Hodge theory for C^*-Hilbert bundles

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Symplectic linear algebra

Symplectic vector space
(V, ω_0) - real $2n$ dimensional vector space, $\omega_0 : V \times V \to \mathbb{R}$ non-degenerate antisymmetric
Symplectic group
$Sp(V, \omega_0) = \{A : V \to V \,|\, \omega_0(Av, Aw) = \omega_0(v, w) \text{ for each } v, w \in V\}$
Retractable onto $U(n)$, of homotopy type of S^1, $\pi_1(Sp(V, \omega_0)) = \mathbb{Z}$.
Possesses a non-universal connected 2-fold covering, the so called **Metaplectic group** $Mp(V, \omega_0)$, $\lambda : Mp(V, \omega_0) \stackrel{2:1}{\to} Sp(V, \omega_0)$
Universal covering would be infinitely many folded over $Sp(V, \omega_0)$.

Properties of the SSW representation

Segal-Shale-Weil representation of the metaplectic group.
Inventors:
David Shale - quantization of solutions to the Klein-Gordon equation, dissertation by I. Segal
André Weil - short after, true rep of $Mp(V, \omega_0)$
Vladimir Berezin - used it at the infinitesimal level

- Underlying vector space $L^2(\mathbb{R}^n)$
- $\rho_0 : Mp(V, \omega_0) \to \mathcal{U}(L^2(\mathbb{R}^n))$ (continuous homomorphism)
- Non-trivial faithful unitary representation of $Mp(V, \omega_0)$
- Splits into 2 irreducible representations, odd and even L^2 functions on $\mathbb{R}^n$.
- There exists $g_0 \in Mp(V, \omega_0)$ such that
$\rho_0(g_0) = \mathcal{F}^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ (continuous on $L^2(\mathbb{R}^n)$)

Properties of the SSW-representation

- Similar to the spinor representation of Spin groups - it is not a representation of the underlying $Sp(V, \omega_0)$.

$$\begin{array}{ccc} Mp(V,\omega_0) & \xrightarrow{\rho_0} & \mathcal{U}(L^2(\mathbb{R}^n)) \\ \downarrow \lambda & \nearrow \nexists & \\ Sp(V,\omega_0) & & \end{array}$$

- Highest weights $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{3}{2})$

Symplectic manifolds

(M, ω) - M manifold, ω non-degenerate differential 2-form and $d\omega = 0$.

Examples:

1) T^*M, where M is any manifold, $\omega_U = \sum_{i=1}^{n} dp_i \wedge dq^i$, q^i local coordinates on the manifold, p_i coordinates at $T_{(q^1,\dots,q^n)}M$
2) S^2 with $\omega = \mathrm{vol} = r^2 \sin\vartheta d\phi \wedge d\vartheta$
3) even dimensional tori $\omega = d\phi_1 \wedge d\vartheta^1 + \ldots + d\phi_n \wedge d\vartheta^n$ (in mechanics: action-angle variables)
4) Kähler manifolds, $\omega(-,-) = h(-, J-)$
5) Kodaira-Thurtson manifold - compact non-Kähler symplectic manifold

Symplectic connections

Darboux theorem: In a neighborhood of any point, one can choose coordinates in which $\omega = \sum_{i=1}^{n} dq^i \wedge dp_i$. In Riemannian, geometry the metric can be transformed into the "canonical" form only point-wise - curvature obstruction. Measured by the curvature tensor. In s.g., due to Darboux theorem, the connection cannot have such meaning.

Definition: A connection on a symplectic manifold (M, ω) equipped with a symplectic form ω is called symplectic if $\nabla\omega = 0$, and it is called Fedosov if in addition, it is torsion-free.

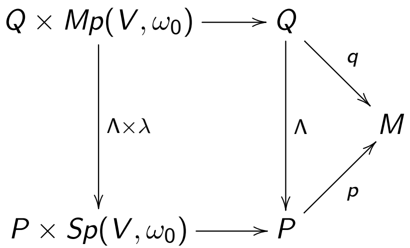
Metaplectic structure

Symplectic structure

(M,ω) symplectic manifold. At any point $m \in M$, consider the set $P_m = \{b = (e_1,\dots,e_{2n}) | \text{b is a symplectic basis of } (T^*_m M, \omega_m)\}$.
$P = \bigcup_{m\in M} P_m$ the space of symplectic repères, $p : P \to M$ ("foot-point" projection).

