Classification of 1^{st} order symplectic spinor operators over contact projective geometries

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Abstract

We give a classification of 1^{st} order invariant differential operators acting between sections of certain bundles associated to Cartan geometries of the so called metaplectic contact projective type. These bundles are associated via representations, which are derived from the so called higher symplectic, harmonic or generalized Kostant spinor modules. Higher symplectic spinor modules are arising from the Segal-Shale-Weil representation of the metaplectic group by tensoring it by finite dimensional modules. We show that for all pairs of the considered bundles, there is at most one 1^{st} order invariant differential operator up to a complex multiple and give an equivalence condition for the existence of such an operator. Contact projective analogues of the well known Dirac, twistor and Rarita-Schwinger operators appearing in Riemannian geometry are special examples of these operators.

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1 Introduction

The operators we would like to classify are 1^{st} order invariant differential operators acting between sections of vector bundles associated to metaplectic contact projective geometries via certain minimal globalizations.

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Metaplectic contact projective geometry on an odd dimensional manifold is first a contact geometry, i.e., it is given by a corank one subbundle of the tangent bundle of the manifold which is nonintegrable in the Frobenius sense in each point of the manifold. Second part of the metaplectic contact projective structure on a manifold is given by a class of projectively equivalent contact partial affine connections. Here, partial contact means that the connections are compatible with the contact structure and that they are acting only on the sections of the contact subbundle. These connections are called projectively equivalent because they have the same class of unparameterized geodesics going in the contact subbundle direction, see, e.g., D. Fox [9], where you can find a relationship between the contact projective geometries and classical path geometries. The adjective "metaplectic" suggests that in addition to contact projective geometries, the metaplectic contact projective structures include some spin phenomena like the spin structures over Riemannian manifolds. Metaplectic contact projective and contact projective geometries have their description also via Cartan geometries. Contact projective geometries could be modeled on a $(2l+1)$-dimensional projective space $\mathbb{PV}$ of a $(2l+2)$-dimensional real symplectic vector space $\mathbb{V}$, which we suppose to be equipped with a symplectic form ω. Here, the projective space is considered as a homogeneous space G/P, where G is the symplectic Lie group $Sp(\mathbb{V}, \omega)$ acting transitively on $\mathbb{PV}$ by the factorization of its defining representation (on $\mathbb{V}$), and P is an isotropy subgroup of this action. In this case, it is easy to see that P is a parabolic subgroup, which turns out to be crucial for our classification. Contact projective geometry, in the sense of É. Cartan, are curved versions $(p : \mathcal{G} \to M, \omega)$ of this homogeneous (also called Klein) model G/P. There exist certain conditions (known as normalization conditions) under which the Cartan's principal bundle approach and the classical one (via the class of connections and the contact subbundle) are equivalent, see, e.g., Čap, Schichl [4] for details. We also remind that contact geometries are an arena for time-dependent Hamiltonian mechanics. Klein model of the metaplectic contact projective geometry consists of two groups $\tilde{G}$ and $\tilde{P}$, where $\tilde{G}$ is the metaplectic group $Mp(\mathbb{V}, \omega)$, i.e., a non-trivial double covering of the symplectic group G, and $\tilde{P}$ is the preimage of P by this covering.

Symplectic spinor operators over projective contact geometries are acting between sections of the so called higher symplectic spinor bundles. These bundles are associated via certain infinite dimensional irreducible admissible representations of the parabolic principal group P. The parabolic group P acts then non trivially only by its Levi factor G_0, while the action of the unipotent part is trivial. The semisimple part $\mathfrak{g}_0^{ss}$ of the Lie algebra of the Levi part of the parabolic group P is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(2l, \mathbb{R})$. Thus to give an admissible representation of P, we have to specify a representation of $\mathfrak{g}_0^{ss}$. Let us recall that the classification of first order invariant operator was done by Slovák, Souček in [24] (generalizing an approach of Fegan in [8]) for all finite dimensional irreducible representations and general parabolic subgroup P of a semisimple G (almost Hermitian structures are studied in detail). Nevertheless, there are some interesting infinite dimensional representations of the complex symplectic Lie algebra, to which we shall focus our attention. These representations form a class consisting of infinite dimensional modules with bounded multiplicities. Modules with bounded multiplicities are representations, for which there is a nonnegative integer, such that the dimension of each weight space of this module is bounded by it from above. Britten, Hooper and Lemire in [2] and Britten, Hooper in [3] showed that each of these modules appear as direct summands in a tensor product of a finite dimensional $\mathfrak{sp}(2l,\mathbb{C})$-module and the so called Kostant (or basic) symplectic spinor module $\mathbb{S}_+$ and vice versa. Irreducible representations in this completely reducible tensor product are called higher symplectic, harmonic or generalized Kostant spinors. It is well known, that all finite dimensional modules over complex symplectic Lie algebra appear as irreducible submodules of a tensor power of the defining representation. Thus the infinite dimensional modules with bounded multiplicities are analogous to the spinor-vector representations of complex orthogonal Lie algebras. Namely, each finite dimensional module over orthogonal Lie algebra is an irreducible summand in the tensor product of a basic spinor representation and some power of the defining module (spinor-vector representations), or in the power of the defining representation itself (vector representations). In order to have a complete picture, it remains to show that the basic (or Kostant) spinors are analogous to the orthogonal ones, even though infinite dimensional. The basic symplectic spinor module $\mathbb{S}_+$ was discovered by Bertram Kostant, when he was introducing half-forms for metaplectic structures over symplectic manifolds in the context of geometric quantization. While in the orthogonal case spinor representations can be realized using the exterior algebra of a maximal isotropic vector space, the symplectic spinor representations are realized using the symmetric algebra of certain maximal isotropic vector space (called Lagrangian in the symplectic setting). This procedure goes roughly as follows: one takes the Chevalley realization of the symplectic Lie algebra C_l by polynomial coefficients linear differential operators acting on polynomials $\mathbb{C}[z^1,\ldots,z^l]$ in l complex variables. The space of polynomials splits into two irreducible summands over the symplectic Lie algebra, namely into the two basic symplectic spinor modules $\mathbb{S}_+$ and $\mathbb{S}_-$. There is a relationship between the modules $\mathbb{S}_+$ and $\mathbb{S}_-$ and the Segal-Shale-Weil or oscillator representation. Namely, the underlying C_l-structure of the Segal-Shale-Weil representation is isomorphic to $\mathbb{S}_+ \oplus \mathbb{S}_-$.

In order to classify 1^{st} order invariant differential operators, one needs to understand the structure of the space of P-homomorphisms between the so called 1^{st} jets prolongation P-module of the domain module and the target representation of P, see chapter 4. Thus the classification problem translates into an algebraic one. In our case, representation theory teaches us, that it is sometimes sufficient to understand our representation at its infinitesimal level. The only thing one needs in this case, is to understand the infinitesimal version of the 1^{st}-jets prolongation module. For our aims, the most important part of the 1^{st} jets prolongation module consists of a tensor product of the defining representation of C_l and a higher symplectic spinor module. In order to describe the space of P-homomorphisms, one needs to decompose the mentioned tensor product into irreducible summands. This was done by Krýsl in [21], where results of Humphreys in [12] and Kac and Wakimoto in [15] were used.

Let us mention that some of these operators are contact analogues of the well known symplectic Dirac operator, symplectic Rarita-Schwinger and symplectic twistor operator. Analytical properties of these operators were studied by many authors, see, e.g., K. Habermann [11] and A. Klein [18]. These symplectic versions were mentioned also by M. B. Green and C. M. Hull, see [10], in the context of covariant quantization of 10 dimensional super-strings and also in the theory of Dirac-Kähler fields, see Reuter [22], where we found a motivation for our studies of this topic.

In the second section, metaplectic contact projective geometries are defined using the Cartan's approach. Basic properties of higher symplectic spinor modules (Theorem 1) together with a theorem on a decomposition of the tensor product of the defining representation of $\mathfrak{sp}(2l, \mathbb{C})$ and an arbitrary higher symplectic spinor module (Theorem 2) are summarized in section 3 . Section 4 is devoted to the classification result. Theorem 3 and Lemmas 1 and 2 in this section are straightforward generalizations of similar results obtained by Slovák and Souček in [24]. Theorem 4 (in section 4) is a well known theorem on the action of a Casimir element on highest weight modules. While in the subsection 4.1. we are interested only in the classification at the infinitesimal level (Theorem 5), we present our classification theorem at the globalized level in subsection 4.2 (Theorem 6). In the fifth section, three main examples of the 1^{st} order symplectic spinor operators over contact projective structures are introduced.

