Doctoral dissertation thesis

Invariant differential operators for projective contact geometries

Svatopluk Krýsl

supervisor: Prof. RNDr. Vladimír Souček, DrSc.

Faculty of Mathematics and Physics
Charles University, Prague

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Chapter 1

Introduction

The main topic of the dissertation belongs to the crossroad of several disciplines - *global analysis* (first order differential operators on manifolds), *geometry* (manifolds with a special geometrical structure) and *representation theory* (finite and infinite dimensional representations of semi-simple and reductive Lie groups and Lie algebras). The principal role of the Lie group theory follows from the fact that we are studying differential operators on manifolds, which are invariant with respect to a suitable action of a semi-simple Lie group on considered functions (or fields).

During last decades, a lot of attention was paid to a special class of geometrical structures on manifolds based on parabolic subgroups of semi-simple Lie groups. Standard name used at present for such structures is a parabolic geometry. Main examples of parabolic geometries include projective, conformal, quaternionic and CR manifolds, further interesting cases are being studied just now in more details. Every parabolic geometry has its homogeneous model $M = G/P$, where P is a parabolic subgroup of a semi-simple Lie group G. Its curved version is, following ideas of Élie Cartan going back to the beginning of the last century, a principal fibre bundle $\mathcal{G}$ over a manifold M the dimension of which equals the dimension of G/P together with the so called Cartan connection ω giving a complete H-equivariant parallelism of the tangent space of $\mathcal{G}$ and reproducing the fundamental vector fields.

One of those geometries is also the so called projective contact geometry. It is a special case of a contact geometry (i.e., a subbundle of the tangent space of codimension 1 is given with suitable properties). The projective and projective contact geometries are the two exceptional parabolic geometries. They form a distinguished set of examples due to the fact that usual geometric data are not sufficient to reconstruct the principal bundle $\mathcal{G}$ and the Cartan connection ω, additional data are needed in these two cases.

The projective contact geometry is described in details in chapter 4, following ideas contained in the forthcoming book Čap, Slovák [7]. We describe in detail the additional structure needed to make the odd dimensional sphere into a homogeneous model of the projective contact geometry. Main data of the curved version are described. Invariant differential operators on manifolds with a given contact projective structure contain a class of standard operators coming in the so called BGG sequences. The form of the sequence and the highest weights of induced bundles are described.

The representations inducing the bundles in the standard BGG sequences are finite dimensional irreducible representations of a reductive group (the Levi factor) with a trivial action of P_+. The projective contact geometry is modelled on a homogeneous space $M = G/P$, where $\mathfrak{g} = \mathfrak{sp}(2k+2, \mathbb{R})$ and $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{sp}(2k, \mathbb{R})$.

There is a very nice and unusual analogy between irreducible modules over $\mathfrak{sp}(2k, \mathbb{R})$ and $\mathfrak{so}(2k, \mathbb{R})$. The standard point of view is to consider the set of finite dimensional representations of $\mathfrak{sp}(2k, \mathbb{R})$ as the most appropriate analogue of the set of finite dimensional representation of $\mathfrak{so}(2k, \mathbb{R})$. There is, however, a substantial difference between the both cases as far as spinor representations concerns. In the symplectic case, there are no finite dimensional representations similar to spinor representations of the orthogonal group. A remarkable and unusual analogue of spinor representations was proposed by Bertram Kostant, see [23]. His spinor representations of the metaplectic group (which is a double cover, as in the orthogonal case, of the symplectic group) are, however, infinite dimensional ones.

Finite dimensional representations are usually classified (and denoted) by their highest weight. Infinite dimensional representations do not have necessarily a highest weight. The Kostant spinor representations are highest weight modules and their highest weight has (as it is the case for spinor representations of the orthogonal group) half-integer entries. The Kostant infinite dimensional representations were used as values for spinor fields on $\mathbb{R}^{2k}$ and the (symplectic) Dirac operator was defined for such spinor fields by Katharina Habermann, see [15].

In the orthogonal case, any finite dimensional representation of $\mathfrak{so}(\mathbb{R}, 2k)$ can be constructed as a spin-tensor, i.e., as an irreducible subrepresentation in the tensor product of a spinor representation with a finite dimensional representation with integral highest weight (a tensor representation). By a Kostant proposal, we know what an analogue of a spinor representation should be. One possible suggestion for an analogue of spin-tensors in the symplectic case hence could be any irreducible subrepresentation of a tensor product of a finite dimensional representation with the Kostant spinor representation. It is not easy, however, to show that such a tensor product is completely reducible and to characterize the corresponding infinite-dimensional representations.

