the homeomorphisms of a Banach space.
- $G_x := p^{-1}(\{x\})$
- $x \mapsto G_x$ (family of Banach e.g. Hilbert spaces parametrized by $x \in M$)
- Let $s : M \to G$ be a section of $G$, i.e., $p \circ s = Id_M$
- $\Gamma(G) = \{ s : M \to G | p \circ s = Id_M \}$
- Any family is a section. Any section is family.
- $\Gamma = \Gamma(G)$ is a vector space
- $M$ compact, one can make a completion of $\Gamma W^{0,2}(G)$
- Defined similarly as the Sobolev spaces but we must know how to integrate Banach valued functions (on a measure space or on a manifold)
\( C^* \)-Hilbert bundles

**Bundles** /Fibrations/Bündeln/Fibré (Champs continus, Dixmier)/Stohy /Snůšky

Not sheaves (= ne svazky). But bundles give rise to sheaves.

- An \( A \)-Hilbert bundle is a Banach bundle the fibers of which are homeomorphic to a fixed Hilbert \( A \)-module \( H \) and the transition functions are into \( \text{Aut}_A(H) \)
- If \( E \to M \) is an \( A \)-Hilbert bundle over a compact \( M \), then \( \Gamma(E) \) is a pre-Hilbert \( A \)-module.
- completions of \( \Gamma(E) \) as in the Banach case, \( \mathcal{W}^{k,2}(\mathcal{H}) \)
- These completions form **Hilbert** \( A \)-modules
- \( \mathcal{W}^{k,2}(\mathcal{H}) \) isomorphic to \( \ell^2(H) \) via Kasparov stabilization - quite complicated
Avoiding the symplectic and the compactness assumption

- $M$ contact manifold with a Riemannian structure
- take Finsler manifold (some necessary compatibilities)
- The group of projective canonical transformation act on the contact repére bundle (projective - can change the time)
- It is the so-called contact parabolic subgroup $P \subseteq Sp(2n + 2, \mathbb{R})$
- It has also "the" Segal-Shale-Weil representation (by inducing)
- Make the association
- You have a Hilbert bundle
- Do the analysis: One define the infinity
- Infinity in the time dimension
- At each time, the universe might be a modeled by a compact manifold and then the analysis above may apply.
De Rham complex tensored by the oscillatory bundle

- $(M^{2n}, \omega)$ symplectic manifold
- admitting metaplectic structure
- $\mathcal{H} \to M$
- $\bigoplus_{k=0}^{2n} \bigwedge^k T^* M \to M$
- $\bigwedge^\bullet T^* M \otimes \mathcal{H} \to M$
- Kuiper ('60): $\mathcal{H}$ is globally trivial; trivializing section defines a flat connection $\nabla$
  
  - $d^\nabla_k (\alpha \otimes h) = d\alpha \otimes h + (-1)^k \epsilon^i \wedge \alpha \otimes \nabla_{e_i} h$ where $(e_i)_{i=1, \ldots, 2n}$ frame and dual coframe
- $d_{k+1}d_k = 0$ since $d$ (de Rham is flat) and $\nabla$ is flat
Cohomology groups are Hausdorff if $A = K(H)$!

**Theorem** (Krýsl, Ann. Glob. Anal. Geom. 2014): Let $M$ be a compact manifold, $A$ a $C^*$-algebra, $(\mathcal{E}^k)_{k \in \mathbb{N}_0}$ a sequence of finitely generated projective $A$-Hilbert bundles over $M$ and $D_k : \Gamma(\mathcal{E}^k) \to \Gamma(\mathcal{E}^{k+1})$, $k \in \mathbb{Z}$, a complex $D$ of differential operators. Suppose that the Laplace operators $\triangle_k$ of $D$ have closed image in the norm topology of $\Gamma(\mathcal{E}^k)$. If $D$ is elliptic, then $D$ is a self-adjoint parametrix possessing complex in $K(H_A^*)$. Moreover, the cohomology groups of $D$ are finitely generated and projective Hilbert $A$-modules.

**Theorem** (Krýsl): If $A$ is a $C^*$-subalgebra of the algebra of compact operators $K(H)$, one may drop the closed image assumption on the Laplacians.