2 Metaplectic contact projective geometry

The aim of this section is neither to serve as a comprehensive introduction into metaplectic contact projective geometries, nor to list all references related to this subject. We shall only present a definition of metaplectic contact projective geometry by introducing its Klein model, and give only a few references, where one can find links to a broader literature on this topic (contact projective geometries, path geometries e.t.c.).

For a fixed positive integer $l \geq 3$, let us consider a real symplectic vector space $(\mathbb{V}, \omega)$ of real dimension $2l+2$ together with the defining action of the symplectic Lie group $G := Sp(\mathbb{V}, \omega)$. The defining action is transitive on $\mathbb{V} - \{0\}$, and thus it defines a transitive action $G \times \mathbb{PV} \to \mathbb{PV}$ on the projective space $\mathbb{PV}$ of $\mathbb{V}$ by the prescription $(g, [v]) \mapsto [gv]$ for $g \in G$ and $v \in \mathbb{V} - \{0\}$. (Here, $[v]$ denotes the one dimensional vector subspace spanned by v.) Let us denote the stabilizer of a point in $\mathbb{PV}$ by P. It is well known that this group is a parabolic subgroup of G, see, e.g., D. Fox [9]. The pair (G, P) is often called Klein pair of contact projective geometry. Let us denote the Lie algebra of P by $\mathfrak{p}$.

Definition 1: Cartan geometry $(p : \mathcal{G} \to M^{2l+1}, \omega)$ is called a contact projective geometry of rank l, if it is a Cartan geometry modeled on the Klein geometry of type (G, P) for G and P introduced above.

It is possible to show that each contact projective geometry defines a contact structure on the tangent bundle TM of the base manifold M and a class $[\nabla]$ of contact projectively equivalent partial affine connections ∇ acting on the sections of the contact subbundle (see the Introduction for some remarks). For more details on this topic, see Fox [9]. In Čap, Schichl [4], one can find a treatment on the equivalence problem for contact projective structures. Roughly speaking, the reader can find a proof there, that under certain conditions, there is an isomorphism between the Cartan approach and the classical one (via contact subbundle and a class of connections). Because we would like to include some spin phenomena, let us consider a slightly modified situation. Fix a nontrivial two-fold covering $q : \tilde{G} \to G$ of the symplectic group $G = Sp(\mathbb{V}, \omega)$ by the metaplectic group $\tilde{G} = Mp(\mathbb{V}, \omega)$, see Kashiwara, Vergne [17]. Let us denote the q-preimage of P by $\tilde{P}$.

Definition 2: Cartan geometry $(p : \tilde{\mathcal{G}} \to M^{2l+1}, \omega)$ is called metaplectic contact projective geometry of rank l, if it is a Cartan geometry modeled on the Klein geometry of type $(\tilde{G}, \tilde{P})$ with $\tilde{G}$ and $\tilde{P}$ introduced above.

Let us remark, that in definition 2, we do not demand metaplectic contact projective structure to be connected to a contact projective structure as one demands in the case of spin structures over Rimannian manifolds or in the case of metaplectic structures over manifolds with a symplectic structure.

3 Higher symplectic spinor modules

Let $C_l \simeq \mathfrak{sp}(2l, \mathbb{C})$, $l \geq 3$, be the complex symplectic Lie algebra. Consider a Cartan subalgebra $\mathfrak{h}$ of C_l together with a choice of positive roots Φ^+. The set of fundamental weights $\{\varpi_i\}_{i=1}^l$ is then uniquely determined. For later use, we shall need an orthogonal basis (with respect to the form dual to the Killing form of C_l), $\{\epsilon_i\}_{i=1}^l$, for which $\varpi_i = \sum_{j=1}^i \epsilon_j$ for $i = 1, \ldots, l$.

For $\lambda \in \mathfrak{h}^*$, let $L(\lambda)$ be the irreducible C_l-module with the highest weight λ. This module is defined uniquely up to a C_l-isomorphism. If λ happens to be integral and dominant (with respect to the choice of $(\mathfrak{h}, \Phi^+)$), i.e., if $L(\lambda)$ is finite dimensional, we shall write $F(\lambda)$ instead of $L(\lambda)$. Let L be an arbitrary (finite or infinite dimensional) weight module over a complex simple Lie algebra. We call L a module with bounded multiplicities, if there is a $k \in \mathbb{N}_0$, such that for each $\mu \in \mathfrak{h}^*$, $\dim L_\mu \leq k$, where L_μ is the weight space of weight μ.

Let us introduce the following set of weights

$$A := \{\lambda = \sum_{i=1}^{l} \lambda_i \varpi_i | \lambda_i \in \mathbb{N}_0, i = 1, \ldots, l-1, \lambda_{l-1} + 2\lambda_l + 3 > 0, \lambda_l \in \mathbb{Z} + \frac{1}{2}\}.$$

Definition 3: For a weight $\lambda \in A$, we call the module $L(\lambda)$ higher symplectic spinor module. We shall denote the module $L(-\frac{1}{2}\varpi_l)$ by $\mathbb{S}_+$ and the module $L(\varpi_{l-1} - \frac{3}{2}\varpi_l)$ by $\mathbb{S}_-$. We shall call these two representations basic symplectic spinor modules.

The next theorem says that the class of higher symplectic spinor modules is quite natural and in a sense broad.

Theorem 1: Let $\lambda \in \mathfrak{h}^*$. Then the following are equivalent

1.) $L(\lambda)$ is an infinite dimensional C_l-module with bounded multiplicities;

2.) $L(\lambda)$ is a direct summand in $\mathbb{S}_+ \otimes F(\nu)$ for some integral dominant $\nu \in \mathfrak{h}^*$;

3.) $\lambda \in A$.

Proof. See Britten, Hooper, Lemire [2] and Britten, Lemire [3]. $\Box$

In the next theorem, the tensor product of a higher symplectic spinor module and the defining representation $\mathbb{C}^{2l} \simeq F(\varpi_1)$ of the complex symplectic Lie algebra C_l is decomposed into irreducible summands. We shall need this statement in the classification procedure. It gives us an important information on the structure of the 1^{st} jets prolongation module for metaplectic contact projective structures.

Theorem 2: Let $\lambda \in A$. Then

$$L(\lambda) \otimes F(\varpi_1) = \bigoplus_{\mu \in A_\lambda} L(\mu),$$

where $A_\lambda := A \cap \{\lambda + \nu | \nu \in \Pi(\varpi_1)\}$ and $\Pi(\varpi_1) = \{\pm\epsilon_i | i = 1, \dots, l\}$ is the set of weights of the defining representation.

Proof. See Krýsl, [21]. $\Box$

Let us remark, that the proof of this theorem is based on the so called Kac-Wakimoto formal character formula published in [15] (generalizing a statement of Jantzen in [14]) and some results of Humphreys, see [12], in which results of Kostant (from [19]) on tensor products of finite and infinite dimensional modules admitting a central character are specified.

4 Classification of first order invariant operators

In this section, we will be investigating first order invariant differential operators acting between sections of certain vector bundles associated to parabolic geometries $(p : \mathcal{G} \to M, \omega)$, i.e., to Cartan geometries modeled on Klein pairs (G, P), where P is an arbitrary parabolic subgroup of an arbitrary semisimple Lie group G.