Metaplectic structure Q

- Formally: (Q,Λ), $q : Q \to M$ is $Mp(V,\omega_0)$-bundle over M
$\Lambda : Q \to P$ bundle morphism
- Compatibility with the symplectic structure:



Exterior forms with valued in the oscillatory bundles

Associated bundles

$\mathcal{S} = (Q \times_{\rho_0} L^2(\mathbb{R}^n))$

Introduced by Bertram Kostant: oscillatory bundle

Associated connections

For a symplectic connection $\nabla \Rightarrow$

$\nabla^S : \Gamma(M, TM) \otimes \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S})$

$$\Omega^i(M, \mathcal{S}) = \Gamma(M, \bigwedge^i T^*M \otimes \mathcal{S})$$

$d_i^{\nabla^S} : \Omega^i(M, \mathcal{S}) \to \Omega^{i+1}(M, \mathcal{S})$ exterior oscillatory derivative

Associated operator of Kath. Habermann

Operators generated by symplectic connections

Symplectic Dirac operators

(M, ω, ∇) with a metaplectic structure

$(e_i.s)(x) = \imath x^i s(x), \quad (e_{i+n}.s)(x) = \frac{\partial s}{\partial x^i}(x)$ *(quantization)*.

$[e_i.e_j., e_j.e_i.] = -\imath\omega(e_i, e_j)$, densely defined

a) $\mathfrak{D} : \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S})$ is the oscillatory or Dirac operator of Habermann

$\mathfrak{D}s = \sum_{i,j=1}^{2n} \omega^{ij} e_i.\nabla^S_{e_j} s$

b) $\mathfrak{D} : \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S})$

$\widetilde{\mathfrak{D}}s = \sum_{i,j=1}^{n} g^{ij} e_i.\nabla_{e_j} s$ for a metric g of a compatible almost complex structure J

Associated second order operator $\mathfrak{P} = \imath[\mathfrak{D}\widetilde{\mathfrak{D}} - \widetilde{\mathfrak{D}}\mathfrak{D}]$

Kernel of $\mathfrak{P}$ on S^2

Operator $\mathfrak{P}$ on S^2
Self-adjoint and elliptic; elliptic = its symbol is a vector bundle isomorphism
$L^2(\mathbb{R}) = \widehat{\bigoplus}_{k=0}^{\infty}\mathbb{C}h_k,\ h_k = e^{x^2/2}\frac{d^k}{dx^k}e^{-x^2}$
at bundle level as $\mathcal{S} = \widehat{\bigoplus}_{k=0}^{\infty}\mathcal{S}_k,$
where $\mathcal{S}_k$ = the line bundle corresponding to the vector space $\mathbb{C}h_k$ irreducible with respect to the group $\lambda^{-1}(U(1)) \subseteq Mp(2,\mathbb{R})$.
$\exists$ monotone sequence $(l_i)_{i=0}^{\infty}$ such that $\mathrm{Ker}\,\mathfrak{P} \cap \Gamma(S^2, \mathcal{S}_{l_i}) \neq 0$ and $\dim(\mathrm{Ker}\,\mathfrak{P} \cap \Gamma(S^2, \mathcal{S}_{l_i})) = 2(i + l_i + 2)$.
In particular, the kernel of $\mathfrak{P}$ is infinite dimensional.

Symbols of operators

For a first order differential operator $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F})$ $[\sigma(D, \xi)s](m) = \imath[D(fs) - fDs](m)$, where $f \in \mathcal{C}^\infty(M)$, $(df)_m = \xi \in T^*_m M$, $s \in \Gamma(M, \mathcal{E})$

Examples:

1) exterior differentiation d, symbol $\sigma(d_i, \xi)\alpha = \imath\xi \wedge \alpha$
2) Laplace-Beltrami operator $\triangle$, symbol $\sigma(\triangle, \xi)f = -(\sum_{i=1}^{n}(\xi_i)^2)f$
3) Dolbeault operator, symbol $\sigma(\overline{\partial}, \xi)\alpha = \imath\xi^{(0,1)} \wedge \alpha$

Hodge theory for elliptic complexes

Definition: For any $m \in M$ and any nonzero co-vector $\xi \in T^*_m M \setminus \{0\}$, the complex

$$0 \to \Gamma(\mathcal{E}^0, M) \stackrel{D_0}{\to} \Gamma(\mathcal{E}^1, M) \stackrel{D_1}{\to} \dots \stackrel{D_{n-1}}{\to} \Gamma(\mathcal{E}^n, M)$$

is called elliptic, iff the symbol sequence

$$0 \to \mathcal{E}^0 \stackrel{\sigma_0^\xi}{\to} \mathcal{E}^1 \stackrel{\sigma_1^\xi}{\to} \dots \stackrel{\sigma_{n-1}^\xi}{\to} \mathcal{E}^n$$

is **exact**.
$\sigma_i^\xi = \sigma(D_i, \xi)$, $i \in \mathbb{N}_0$
Elliptic operator $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F}) \stackrel{\text{def}}{\Leftrightarrow}$
$0 \to \Gamma(M, \mathcal{E}) \stackrel{D}{\to} \Gamma(M, \mathcal{F}) \to 0$ is an elliptic complex.