Recently, these facts were proved by Britten, Hooper and Lemire, see [3]. They proved the complete reducibility of such tensor products and characterized the corresponding irreducible components by their (half-integral) highest weights. They also showed their another nice characterization. They are exactly those infinite-dimensional representations, which have uniformly bounded dimension of their weight spaces. The Kostant spinor representations are characterized among them as the unique representations, for which dimensions of all their weight spaces are equal to one! In the thesis, these representations are called harmonic representations.

It hence seems to be clear that finite-dimensional spin-tensor representations of the orthogonal group have as an appropriate analogue the union of set of all finite dimensional representations (which is an analogue of tensor representations) and the set of all harmonic representations (an analogue of spin-tensor representations). All the suggests that for the symplectic group, we should study invariant differential operators acting on fields with values in harmonic representations, resp. in the corresponding induced bundles.

For this purpose, the key fact is to prove that the tensor product of a harmonic representation with the defining representation is completely reducible and its components are again harmonic modules. These facts are proved in the third chapter and it is one of the main results in the thesis. For every projection from the tensor product of a harmonic representation with the defining representation, the analogue of a conformal weight can be uniquely computed in such a way that the chosen projection is a $\mathfrak{p}$-module homomorphism. This is a basic fact, see the chapter 6, needed for an investigation of first order invariant differential operators on functions with values in a harmonic representation, its proof can be found in chapter 6.

A part of the thesis is devoted to another special case of a parabolic geometry. It is a contact orthogonal geometry in odd dimension. In chapter 4, the BGG diagram is computed for this parabolic geometry in a general case. This is another new result in the thesis. It means that we give a full description of all weights for inducing bundles for Lie algebra cohomology for any irreducible G-modules. As an application of this result, it is possible to determine a form of singular orbits for the affine action of the Weyl group. In particular, potential applications include a quite interesting case of the complex, which starts with an overdetermined first order system, which reduces in the flat case to the system of two Dirac operators on $\mathbb{R}^n$. The resulting complex is then an analogue of the Dolbeault complex in two Clifford variables. For the mentioned applications, it is also important to understand real versions of the Hasse and BGG diagrams.

Now, let us come to a more detailed description of chapters of this dissertation thesis. The first chapter includes this introduction.

In the second chapter some introductory notions and basic facts are presented. In the first section, the definitions of a $|k|$-grading and standard parabolic subalgebras are given. We have mentioned the relationship between $|k|$-gradings and parabolic subalgebras of a simple Lie algebra in the real and complex case. In the second section, standard cyclic modules over simple Lie algebras are sketched and real representations of real forms of simple Lie algebras explained. Some combinatorial and algebraic structures like Hasse and Bernstein-Gelfand-Gelfand (BGG) diagrams and cohomologies of Lie algebras with values in finite-dimensional $\mathfrak{g}$-modules are introduced in the third section. In this section, we have also presented the Kostant theorem on cohomologies together with the relationship between cohomologies of Lie algebras and the BGG-diagrams. The fourth section is devoted to symplectic Clifford algebras, Heisenberg algebras and groups together with their representation. In this section the Segal-Shale-Weil representation of a metaplectic group is introduced. The last two sections of this chapter involve some basic facts on Cartan geometries (especially the parabolic ones), first jet-prolongation of a P-module, absolutely invariant derivative and invariant differential operators.

The third chapter is devoted to the study of certain class of irreducible infinite-dimensional standard cyclic modules over symplectic algebras, the so called harmonic modules. In the first section we have summarized some results of Kostant on tensor products of finite and infinite dimensional $\mathfrak{g}$-modules admitting a central character. In the second section, we present a generalization of some theorems on completely reducible modules, which are well known in the finite dimensional case, to the infinite dimensional one. In the third section the half-spinor representations of the symplectic algebra are introduced together with a description of an analogy between this modules and the spinor modules for orthogonal algebras. We have summarized some results of Britten, Hooper and Lemire written in articles [3], [2] in the fourth section. In the fifth section, a theorem is proved, in which the modules occurring in the tensor product of a harmonic module with the defining representation are described. We shall need this theorem for future application in the theory of invariant differential operators for projective contact geometries.

In the fourth chapter the projective contact geometries are studied. We present a theorem in which the bijective relationship between contact projections and quotient connection is described. We have also derived some transformation formulas for contact projections and quotient connections. Further, we have introduced the projective contact sphere as a homogeneous model of projective contact geometry. This introduction can be found in the book of Čap, Slovák [7]. The last part of this chapter involves a computation of Hasse and BGG diagram for the contact graded symplectic algebra of general rank and for a general dominant integral weight. The mentioned diagrams for the real form of this algebra are also presented.

We study the Hasse and BGG diagrams for contact graded orthogonal algebras of a general odd rank in the fifth chapter. For this type of grading the saturated sets are computed and the prescription for weights in the BGG diagram is given.