We first consider a general real semisimple Lie group G together with its parabolic subgroup P and then we restrict our attention to the metaplectic contact projective case. Let us suppose that the Lie algebra $\mathfrak{g}$ of the group G is equipped with a $|k|$-grading $\mathfrak{g} = \bigoplus_{i=-k}^{k} \mathfrak{g}_i$, i.e., $\mathfrak{g}_1$ generates $\bigoplus_{i=1}^{k} \mathfrak{g}_i$ as a Lie algebra and $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for $i, j \in \{-k, \dots, k\}$. [1] Denote the semisimple part and the center of the reductive Lie algebra $\mathfrak{g}_0 \subset \mathfrak{g}$ by $\mathfrak{g}_0^{ss}$ and $\mathfrak{z}(\mathfrak{g}_0)$, respectively. The subalgebra $\bigoplus_{i=0}^{k} \mathfrak{g}_i$ forms a parabolic subalgebra of $\mathfrak{g}$ and will be denoted by $\mathfrak{p}$. Let us suppose that $\mathfrak{p}$ is isomorphic to the Lie algebra of the fixed parabolic subgroup P of G. The nilpotent part $\bigoplus_{i=1}^{k} \mathfrak{g}_i$ of $\mathfrak{p}$ is usually denoted by $\mathfrak{g}_+$ and the negative $\bigoplus_{i=-k}^{-1} \mathfrak{g}_i$ part of $\mathfrak{g}$ by $\mathfrak{g}_-$. Let us consider Killing forms $(,)_\mathfrak{g}$ and $(,)_{\mathfrak{g}_0^{ss}}$ of $\mathfrak{g}$ and $\mathfrak{g}_0^{ss}$, respectively. Further, fix a basis $\{\xi^i\}_{i=1}^r$ of $\mathfrak{g}_+$, such that $\{\xi^i\}_{i=1}^s$ is a basis of $\mathfrak{g}_1$ and $\{\xi^i\}_{i=s+1}^r$ is a basis of $\bigoplus_{i=2}^{k} \mathfrak{g}_i$. The second basis,

[1] By definition, $\mathfrak{g}_i = 0$ for $|i| > k$ is to be understood.

we will use, is a basis of $\mathfrak{g}_0^{ss}$, which will be denoted by $\{\eta^i\}_{i=1}^t$. The $|k|$-grading of $\mathfrak{g}$ uniquely determines the so called grading element $Gr \in \mathfrak{z}(\mathfrak{g}_0)$. The defining equation for this element is $[Gr, X] = jX$ for $X \in \mathfrak{g}_j$ and each $j \in \{-k, \dots, k\}$. It is known that for each $|k|$-grading of a real (or complex) semisimple Lie algebra the grading element exists, see, e.g., Yamaguchi [28]. Sometimes, we will denote the grading element Gr by η^{t+1}. The set $\{\eta^i\}_{i=1}^{t+1}$ is then a basis of $\mathfrak{g}_0$. Let us denote the basis of $\mathfrak{g}_-$ dual to $\{\xi^i\}_{i=1}^r$ with respect to the Killing form $(,)_\mathfrak{g}$ by $\{\xi_i\}_{i=1}^r$ and the basis of $\mathfrak{g}_0$ dual to the basis $\{\eta^i\}_{i=1}^{t+1}$ with respect to the Killing form $(,)_\mathfrak{g}$ by $\{\eta_i\}_{i=1}^{t+1}$.

At the beginning, let us consider two complex irreducible representations $(\sigma, \mathbf{E})$ and $(\tau, \mathbf{F})$ of P in the category $\mathcal{R}(P)$, the objects of which are locally convex, Hausdorff vector spaces with a continuous linear action of P, which is admissible, of finite length. Here, admissible action means that the restriction of this action to the Levi subgroup G_0 of P is admissible, see Vogan [27]. The morphisms in the category $\mathcal{R}(P)$ are linear continuous P-equivariant maps between the objects. It is well known that the unipotent part of the parabolic group acts trivially on both $\mathbf{E}$ and $\mathbf{F}$. We shall call $\mathbf{E}$ and $\mathbf{F}$ the domain and the target module, respectively and we shall specify further conditions on these representations later. Generally, for a Lie group G and its admissible representation $\mathbf{E}$, we shall denote the corresponding Harish-Chandra $(\mathfrak{g}, K)$-module (K is maximal compact in G) by E and when we will only be considering the $\mathfrak{g}$-module structure, we shall use the symbol $\mathbb{E}$ for it. Further, we will denote the corresponding actions of an element X from the Lie algebra of G on a vector v simply by $X.v$, and the action of $g \in G$ on a vector v by $g.v$ - the considered representation will be clear from a context.

Let us stress that most our proofs are formally almost identical to that ones written by Slovák, Souček in [24], but we formulate them also for infinite dimensional admissible irreducible $\mathbf{E}$ and $\mathbf{F}$, and use the decomposition result in Krýsl [21] when we will be treating the metaplectic contact projective case.

Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry modeled on the Klein pair (G, P). Because $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism for each $u \in \mathcal{G}$ by definition, we can define a vector field $\omega^{-1}(X)$ for each $X \in \mathfrak{g}$ by the equation $\omega_u(\omega^{-1}(X)_u) = X$, the so called constant vector field. For later use, consider two associated vector bundles $\mathbf{E}M := \mathcal{G} \times_\sigma \mathbf{E}$ and $\mathbf{F}M := \mathcal{G} \times_\tau \mathbf{F}$ - the so called domain and target bundle, respectively. To each Cartan geometry, there is an associated derivative ∇^ω defined as follows. For any section $s \in \Gamma(M, \mathbf{E}M)$ considered as $s \in \mathcal{C}^\infty(\mathcal{G}, \mathbf{E})^P$ under the obvious isomorphism, we obtain a mapping $\nabla^\omega s : \mathcal{G} \to \mathfrak{g}_-^* \otimes \mathbf{E}$, defined by the formula

$$(\nabla^\omega s(u))X := \mathcal{L}_{\omega^{-1}(X)} s(u),$$

where $X \in \mathfrak{g}_-$, $u \in \mathcal{G}$ and $\mathcal{L}$ is the Lie derivative. The associated derivative ∇^ω is usually called absolute invariant derivative. The 1^{st} jets prolongation module $J^1\mathbf{E}$ of $\mathbf{E}$ is defined as follows. As a vector space, it is simply the space $\mathbf{E} \oplus (\mathfrak{g}_+ \otimes \mathbf{E})$. To be specific, let us fix the Grothendieck's projective tensor product topology on 1^{st} jets prolongation module, see Treves [26] or/and D.

Vogan [27]. The vector space $J^1\mathbf{E}$ comes up with an inherited natural action of the group P, forming the 1^{st} jets prolongation P-module, see Čap, Slovák, Souček [6]. Let us remark that the function $u \mapsto (s(u), \nabla^\omega s(u))$ defines a P-equivariant function on $\mathcal{G}$ with values in $J^1\mathbf{E}$ and thus a section of the first jet prolongation bundle $J^1(\mathbf{E}M)$ of the associated bundle $\mathbf{E}M$. For details, see Čap, Slovák Souček [6].

By differentiation of the P-action on $J^1\mathbf{E}$, we can obtain a $\mathfrak{p}$-module structure, the so called infinitesimal 1^{st} jets prolongation $\mathfrak{p}$-module $J^1\mathbb{E}$, which is as a vector space isomorphic to $\mathbb{E} \oplus (\mathfrak{g}_+ \otimes \mathbb{E})$. The $\mathfrak{p}$-representation is then given by the formula

$$R.(v', S \otimes v'') := (R.v', S \otimes R.v'' + [R, S] \otimes v'' + \sum_{i=1}^{r} \xi^i \otimes [R, \xi_i]_{\mathfrak{p}}.v') \qquad (1)$$

where $R \in \mathfrak{p}$, $S \in \mathfrak{g}_+$, $v', v'' \in \mathbb{E}$ and $[R, \xi_i]_{\mathfrak{p}}$ denotes the projection of $[R, \xi_i]$ to $\mathfrak{p}$. For a derivation of the above formula, see Čap, Slovák, Souček [6] for more details. Obviously, this action does not depend on a choice of the vector space basis $\{\xi^i\}_{i=1}^r$. We will call this action the induced action of $\mathfrak{p}$.

Definition 4: We call a vector space homomorphism $\mathfrak{D} : \Gamma(M, \mathbf{E}M) \to \Gamma(M, \mathbf{F}M)$ first order invariant differential operator, if there is a P-module homomorphism[2] $\mathbf{D} : J^1\mathbf{E} \to \mathbf{F}$, such that $\mathfrak{D}s(u) = \mathbf{D}(s(u), \nabla^\omega s(u))$ for each $u \in \mathcal{G}$ and each section $s \in \Gamma(M, \mathbf{E}M)$ (considered as a P-equivariant $\mathbf{E}$-valued smooth function on $\mathcal{G}$).

Let us remark, that this definition could be generalized for an arbitrary order. The corresponding operators are called strongly invariant. There exist also operators which are invariant in a broader sense (see Čap, Slovák, Souček [5]) and not strongly invariant.