Examples of elliptic complexes

1) de Rham complex is elliptic
2) Dolbeault complex is elliptic
3) $0 \to \mathcal{C}(M) \overset{\triangle}{\to} \mathcal{C}(M) \to 0$ is elliptic

Theorem (quadratic algebra):
$D^\bullet = (D_i, \Gamma(M, \mathcal{E}^i))_{i \in \mathbb{N}_0}$ elliptic complex $\Rightarrow$ each associated Laplacian $\triangle_i = D_{i-1} D_{i-1}^* + D_i^* D_i$ is elliptic

The order of $\triangle_i$ denoted by r_i.

C^*-algebras

A associative algebra over $\mathbb{C}$ with a norm $||: A \to \mathbb{R}_0^+$, i.e.,

1) $^* : A \to A$ is an antiinvolution,
2) $|a|^2 = |aa^*|$ for all $a \in A$ and
3) $(A, ||)$ is a Banach space.

Examples:

1) $\mathcal{C}_c^0(X) = \{f : X \to \mathbb{C}; \lim_{x\to\infty} f(x) = 0\}$, where X is a Hausdorff topological space
2) H a Hilbert space, $A = \{a : X \to X; a \text{ is bounded }\}$, $*A := A^*$, $|A| = \sup\{\frac{|Ax|}{|x|}, x \neq 0\}$.
3) $\mathrm{Mat}(\mathbb{C}^n)$, $*A = A^\dagger$, $|A| = \max\{|\lambda|, \lambda \in \mathrm{spec}(A)\}$ (the norms 2) and 3) are equal)

Pre-Hilbert C^*-modules

A a unital C^*-algebra, 1 unit
$\text{spec}(a) = \{\lambda \in \mathbb{C} | a - \lambda 1$ does not possesses inverse (in A)$\}$
$a = a^* \implies \text{spec}(a) \subseteq \mathbb{R}$
$A_0^+ = \{a \in A | a = a^*$ and $\text{spec}(a) \subseteq \mathbb{R}_0^+\}$ - positive elements.
U a vector space with a left action on A equipped with
$(,) : U \times U \to A$ (mimics the Hilbert product) such that for each $u, v, w \in U$, $r \in \mathbb{C}$, $a \in A$

1) $(u + rv, w) = (u, w) + r(u, w)$
2) $(a.u, v) = a(u, v)$
3) $(u, v) = (v, u)^*$
4) $(u, u) \in A_0^+$ and $(u, u) = 0 \Rightarrow u = 0$

is called pre-Hilbert module. If $U \ni u \mapsto |(u, u)|^{1/2}$ makes U a **complete** normed space is called an A-Hilbert module.

Homomorphisms

$L : U \to V$, pre-Hilbert A-modules U, V - $a \in A$
$u \in U$, $L(a.u) = a.L(u)$
and continuous with respect to the topologies induced by $| \, |_U$ and $| \, |_V$.

adjoint of $L : U \to V$ is a map $L^* : V \to U$ satisfying
$(Lu, v)_V = (u, L^* v)_U$ for each $u \in U, v \in V$
Adjoint need not exist. If it exists, it is unique and moreover, a pre-Hilbert module homomorphism

Properties of Hilbert C^*-modules

For any pre-Hilbert A-submodule $V \subseteq U$, we set
$V^\perp = \{u \in U | \, (u, v)_U = 0 \text{ for all } v \in V\}$.
Not in general true that $V \oplus V^\perp = U$
U is finitely generated projective, if $U \oplus U^\perp \cong A^n$, where A^n is the direct sum of n copies of A.
In more detail, $A^n = \underbrace{A \oplus \ldots \oplus A}_{n}$ as a vector space, the action is given by $a.(a_1, \ldots, a_n) = (aa_1, \ldots, aa_n)$ and the A-Hilbert product $(,)_{A^n}$ is defined by the formula

$$((a_1', \ldots, a_n'), (a_1, \ldots, a_n))_{A^n} = \sum_{i=1}^{n} a_i' a_i^*,$$

where $a, a_i, a_i' \in A$, $i = 1, \ldots, n$.

A-Hilbert bundles

Let $(U, (,)_U)$ be a Hilbert A-module. A Banach bundle $p : \mathcal{E} \to M$ is called an A-Hilbert bundle with typical fiber $(U, (,)_U)$ if

1) p is a Banach bundle with fiber $(U, | \, |_U)$,
2) each fiber $p^{-1}(\{m\})$ is equipped with a Hilbert A-product $(,)_m$ such that $(p^{-1}(\{m\}), (,)_m)$ isomorphic to the fixed $(U, (,)_U)$ as a Hilbert A-module via a bundle chart of p,
3) there exists an atlas of bundle charts of p the elements of which satisfy the above item and such that its transition maps are Hilbert A-module automorphisms of $(U, (,)_U)$, i.e., elements of $\mathrm{Aut}_A(U)$.