In the sixth chapter, the theory of first order invariant differential operators for parabolic geometries is introduced. We have generalized this theory to the case of some infinite-dimensional standard cyclic modules. The main ingredient needed for this generalization is the multiplicity freeness of the tensor product of such a standard cyclic module with the defining representation.

Contents

Chapter 2

Lie algebras and their representations, symplectic Clifford algebras and parabolic geometries

2.1 $|k|$-graded Lie algebras

In this section we summarize some basic concepts of real or complex $|k|$-graded semisimple Lie algebras, parabolic subalgebras, contact gradings and some groups associated to a $|k|$-grading. More information on this topic and related geometrical ideas (prolongations of systems of differential equations) can be found in Yamaguchi [31].

At first let us recall some basic facts on complex and real simple Lie algebras. It is well known that the complex simple Lie algebras are classified by their Dynkin diagrams. We denote the Dynkin diagram of a simple complex Lie algebra $\mathfrak{g}$ by X_l if the rank of $\mathfrak{g}$ is l. In this case $X_l \in \{A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2\}$.

In the real case, there is a similar result for real forms of complex simple Lie algebras. A real Lie algebra $\mathfrak{g}$ is simple if and only if it is a real form of a complex simple Lie algebra $\mathfrak{f}$, i.e., $\mathfrak{f} = \mathfrak{g} \otimes \mathbb{C}$, or it is a complex simple Lie algebra considered as a real Lie algebra (we simply forget the multiplication by complex numbers). The real forms (in some books only the noncompact real forms) of complex simple Lie algebras are classified by the so called *Satake* (or *Vogan*) *diagrams.* Satake diagrams look like the Dynkin diagrams, but some nodes of the Satake diagram are black and some white nodes of this diagram are joined by an arrow, see Šilhan [28] or Knapp [20], pp. 339. We denote the Satake diagram of a (noncompact) real form $\mathfrak{g}$ by S_l if the rank of $\mathfrak{g}$ is l. The arrows joining some white nodes of a Satake diagram induce a map of the system of simple roots Δ. This map will be called symmetry and will be denoted by $s\nu$, see Šilhan [28]. We define $s\nu(\alpha_i) = \alpha_j$ if and only if $\alpha_i \in \Delta$ is joined by an arrow with $\alpha_j \in \Delta$ (α_i, α_j are white nodes).

Definition 2.1.1. A $|k|$-*graded Lie algebra* is a complex or real Lie algebra equipped with the vector space decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for $i, j \in \{-k, \ldots, k\}$[1] and such that the *negative part* $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$ as a Lie algebra. We call such a grading $|k|$-*grading.*

Let $\mathfrak{p}$ denote the subalgebra $\mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ and $\mathfrak{p}_+$ denote the subalgebra $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ called the *positive part* of a $|k|$-graded Lie algebra. In the case $\mathfrak{g}$ is a semisimple Lie algebra, the subalgebra $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. We will call this parabolic subalgebra *associated parabolic subalgebra.*[2] From the definition 2.1.1 it follows that $\mathfrak{p}_+$ is a nilpotent Lie algebra. To any $|k|$-grading of a simple Lie algebra, we can associate a unique element $E \in \mathfrak{g}$ such that $[E, X] = jX$ for each $X \in \mathfrak{g}_j$ for $j = -k, \ldots, k$. For proof see the book Čap, Slovák [7]. We call such an element the *grading element.* It is also well known that $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$, thus we have a decomposition $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}_0$ and $\mathfrak{g}_0^{ss} := [\mathfrak{g}_0, \mathfrak{g}_0]$ is the semisimple part of $\mathfrak{g}_0$, see the book Čap, Slovák [7] for details.

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h}$ any Cartan subalgebra of $\mathfrak{g}$ and Φ the system of roots for $(\mathfrak{g}, \mathfrak{h})$. Chose a system Φ^+ of positive roots. To the choice $(\mathfrak{h}, \Phi^+)$, we can associate the so called *standard Borel subalgebra*, i.e., certain maximal solvable subalgebra of $\mathfrak{g}$. The standard Borel subalgebra $\mathfrak{b}$ is defined by

$$\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+,$$

[1] for $|i| > k$, $\mathfrak{g}_i = \{0\}$ is to be understood

[2] associated to the $|k|$-graded Lie algebra $\mathfrak{g}$

where $\mathfrak{n}_+ := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$.

Definition 2.1.2 (Standard parabolic subalgebra). Let $\mathfrak{g}$ be a simple Lie algebra. We call any subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ *standard parabolic subalgebra* with respect to $(\mathfrak{h}, \Phi^+)$ if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and a Φ^+ is a system of positive roots such that $\mathfrak{p} \supseteq \mathfrak{b}$, where $\mathfrak{b}$ is the standard Borel subalgebra with respect to $(\mathfrak{h}, \Phi^+)$.