We shall denote the vector space of first order invariant differential operators by $\mathrm{Diff}(\mathbf{E}M, \mathbf{F}M)^1_{(p:\mathcal{G}\to M)}$. It is clear that $\mathrm{Diff}(\mathbf{E}M, \mathbf{F}M)^1_{(p:\mathcal{G}\to M)} \simeq \mathrm{Hom}_P(J^1\mathbf{E}, \mathbf{F})$ as complex vector spaces. Let us denote the restricted 1^{st} jets prolongation P-module, i.e., the quotient P-module

$$[\mathbf{E} \oplus (\mathfrak{g}_+ \otimes \mathbf{E})]/[\{0\} \oplus (\bigoplus_{i=2}^{k} \mathfrak{g}_i \otimes \mathbf{E})],$$

by $J^1_R\mathbf{E}$. According to our notation, the meanings of $J^1_R E$ and $J^1_R\mathbb{E}$ are also fixed. Now, let us introduce a linear mapping $\Psi : \mathfrak{g}_1 \otimes \mathbb{E} \to \mathfrak{g}_1 \otimes \mathbb{E}$ given by the following formula

$$\Psi(X \otimes v) := \sum_{i=1}^{s} \xi^i \otimes [X, \xi_i].v.$$

Obviously, mapping Ψ does not depend on a choice of the basis $\{\xi^i\}_{i=1}^s$.

First, let us derive the following

Theorem 3: Let $\mathbb{E}$ and $\mathbb{F}$ be two $\mathfrak{p}$-modules such that the nilpotent part $\mathfrak{g}_+$ acts trivially on them. If $D \in \mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E}, \mathbb{F})$ is a $\mathfrak{p}$-homomorphism, then

[2] By a P-module homomorphism, we mean a morphism in $\mathcal{R}(P)$.

D vanishes on the image of Ψ and D factors through the restricted jets, i.e., $D(0, Z \otimes v'') = 0$ for each $v'' \in \mathbb{E}$ and $Z \in \bigoplus_{i=2}^k \mathfrak{g}_i$. Conversely, suppose $D \in \mathrm{Hom}_{\mathfrak{g}_0}(J^1\mathbb{E}, \mathbb{F})$ is a $\mathfrak{g}_0$-homomorphism, D factors through the restricted jets, and D vanishes on the image of Ψ, then D is a $\mathfrak{p}$-module homomorphism.

Proof. Let $D \in \mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E}, \mathbb{F})$ be a $\mathfrak{p}$-homomorphism. Take an element $\tilde{v} \in \mathfrak{g}_+ .J^1\mathbb{E}$. Then $D(\tilde{v}) = D(X.v)$ for some $X \in \mathfrak{g}_+$ and $v \in \mathbb{E}$. Using the fact, that D is a $\mathfrak{p}$-homomorphism, we can write $D(\tilde{v}) = X.D(v) = 0$, because the nilpotent algebra $\mathfrak{g}_+$ acts trivially on the module $\mathbb{F}$. Thus D vanishes on the image of $\mathfrak{g}_+$ on $J^1\mathbb{E}$.

Now, we would like to prove, that D factors through $J^1_R\mathbb{E}$. Take an arbitrary element $Z \in \bigoplus_{i=2}^k \mathfrak{g}_i$ and $v'' \in \mathbb{E}$. Because $\mathfrak{g}$ is a $|k|$-graded algebra, there are $n \in \mathbb{N}$ and $X_i, Y_i \in \mathfrak{g}_+$ for $i = 1, \dots, n$, such that $Z = \sum_{i=1}^n [X_i, Y_i]$. It is easy to compute that $\sum_{i=1}^n X_i.(0, Y_i \otimes v'') = (0, \sum_{i=1}^n Y_i \otimes X_i.v'' + [X_i, Y_i].v'' + 0) = (0, \sum_{i=1}^n [X_i, Y_i] \otimes v'') = (0, Z \otimes v'')$. Thus we may write $D(0, Z \otimes v'') = D(\sum_{i=1}^n X_i.(0, Y_i \otimes v'')) = 0$, because D acts trivially on $\mathfrak{g}_+ .J^1\mathbb{E}$, as we have already proved.

Second, we shall prove that D vanishes on the image of Ψ. Substituting $v'' = 0$ into formula (1) for the induced action, we get that $X.(v', 0) = (X.v', \sum_{i=1}^r \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v')$ for $v' \in \mathbb{E}$ and $X \in \mathfrak{g}_1$. Assuming that the nilpotent subalgebra $\mathfrak{g}_+$ acts trivially on $\mathbb{E}$, one obtains $X.(v', 0) = (0, \sum_{i=1}^r \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v') = (0, \sum_{i=1}^s \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v' + \sum_{i=s+1}^r \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v')$. The last summand is zero, because $[X, \xi_i]_{\mathfrak{p}} = 0$ for $i > s$. Thus we have $X.(v', 0) = (0, \sum_{i=1}^s \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v')$. Because D vanishes on the image of the action of $\mathfrak{g}_+$ on $J^1\mathbb{E}$, we know that $0 = D(X.(v', 0)) = D(0, \sum_{i=1}^s \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v')$. Since one can omit the restriction of the Lie bracket in the last term to the subalgebra $\mathfrak{p}$ (we are considering ξ_i only for $i = 1, \dots, s$), D vanishes on the image of Ψ.

Now, we would like to prove the opposite direction. Hence suppose, a $\mathfrak{g}_0$-homomorphism D is given. Let us take an element $S \in \mathfrak{g}_+$ (for $S \in \mathfrak{g}_0$ it is clear) and an arbitrary element $\tilde{v} = (v', Y \otimes v'') \in J^1\mathbb{E}$. Thus $D(S.\tilde{v}) = D(S.(v', Y \otimes v'')) = D(S.v', Y \otimes S.v'' + [S, Y] \otimes v'' + \sum_{i=1}^r \xi^i \otimes [S, \xi_i]_{\mathfrak{p}}.v') = D(0, \sum_{i=1}^r \xi^i \otimes [S, \xi_i]_{\mathfrak{p}}.v') = 0 = S.D(\tilde{v})$, where we have used that the action of $\mathfrak{g}_+$ is trivial on $\mathbb{E}$, D factors through the restricted jets, vanishes on the image of Ψ, and the fact that $\mathfrak{g}_+$ acts trivially on $\mathbb{F}$. $\square$

Now, we derive the following

Lemma 1: For the mapping Ψ, we have

$$\Psi(X \otimes v) = \sum_{j=1}^{t+1} [\eta_j, X] \otimes \eta^j.v$$

for each $X \in \mathfrak{g}_1$ and $v \in \mathbb{E}$.

Proof. Take an element $X \in \mathfrak{g}_1$. Using the invariance of the Killing form $(,)_{\mathfrak{g}}$, expressed by $[X, \xi_i] = \sum_{i=1}^{t+1} (\eta_i, [X, \xi_i])_{\mathfrak{g}} \eta^i = \sum_{i=1}^{t+1} ([\eta_i, X], \xi_i)_{\mathfrak{g}} \eta^i$, we compute the value $\Psi(X \otimes v)$ as

$$\Psi(X \otimes v) \quad = \quad \sum_{i=1}^{s} \xi^i \otimes [X, \xi_i].v$$

$$
\begin{aligned}
&= \sum_{i=1}^{s} \xi^i \otimes \sum_{j}^{t+1} (\eta_j, [X, \xi_i])_{\mathfrak{g}} \eta^j.v \\
&= \sum_{i=1}^{s} \xi^i \otimes \sum_{j=1}^{t+1} ([\eta_j, X], \xi_i)_{\mathfrak{g}} \eta^j.v \\
&= \sum_{i=1}^{s} \sum_{j=1}^{t+1} ([\eta_j, X], \xi_i)_{\mathfrak{g}} \xi^i \otimes \eta^j.v \\
&= \sum_{j=1}^{t+1} [\eta_j, X] \otimes \eta^j.v.
\end{aligned}
$$

□

For any real Lie algebra $\mathfrak{g}$, let us denote its complexification over reals by $\mathfrak{g}^{\mathbb{C}}$, i.e., $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathfrak{h}$ be a (complex) Cartan subalgebra of $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$. For each $\lambda, \mu, \alpha \in \mathfrak{h}^*$, we define a complex number

$$
c_{\lambda\alpha}^{\mu} = \frac{1}{2}[(\lambda, \lambda + 2\delta)_{\mathfrak{g}_0^{ss}} + (\alpha, \alpha + 2\delta)_{\mathfrak{g}_0^{ss}} - (\mu, \mu + 2\delta)_{\mathfrak{g}_0^{ss}}],
$$

where δ denotes the sum of fundamental weights with respect to a choice of positive roots. [3]

From now on, we shall suppose that the semisimple part of the Levi factor of P is actually simple and the center of the Levi factor is one dimensional. These assumptions are rather technical and introduced only in order to simplify formulations of our statements. Until yet, we have demanded the considered modules to be admissible irreducible P-modules. In particular, we have used the fact that the unipotent part of P acts trivially on them. From now on, we will suppose in addition that the modules $\mathbb{E}$ and $\mathbb{F}$ are irreducible highest weight modules over the complexification $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}_0^{ss}$ of the semisimple part of the Levi factor G_0 of P. Further we shall suppose, that the grading element acts by a complex multiple on each of the modules $\mathbb{E}$ and $\mathbb{F}$. We call a pair $(\lambda, c) \in \mathfrak{h}^* \times \mathbb{C}$ a highest weight of a representation $\mathbb{E}$ over the reductive Lie algebra $(\mathfrak{g}_0)^{\mathbb{C}}$, if the restriction of the representation of $\mathfrak{g}_0$ on $\mathbb{E}$ to the simple part $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$ has highest weight λ and the grading element Gr acts by a complex number c. The complex number c is often called generalized conformal weight of the $\mathfrak{p}$-module $\mathbb{E}$.