Sections and completions

$p : \mathcal{E} \to M$ be an A-Hilbert bundle
space of smooth sections $\Gamma(M, \mathcal{E})$
The space of sections admits a left action of A
$(a.s)(m) = a.(s(m))$ for $a \in A$, $s \in \Gamma(M, \mathcal{E})$ and $m \in M$.
M is compact
Riemannian metric g on M and a volume element vol_g for this metric
An A-product on $\Gamma(M, \mathcal{E})$ is defined by

$$(s', s)_0 = \int_{m \in M} (s(m), s'(m))_m (\mathrm{vol}_g)_m,$$

where $(,)_m$ denotes the Hilbert A-product in fiber $p^{-1}(\{m\})$
$\Gamma(M, \mathcal{E})$ pre-Hilbert A-module
We denote the completion of the normed space $(\Gamma(M, \mathcal{E}), |\,|_0)$ by $W^0(\mathcal{E})$ and call it the zeroth Sobolev type completion

Sobolev completion

Let us denote the Laplace-Beltrami operator for g by $\triangle_g$.
For each $t \in \mathbb{N}_0$, we define an A-product $(\,,\,)_t$ on $\Gamma(M, \mathcal{E})$

$$(s', s)_t = \int_{m \in M} (s'(m), (1 + \triangle_g)^t s(m))_m (\mathrm{vol}_g)_m \quad s', s \in \Gamma(M, \mathcal{E}).$$

We denote the completion of $\Gamma(M, \mathcal{E})$ with respect to the norm $|\,|_t$ induced by $(\,,\,)_t$ by $W^t(\mathcal{E})$ and call it the Sobolev type completion (of order t).
Sobolev type completions form Hilbert A-modules.

Differential operators

1) Differential operators in A-Hilbert bundles, in local coordinates $D = c_\alpha \partial^\alpha$, $c_\alpha \in \mathrm{End}_A(U)$
2) Possess continuous extensions to the Sobolev type completions
3) Their extensions to the Sobolev type completions are adjointable
4) Ellipticity (of complexes) is defined as in the finite rank case and is called the A-ellipticity (Mishchenko, Fomenko; Solovyov, Troitsky)

Hodge Theory for C^*-bundles

Theorem: Let A be a unital C^*-algebra and $(p_i : \mathcal{E}^i \to M)_{i \in \mathbb{N}_0}$ be a sequence of finitely generated projective A-Hilbert bundles over a compact manifold M. If $D^\bullet = (\Gamma(M, \mathcal{E}^i), D_i)_{i \in \mathbb{N}_0}$ is an A-elliptic complex of differential operators and for each $k \in \mathbb{N}_0$, the image of the r_k-th extension of the associated Laplacian $\triangle_k$ to the Sobolev type completion $W^{r_k}(\mathcal{E}^k)$ is closed, then for any $i \in \mathbb{N}_0$,

1) $H^i(D^\bullet, A) \cong \operatorname{Ker} \triangle_i$ as Hilbert-A-modules
2) $H^i(D^\bullet, A)$ is a finitely generated projective Hilbert A-module.

Theorem: Let (M^{2n}, ω) be a compact symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection. Then the kernel of $\mathfrak{P}$ is a finitely generated projective Hilbert $\operatorname{End}(S)$-module.

Literature

[1] Shale D., Linear symmetries of free boson fields, Trans. Amer. Math. Soc., 1962.
[2] Weil A., Sur certains groups d'opé rateurs unitaires, Acta Math. 111, 1964.
[3] Kostant B., Symplectic spinors, Symposia Mathematica, Vol. XIV, pp. 139–152.
[4] Habermann K., Habermann L., Introduction to symplectic Dirac operators, Springer-Verlag.
[5] Fomenko, A., Mishchenko, A., Indeks elipticheskich operatorau nad C^*-algebrami, Izv. Akad. Nauk SSSR, Ser. Mat., 43, 1979.
[6] Krýsl, S., Cohomology of elliptic complexes with coefficient in a C^* algebra, arxiv.org, Annals Glob. Anal. Geom.
[7] Krýsl, S., Cohomology of the de Rham complex twisted by the oscillatory module, arxiv.org, Diff. Geom. Appl.
[8] Krýsl, S., Hodge theory for elliptic complexes over unital C^*-algebras, arxiv.org, subm. Banach Journal of Math. Analysis