Recall a well known theorem on the action of the universal Casimir element on highest weight modules.

Theorem 4: Let $\mathbb{E}$ be a highest weight module over the simple complex Lie algebra $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$ with a highest weight $\lambda \in \mathfrak{h}^*$ and $C \in \mathfrak{U}((\mathfrak{g}_0^{ss})^{\mathbb{C}})$ be the universal Casimir element of $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$. Then

$$
C.v = (\lambda, \lambda + 2\delta)_{\mathfrak{g}_0^{ss}} v,
$$

[3] We are denoting the Killing form on $\mathfrak{g}_0^{ss}$ as well as the dual form on $(\mathfrak{g}_0^{ss})^*$ by the same symbol $(,)_{\mathfrak{g}_0^{ss}}$. We shall also not distinguish between the Killing form of a real algebra and that one of the complexification of this algebra. We hope that this will cause no confusion.

where $v \in \mathbb{E}$.

Proof. See, e.g., Humphreys [13]. $\square$

Before we state the next lemma, let us do some comments on the relationship between the Killing forms $(,)_{\mathfrak{g}_0^{ss}}$ and $(,)_{\mathfrak{g}}$. It is well known that the restriction of $(,)_{\mathfrak{g}}$ to $\mathfrak{g}_0^{ss}$ is a non degenerate and obviously an invariant bilinear form, and therefore there is a constant $\kappa \in \mathbb{C}^{\times}$, such that for $X, Y \in \mathfrak{g}_0^{ss}$ we have $(X,Y)_{\mathfrak{g}_0^{ss}} = \kappa(X,Y)_{\mathfrak{g}}$ - due to the uniqueness of invariant non-degenerate forms up to a non zero complex multiple. The bases $\{\eta^i\}_{i=1}^t$ and $\{\eta_i\}_{i=1}^t$ of $\mathfrak{g}_0^{ss}$ are not dual with respect to the Killing form $(,)_{\mathfrak{g}_0^{ss}}$ in general. For further purposes, we can consider these bases being also bases of the appropriate complexified Lie algebras. According to the relationship between the Killing forms in question, we know that $\{\eta^i\}_{i=1}^t$ and $\{\kappa^{-1}\eta_i\}_{i=1}^t$ are dual with respect to $(,)_{\mathfrak{g}_0^{ss}}$. We would like to compute $(\sum_{i=1}^t \eta^i\eta_i).v$. Due to Theorem 4, we can write $(\sum_{i=1}^t \eta^i\kappa^{-1}\eta_i).v = (\lambda, \lambda+2\delta)_{\mathfrak{g}_0^{ss}} v$, if $v \in L(\lambda)$. Therefore $(\sum_{i=1}^t \eta^i\eta_i).v = \kappa(\lambda,\lambda+2\delta)_{\mathfrak{g}_0^{ss}} v$. Let us denote $(Gr, Gr)_{\mathfrak{g}} =: \rho^{-1}$, i.e., $\eta^{t+1} = Gr$ whereas $\eta_{t+1} = \rho Gr$. Thus if Gr acts by a complex number c, we have that the action of $\eta^{t+1}\eta_{t+1}$ is by ρc^2. We will use these computations in the proof of the following

Lemma 2: Suppose $\mathbb{E}$ is an irreducible $\mathfrak{p}^{\mathbb{C}}$-module, the action of $(\mathfrak{g}_+)^{\mathbb{C}}$ being trivial and the highest weight of $\mathbb{E}$ over $(\mathfrak{g}_0)^{\mathbb{C}}$ is $(\lambda, c) \in \mathfrak{h}^* \times \mathbb{C}$. Let us further suppose that $\mathbb{E} \otimes (\mathfrak{g}_1)^{\mathbb{C}}$ decomposes into a finite direct sum $\mathbb{E} \otimes \mathfrak{g}_1 = \bigoplus_{\mu} \mathbb{E}^{\mu}$ of irreducible $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-modules, where $\mathbb{E}^{\mu}$ is an irreducible $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-module with a highest weight μ. Let us fix a set of projections π_{μ} onto the irreducible summands in $\mathbb{E} \otimes (\mathfrak{g}_1)^{\mathbb{C}}$. Assume further that $(\mathfrak{g}_1)^{\mathbb{C}}$ is an irreducible $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-module with a highest weight α. Then

$$\Psi = \sum_{\mu} (\rho c - \kappa c^{\mu}_{\lambda\alpha})\pi_{\mu} \tag{2}$$

Proof. Let us do the following computation with "Casimir" operators $\sum_{i=1}^{t+1} \eta^i\eta_i \in \mathfrak{U}(\mathfrak{g}_0)$. For $X \in \mathfrak{g}_1$ and $v \in \mathbb{E}$, we have:

$$\begin{aligned}\sum_{i=1}^{t+1}(\eta^i\eta_i).(X\otimes v) &= \sum_{i=1}^{t+1}(\eta^i\eta_i).X\otimes v + X\otimes\sum_{i=1}^{t+1}(\eta^i\eta_i).v + \\ &\quad 2\Psi(X\otimes v),\end{aligned} \tag{3}$$

where we have used Lemma 1. Now, we would like to compute the first two terms of the R.H.S. of the last written equation using the universal Casimir element of $\mathfrak{g}_0^{ss}$, see Theorem 4.

$$\sum_{i=1}^{t+1}(\eta^i\eta_i).X\otimes v = \kappa(\alpha,\alpha+2\delta)_{\mathfrak{g}_0^{ss}} X\otimes v + \rho X\otimes v \tag{4}$$

$$X\otimes\sum_{i=1}^{t+1}(\eta^i\eta_i).v = \kappa(\lambda,\lambda+2\delta)_{\mathfrak{g}_0^{ss}} X\otimes v + \rho c^2 X\otimes v \tag{5}$$

Let us compute the L.H.S. of (3)

$$\sum_{i=1}^{t+1}(\eta^i\eta_i).(X\otimes v) = \sum_\mu \kappa(\mu,\mu+2\delta)_{\mathfrak{g}_0^{ss}}\pi_\mu(X\otimes v) + \sum_\mu \pi_\mu[\rho X\otimes v + 2\rho c X\otimes v + \rho c^2 X\otimes v] \qquad (6)$$

Substituting equations (4), (5) and (6) into equation (3) we obtain

$$\sum_\mu \kappa(\mu,\mu+2\delta)_{\mathfrak{g}_0^{ss}}\pi_\mu(X\otimes v) + 2\sum_\mu \rho c\pi_\mu(X\otimes v) + \sum_\mu \rho c^2\pi_\mu(X\otimes v) + \rho X\otimes v =$$
$$= 2\Psi(X\otimes v) + \kappa(\alpha,\alpha+2\delta)_{\mathfrak{g}_0^{ss}} X\otimes v + \rho X\otimes v + \kappa(\lambda,\lambda+2\delta)_{\mathfrak{g}_0^{ss}} X\otimes v + \rho c^2 X\otimes v.$$

As a result we obtain

$$\Psi(X\otimes v) = \sum_\mu (\rho c - \kappa c^\mu_{\lambda\alpha})\pi_\mu(X\otimes v).$$

□

4.1 Infinitesimal level classification

Let $(\mathbb{V},\omega)$ be a real symplectic vector space of dimension $2l+2$, $l\geq 3$. In this subsection, we shall focus our attention to the specific case of symplectic Lie algebra $\mathfrak{sp}(\mathbb{V},\omega)\simeq\mathfrak{sp}(2l+2,\mathbb{R})$ and its parabolic subalgebra $\mathfrak{p}$ introduced in section 2. We shall be investigating the vector space $\mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E},\mathbb{F})$ for suitable $\mathfrak{p}$-modules $\mathbb{E},\mathbb{F}$, i.e., classify the first order invariant differential operator at the infinitesimal level. For a moment, we shall consider a complex setting.

The complex symplectic Lie algebra $\mathfrak{g}^{\mathbb{C}}=\mathfrak{sp}(2l+2,\mathbb{C})$ possesses a $|2|$-grading,

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}_{-2}\oplus\mathfrak{g}^{\mathbb{C}}_{-1}\oplus\mathfrak{g}^{\mathbb{C}}_{0}\oplus\mathfrak{g}^{\mathbb{C}}_{1}\oplus\mathfrak{g}^{\mathbb{C}}_{2},$$

such that $\mathfrak{g}^{\mathbb{C}}_2\simeq\mathbb{C}$, $\mathfrak{g}^{\mathbb{C}}_1\simeq\mathbb{C}^{2l}$, $\mathfrak{g}^{\mathbb{C}}_0=(\mathfrak{g}_0^{ss})^{\mathbb{C}}\oplus(\mathfrak{z}(\mathfrak{g}_0))^{\mathbb{C}}\simeq\mathfrak{sp}(2l,\mathbb{C})\oplus\mathbb{C}$. This splitting could be displayed as follows. Choose a basis B of $\mathbb{V}$ such that ω, expressed in coordinates with respect to B, is given by $\omega((z^1,\dots,z^{2l+2}),(w^1,\dots,w^{2l+2})) = w^1z^{2l+2}+\dots+w^{l+1}z^{l+2}-w^{l+2}z^{l+1}-\dots-w^{2l+2}z^1$. For $A\in\mathfrak{sp}(2l+2,\mathbb{C})$ we have:

$$A = \left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right)$$

with respect to B. As one can easily compute, the parabolic subalgebra $\mathfrak{p}^{\mathbb{C}} = (\mathfrak{g}_0)^{\mathbb{C}}\oplus(\mathfrak{g}_1)^{\mathbb{C}}\oplus(\mathfrak{g}_2)^{\mathbb{C}}$ is a complexification of the Lie algebra of the group P

introduced in section 2, where we have defined the metaplectic contact projective geometry. Before we state the next theorem, we should compute the coefficients ρ and κ for the case $\mathfrak{g} = \mathfrak{sp}(2l+2,\mathbb{C})$ considered with the grading given above. One can easily realize, that

$$Gr = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0_{2l} & 0 \\ \hline 0 & 0 & -1 \end{array}\right)$$

is the grading element, and that $(Gr, Gr)_{\mathfrak{g}} = 8l+8$. Computing the square-norm of an element of $\mathfrak{g}_0^{ss}$ via $(,)_{\mathfrak{g}}$ and $(,)_{\mathfrak{g}_0^{ss}}$, one obtains for the ratio $\kappa = \frac{l}{l+1}$. Further, let us introduce a bilinear form $<,>$ on $\mathfrak{h}^*$, in which the orthogonal basis $\{\epsilon_i\}_{i=1}^l$ is orthonormal. The relation between the Killing form $(,)_{\mathfrak{g}_0^{ss}}$ and $<,>$ is given by $(X,Y)_{\mathfrak{g}_0^{ss}} = \frac{1}{8l+8} < X, Y >$ for $X, Y \in \mathfrak{h}^*$. For each $\lambda, \mu, \alpha \in \mathfrak{h}^*$, let us define a complex number

$$\tilde{c}^{\mu}_{\lambda\alpha} = \frac{1}{2}(< \mu, \mu + 2\delta > - < \lambda, \lambda + 2\delta > - < \alpha, \alpha + 2\delta >).$$

Substituting the computed values of ρ and κ and the relation between $(,)_{\mathfrak{g}_0^{ss}}$ and $<,>$ into formula (2), we obtain a prescription for mapping Ψ (in the metaplectic contact projective case)

$$\Psi = \frac{1}{8l+8}\sum_{\mu}(c - \tilde{c}^{\mu}_{\lambda\alpha})\pi_{\mu}.$$

Theorem 5: For $(\lambda, c), (\mu, d) \in A \times \mathbb{C}$, let $\mathbb{E}$ and $\mathbb{F}$ be two $\mathfrak{p}^{\mathbb{C}}$-modules such that $\mathbb{E}$ and $\mathbb{F}$ are irreducible if considered as $(\mathfrak{g}_0)^{\mathbb{C}}$-modules with highest weight (λ, c) and (μ, d), respectively, and let $(\mathfrak{g}_+)^{\mathbb{C}}$ has a trivial action on each of these modules. Further, suppose $\lambda \neq \mu$. Then

$$\mathrm{Hom}_{\mathfrak{p}^{\mathbb{C}}}(J^1\mathbb{E}, \mathbb{F}) \simeq \begin{cases} \mathbb{C}, & \text{if} \quad \mu \in A_\lambda \quad \text{and} \quad d - 1 = c = \tilde{c}^{\mu}_{\lambda\varpi_1} \\ 0 & \text{in other cases.} \end{cases}$$

Proof. Let us start with the second part of the statement, i.e., $\mu \notin A_\lambda$ or $c \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$ or $d-1 \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$, and consider an element $T \in \mathrm{Hom}_{\mathfrak{p}^{\mathbb{C}}}(J^1\mathbb{E}, \mathbb{F})$. Then $T \in \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1\mathbb{E}, \mathbb{F})$. Because T is a $\mathfrak{p}^{\mathbb{C}}$-homomorphism, we have that $T \in \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E}, \mathbb{F})$ due to Theorem 3 (used in the complexified setting). We also know that

$$\mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E}, \mathbb{F}) = \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(\mathbb{E}, \mathbb{F}) \oplus \bigoplus_{\nu \in A_\lambda} \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(L(\nu), L(\mu))$$

due to Theorem 2. If we suppose $\mu \notin A_\lambda$ and $\lambda \neq \mu$, then due to Theorems 2.6.5, 2.6.6 in Dixmier [7], each member of the direct sum is zero. Now suppose that $\mu \in A_\lambda$. Thus $c \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$ or $d-1 \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$. First suppose that $c \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$. Using

Theorem 2 and the cited theorems of Dixmier, we see that $\mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E},\mathbb{F}) \simeq \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(\mathbb{E},\mathbb{F}) \oplus \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(L(\mu),L(\mu)) \simeq \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(L(\mu),L(\mu))$, because the decomposition of $(\mathfrak{g}_1)^{\mathbb{C}} \otimes \mathbb{E}$ is multiplicity-free and $\lambda \neq \mu$. Thus we can consider T to be a $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-intertwining operator acting on the irreducible highest weight module $L(\mu)$. We have two possibilities: $T : L(\mu) \to L(\mu)$ is either zero and we are done, or $\mathrm{Ker}\, T = \{0\}$. We will suppose the latter possibility. Take a nonzero element $0 \neq v \in L(\mu)$. Using the formula $\Psi = (8l+8)^{-1}\sum_\nu (c - \tilde{c}^\nu_{\lambda\varpi_1})\pi_\nu$, we obtain under the assumption $c \neq \tilde{c}^\mu_{\lambda\varpi_1}$ that $\Psi(v) = (8l+8)^{-1}(c - \tilde{c}^\mu_{\lambda\alpha})v \neq 0$. Because $\mathrm{Ker}\, T = \{0\}$, we have that $T\Psi(v) \neq 0$ and thus, according to Theorem 3, T it is not a $\mathfrak{p}^{\mathbb{C}}$-module homomorphism because it does not vanish on the image of Ψ. Secondly, consider the case $d \neq \tilde{c}^\mu_{\lambda\alpha} + 1$. We can make the following easy computation. $d(S_1 \otimes v'') = Gr.(S_1 \otimes v'') = [Gr, S_1] \otimes v'' + S_1 \otimes Gr.v'' = (1+c)S_1 \otimes v''$ for $S_1 \in (\mathfrak{g}_1)^{\mathbb{C}}$ and $v'' \in \mathbb{E}$. Thus $c = d-1$ and we are obtaining the case $c \neq \tilde{c}^\mu_{\lambda\varpi_1}$, which was already handled.

Now, consider the case $\mu \in A_\lambda$, $c = \tilde{c}^\mu_{\lambda\alpha}$ and $d - 1 = \tilde{c}^\mu_{\lambda\alpha}$ and take a $T \in \mathrm{Hom}_{\mathfrak{p}^{\mathbb{C}}}(J^1\mathbb{E},\mathbb{F})$. As in the previous case, this implies $T \in \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E},\mathbb{F})$. Decomposing $J^1_R\mathbb{E} = L(\lambda) \oplus F(\varpi_1) \otimes L(\lambda)$ into irreducible modules and substituting this decomposition into $\mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E},\mathbb{F})$, we obtain a direct sum

$$\mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(\mathbb{E},\mathbb{F}) \oplus \bigoplus_{\nu \in A_\lambda} \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(L(\nu),L(\mu)).$$

According to our assumptions $\mu \in A_\lambda$ and $\lambda \neq \mu$, and due to the structure of the set A_λ, we know that the direct sum simplifies into a space isomorphic to $\mathbb{C}$ (using the above cited theorem of Dixmier once more). Thus we know that $\mathrm{Hom}_{\mathfrak{p}^{\mathbb{C}}}(J^1\mathbb{E},\mathbb{F}) \subseteq \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E},\mathbb{F}) \simeq \mathbb{C}$. To obtain an equality in the previous inclusion, consider the one dimensional vector space of $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-homomorphisms $\{w\widetilde{\pi_\mu} | w \in \mathbb{C}\}$, where $\widetilde{\pi_\mu}$ is a trivial extension of the projection $(\mathfrak{g}_1)^{\mathbb{C}} \otimes \mathbb{E} \to L(\mu)$. The elements of this vector space are clearly $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-homomorphisms, which vanish on the image of Ψ, if $c = \tilde{c}^\mu_{\lambda\alpha}$, and they factorize through the restricted jets. What remains is to show that for each $w \in \mathbb{C}$, mappings $w\widetilde{\pi_\mu}$ are not only $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-homomorphisms, but also $(\mathfrak{g}_0)^{\mathbb{C}}$-homomorphisms. Notice that it is sufficient to test the condition only on $(\mathfrak{g}_1)^{\mathbb{C}} \otimes \mathbb{E}$ because $Gr \in (\mathfrak{g}_0)^{\mathbb{C}}$, and $\widetilde{\pi_\mu}$ is the trivial extension, see formula (1). For $S_1 \in (\mathfrak{g}_1)^{\mathbb{C}}$ and $v'' \in \mathbb{E}$, we have $Gr.\widetilde{\pi_\mu}(S_1 \otimes v'') = d\widetilde{\pi_\mu}(S_1 \otimes v'')$ by definition. Now, let us evaluate $\widetilde{\pi_\mu}Gr.(S_1 \otimes v'') = \widetilde{\pi_\mu}([Gr, S_1] \otimes v'' + S_1 \otimes Gr.v'') = \widetilde{\pi_\mu}(S_1 \otimes v'' + cS_1 \otimes v'') = (1+c)\widetilde{\pi_\mu}(S_1 \otimes v'') = d\widetilde{\pi_\mu}(S_1 \otimes v'') = Gr.\widetilde{\pi_\mu}(S_1 \otimes v'')$, thus $\widetilde{\pi_\mu}$ commutes with the action of Gr. Therefore $\widetilde{\pi_\mu}$ is a $(\mathfrak{g}_0)^{\mathbb{C}}$-homomorphism and the statement follows using Theorem 3. $\square$

Let us remark, that for $\lambda = \mu$, the space of homomorphisms is also one dimensional. But this case leads to zeroth order operators, which are not interesting from the point of view of our classification. Let us derive an easy corollary of the above theorem.

Corollary 1: The preceding theorem remains true for a real form $\mathfrak{f}$ of $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$, if one considers complex representations and complex linear homomorphisms. In particular, it remains true for the split real form $\mathfrak{f} = \mathfrak{g}_0^{ss} \simeq \mathfrak{sp}(2l,\mathbb{R})$.

Proof. First, observe that the decomposition of $F(\varpi_1) \otimes L(\lambda)$ remains the same also over $\mathfrak{f}$. For it, let us take an irreducible summand $\mathbb{M}$ in the decomposition and suppose there is a proper nontrivial complex submodule $\mathbb{M}'$ of $\mathbb{M}$. For $v \in \mathbb{M}'$ and $X + iY \in \mathfrak{f} + i\mathfrak{f}$, we get that $(X + iY).v = X.v + iY.v$. Using the fact that $\mathbb{M}'$ is closed under complex number multiplication and $X.v, Y.v \in \mathbb{M}'$, we would obtain that $\mathbb{M}'$ is $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-invariant, which is a contradiction.

Second, we would like to prove that each $\mathfrak{f}$-invariant complex linear endomorphisms of an irreducible module, say $\mathbb{F}$, is a scalar. It is easy to observe, that such an endomorphism is actually $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-endomorphism, i.e., the theorem of Dixmier used in the proof of the previous theorem, could be applied and the corollary follows. $\square$

4.2 Globalized level classification

In this subsection, we shall extend the results obtained in the previous one to the group level. We will do it using some basic facts on globalization techniques.

Let $(\mathbb{V}, \omega)$ be a real symplectic vector space of real dimension $2l + 2$, $l \geq 3$, $G = Sp(\mathbb{V}, \omega)$ and P as described in section 2. First, we introduce the groups, we shall be considering. Let G_+, G_0, G_0^{ss}, K be the unipotent part, the Levi factor, the semisimple part of P and the maximal compact subgroup of G, respectively. Recall that we have fixed a non-trivial 2-fold covering $q : \tilde{G} \to G$ of the symplectic group G by the metaplectic group $\tilde{G} = Mp(\mathbb{V}, \omega)$. Let us denote the respective q-preimages by $\tilde{G}_+, \tilde{G}_0, \tilde{G}_0^{ss}, \tilde{K}$. Further, let us denote the maximal compact subgroup of the semisimple part G_0^{ss} of the Levi factor by K_0^{ss} and its q-preimage by $\tilde{K}_0^{ss}$. We have

$$\tilde{K}_0^{ss} \simeq \widetilde{U(l)} = \{(u, z) \in U(l) \times \mathbb{C}^\times | \det u = z^2\},$$

which is obviously connected, see Tirao, Vogan and Wolf [25].

Second, let us introduce a class of $\tilde{P}$-modules we shall be dealing with. In Kashiwara, Vergne [17], the so called metaplectic (or Segal-Shale-Weil or oscillator) representation over $\tilde{G}_0^{ss}$ is introduced. Let $\mathbf{S}_+$ be the irreducible submodule of the Segal-Shale-Weil representation consisting of even functions. Let us take the underlying $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module and denote it by S_+. The $\mathfrak{g}_0^{ss}$-module structure of this representation coincides with the irreducible highest weight module structure of $\mathbb{S}_+$, which was introduced in section 3. For a choice of a weight $\lambda \in A$, we know that there exists a dominant integral weight ν (with respect to choices made in section 3), such that $\mathbb{L} := L(\lambda) \subseteq \mathbb{S}_+ \otimes F(\nu)$. Because $\mathbb{S}_+ \otimes F(\nu)$ decomposes without multiplicities, we have an identification of $L(\lambda)$ with its isomorphic module in $\mathbb{S}_+ \otimes F(\nu)$. Now we would like to make $\mathbb{L}$ a $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module. Using a result of Baldoni [1], this could be done as follows. Because S_+ and $F(\nu)$ are $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-modules, their tensor product is a $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module as well. Using the fact that $\tilde{K}_0^{ss} = \widetilde{U(l)}$ is connected, we are obtaining a $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module structure on each irreducible summand in $\mathbb{S}_+ \otimes F(\nu)$, in particular on $\mathbb{L}$. Denote the resulting $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module by L. Using globalization results of Kashiwara and Schmid in [16], there exists a minimal globalization for this $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module, which will be denoted by $\mathbf{L} =: \mathbf{L}(\lambda)$. (For this topic, see also Vogan [27] and Schmid [23].) Thus $\mathbf{L}(\lambda)$ is a complex $\tilde{G}_0^{ss}$-module. Further, we need to specify the action of the center of $\tilde{G}_0$ and that one of the unipotent part $\tilde{G}_+$. For each $(\lambda, c) \in A \times \mathbb{C}$ we suppose, that the unipotent $\tilde{G}_+$ acts trivially on $\mathbf{L}(\lambda)$ and the grading element Gr in the Lie algebra of the center of the Levi factor $\tilde{G}_0$ acts by multiplication by a complex number $c \in \mathbb{C}$. Since the center is isomorphic to $\mathbb{R}^\times$ we need to specify the action of, e.g., $-1 \in \mathbb{R}^\times$. This action should be any $\gamma \in \mathbb{R}$ satisfying $\gamma^2 = 1$. So we have obtained a $\tilde{P}$-module structure on $\mathbf{L}(\lambda)$ which we will refer to as $\mathbf{L}(\lambda, c)_\gamma$. Let us remark, that defining the action of $\tilde{G}_+$ to be trivial, is actually no restriction, when one considers only irreducible admissible $\tilde{P}$-modules. We shall call the corresponding associated bundles higher symplectic bundles and the corresponding 1^{st} order invariant differential operators symplectic spinor operators, stressing the fact that the representations of $\tilde{P}$ we are considering are coming from higher symplectic spinor modules.

Theorem 6. Let $(\lambda, c, \gamma), (\mu, d, \gamma') \in A \times \mathbb{C} \times \mathbb{Z}_2$ [4], $\lambda \neq \mu$ and $(p : \tilde{\mathcal{G}} \to M^{2l+1}, \omega)$ be a metaplectic contact projective geometry of rank l. Consider the $\tilde{P}$-modules $\mathbf{E} := \mathbf{L}(\lambda, c)_\gamma$ and $\mathbf{F} := \mathbf{L}(\mu, d)_{\gamma'}$. Then for the vector space of invariant differential operators up to a zeroth order we have

$$\mathrm{Diff}(\mathbf{E}M, \mathbf{F}M)^1_{(p:\tilde{\mathcal{G}} \to M^{2l+1}, \omega)} \simeq \begin{cases} \mathbb{C} & \text{if } \mu \in A_\lambda,\ d-1 = c = \tilde{c}^{\mu}_{\lambda\varpi_1} \text{ and } \gamma = \gamma' \\ 0 & \text{in other cases.} \end{cases}$$

Proof. According to the definition of first order invariant differential operators between sections of associated vector bundles over Cartan geometries, the vector space $\mathrm{Diff}(\mathbf{E}M, \mathbf{F}M)^1_{(p:\tilde{\mathcal{G}} \to M, \omega)}$ is isomorphic to the space $\mathrm{Hom}_{\tilde{P}}(J^1\mathbf{E}, \mathbf{F})$. From the definition of the minimal globalization, it follows that it gives a natural bijection between Hom's of respective categories: the Harish-Chandra category of $(\mathfrak{p}, \widetilde{K \cap G_0})$-modules and the category of admissible $\tilde{P}$-modules, see Kashiwara, Schmid [16]. Thus we have $\mathrm{Hom}_{\widetilde{P}}(J^1\mathbf{E}, \mathbf{F}) \simeq \mathrm{Hom}_{(\mathfrak{p}, \widetilde{K\cap G_0})}(J^1E, F)$. Because the identity component $(\widetilde{K \cap G_0})_1$ is connected by definition, we can write $\mathrm{Hom}_{(\mathfrak{p}, (\widetilde{K\cap G_0})_1)}(J^1E, F) \simeq \mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E}, \mathbb{F})$, see W. Baldoni [1]. It remains to show that each $\mathfrak{p}$-module homomorphism is actually a $(\mathfrak{p}, (\widetilde{K \cap G_0})_{-1})$-module homomorphism, where $(\widetilde{K \cap G_0})_{-1}$ denotes the component of the group $\widetilde{K \cap G_0}$ to which -1 belongs. Let us parameterize the elements of the (-1)-component of $(\widetilde{K \cap G_0}) \simeq \widetilde{U(l)} \times \mathbb{Z}_2$ by pairs $(k, -1)$, $k \in \widetilde{U(l)}$, and denote the appropriate $\tilde{P}$-representation on E by ρ. We can easily check that for $(v', S \otimes v'') \in J^1\mathbf{E}$, we have $(k, -1).(v', S \otimes v'') = (\rho(k, -1)v', Ad(k, -1)S \otimes \rho(k, -1)v'') = (\gamma\rho(k, 1)v', Ad(k, 1)S \otimes \gamma\rho(k, 1)v'') = \gamma(k, 1).(v', S \otimes v'')$. Further for a $\mathfrak{p}$-homomorphism $T \in \mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E}, \mathbb{F})$, we can write $T(k, -1).(v', S \otimes v'') = \gamma T(k, 1).(v', S \otimes v'') = \gamma(k, 1).T(v', S \otimes v'') = \gamma\gamma'(k, -1).T(v', S \otimes v'')$. Thus we have also $\mathrm{Hom}_{(\mathfrak{p}, \widetilde{K\cap G_0})}(J^1E, F) \simeq \mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E}, \mathbb{F})$ if $\gamma = \gamma'$. The Hom at the

[4] The group $\mathbb{Z}_2$ is considered as multiplicative, i.e., $\mathbb{Z}_2 = \{-1, 1\}$.

right hand side was determined in corollary 1. In the case $\gamma \neq \gamma'$, we have that $T = 0$ and the proof is finished. $\square$

5 Examples: contact projective Dirac, twistor and Rarita-Schwinger operators

In this section, we shall introduce three main examples of contact projective analogues of Dirac, twistor and Rarita-Schwinger operators known from Riemannian and partly from symplectic geometry. In each of the next paragraphs, we suppose that a metaplectic contact projective geometry $(p : \tilde{\mathcal{G}} \to M^{2l+1}, \omega)$ of rank l is fixed.

Contact projective Dirac operator. For $\lambda = -\frac{1}{2}\varpi_l$, we have $A_\lambda = \{\varpi_1 - \frac{1}{2}\varpi_l, \varpi_{l-1} - \frac{3}{2}\varpi_l\}$ according to Theorem 2. Take $\mu = \varpi_{l-1} - \frac{3}{2}\varpi_l \in A_\lambda$. Using $\delta = l\epsilon_1 + (l-1)\epsilon_2 + \ldots + \epsilon_l$, we obtain that $\tilde{c}^{\mu}_{\lambda\alpha} = \frac{1+2l}{2}$. Thus for conformal weight $c = \frac{1+2l}{2}$ and $\gamma \in \mathbb{Z}_2$ there is an invariant differential operator $\mathfrak{D}^{\frac{1}{2}} : \Gamma(M, \mathbf{L}(\lambda, \frac{1+2l}{2})_\gamma M) \to \Gamma(M^{2l+1}, \mathbf{L}(\mu, \frac{3+2l}{2})_\gamma M)$. This operator could be called contact projective Dirac operator because of the analogy with the orthogonal case.

Contact projective twistor operator. Taking the same $\lambda = -\frac{1}{2}\varpi_l$ as in the previous example and $\mu = \varpi_1 - \frac{1}{2}\varpi_l$, we obtain $c = \frac{1}{2}$ and the corresponding operator $\mathfrak{T} : \Gamma(M, \mathbf{L}(\lambda, \frac{1}{2})_\gamma M) \to \Gamma(M, \mathbf{L}(\mu, \frac{3}{2})_\gamma M)$ ($\gamma \in \mathbb{Z}_2$) is called contact projective twistor operator also due to the analogy with the orthogonal case.

Contact projective Rarita-Schwinger operator. Here, take $\lambda = \varpi_1 - \frac{1}{2}\varpi_l$. $A_\lambda = \{\varpi_2 - \frac{1}{2}\varpi_l, 2\varpi_1 - \frac{1}{2}\varpi_l, -\frac{1}{2}\varpi_l, \varpi_1 + \varpi_{l-1} - \frac{3}{2}\varpi_l\}$. For $\mu = \varpi_1 + \varpi_{l-1} - \frac{3}{2}\varpi_l$, we obtain $c = \frac{1+2l}{2}$, and we shall call this operator contact projective Rarita-Schwinger operator,
$\mathfrak{D}^{\frac{3}{2}} : \Gamma(M, \mathbf{L}(\lambda, \frac{1+2l}{2})_\gamma M) \to \Gamma(M, \mathbf{L}(\mu, \frac{3+2l}{2})_\gamma M)$, where again $\gamma \in \mathbb{Z}_2$.

Remark: It may be interesting to mention, that computing formally the conformal weights using a Lepowsky generalization of a result of Bernstein-Gelfand-Gelfand on homomorphism of non-true Verma-modules, one gets exactly the same weights, although Lepowsky is considering only Verma modules induced by finite dimensional representations.

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