Gradings of complex simple Lie algebra. Let $\mathfrak{g}$ be a complex simple Lie algebra. For a given tuple $(\mathfrak{h}, \Phi^+)$ of a Cartan subalgebra and a choice of the system of positive roots, one can show that there is a bijective correspondence between the set of all standard parabolic subalgebras $\mathfrak{p} \subseteq \mathfrak{g}$ with respect to $(\mathfrak{h}, \Phi^+)$ and subsets Σ of the system of simple roots $\Delta (\subseteq \Phi^+)$.

This correspondence is given as follows. Let $(\mathfrak{h}, \Phi^+)$ be a choice of a Cartan subalgebra and a system of positive roots. From the structure theory of Lie algebras it follows that any standard parabolic $\mathfrak{p}$ subalgebra with respect to $(\mathfrak{h}, \Phi^+)$ is of the form $\mathfrak{p} = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_{-\alpha}$ for a subset $\Psi \subseteq \Phi^+$, see the book Čap, Slovák [7]. To such a parabolic subalgebra, we associate a subset Σ of the system of simple roots Δ given by $\Sigma = \{\alpha \in \Delta; \mathfrak{g}_{-\alpha} \nsubseteq \mathfrak{p}\}$. Conversely, for a choice $(\mathfrak{h}, \Phi^+)$ of a Cartan subalgebra and a system of positive roots, let a subset Σ of the system of simple roots be given. Let $\mathfrak{b}$ be the standard Borel subalgebra with respect to $(\mathfrak{h}, \Phi^+)$. Then we define $\mathfrak{p}$ to be the direct sum $\mathfrak{p} = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_{-\alpha}$, where $\Psi \subseteq \Phi^+$ is defined to consist of those $\alpha \in \Phi^+$ with the property that in the decomposition of α into simple roots no element of Σ occurs in it with a nonzero coefficient.

The next construction defines a $|k|$-grading from a given subset Σ of the system of simple roots. Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Φ^+ a choice of positive roots and Δ the system of positive roots according to $(\mathfrak{h}, \Phi^+)$. Let Σ be a subset of Δ, $\Sigma \subseteq \Delta$. Put

$$\Phi_j^+ := \{\alpha = \sum_{i=1}^{l} n_i \alpha_i \in \Phi^+; \sum_{\alpha_i \in \Sigma} n_i = j\}$$

for $j \geq 0$, where $\Delta =: \{\alpha_1, \dots, \alpha_l\}$. A nonnegative number j is called the

Σ-*height* of a root $\alpha \in \Phi^+$, if $\alpha \in \Phi_j^+$. We can construct

$$\mathfrak{g}_0 := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

$$\mathfrak{g}_j := \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_{-j} := \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_{-\alpha}$$

for $j > 0$. Then it can be shown that $\mathfrak{g} = \oplus_{j=-k}^{k} \mathfrak{g}_j$ is a $|k|$-graded Lie algebra, where k is the largest nonnegative number such that there is a positive root $\alpha \in \Phi^+$ the Σ-height of which equals k. Conversely, for any $|k|$-grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k$ a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a system of positive roots Φ^+ can be chosen such that the associated parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k$ is a standard parabolic subalgebra with respect to $(\mathfrak{h}, \Phi^+)$. The subset Σ of the system of simple roots Δ corresponding to $\mathfrak{p}$ is such that the $|k|$-grading is given by the Σ-height, described above. For details, see the book of Čap, Slovák [7]. The *depth* k of this $|k|$-grading can be read off the expansion of the highest root θ (i.e., the highest weight of the adjoint representation of $\mathfrak{g}$). Let $\theta = \sum_{i=1}^{l} n_i(\theta)\alpha_i$ then $k = \sum_{\alpha_i \in \Sigma} n_i(\theta)$, see Yamaguchi [31].

For a subset $\Sigma \subseteq \Delta$ we denote by (X_l, Σ) the corresponding $|k|$-grading or the standard parabolic Lie algebra associated to Σ, if X_l is the Dynkin diagram of $\mathfrak{g}$. Sometimes we will write the symbol (X_l, Σ) as the so called *crossed Dynkin diagram* which is the Dynkin diagram X_l with nodes in Σ being crossed.

Example: Let $\mathfrak{g} = A_2 = \mathfrak{sl}(3, \mathbb{C})$. Let us denote the algebra of diagonal matrices in $\mathfrak{g}$ by $\mathfrak{h}$. This algebra is a Cartan subalgebra of $\mathfrak{g}$. Let $\{\varepsilon_i\}_{i=1}^{3}$ be elements of $\mathfrak{h}^*$ defined by their action on an element $d = \text{diag}\ [a_1, a_2, a_3]$ for $a_i \in mathbbC, i = 1, \ldots, 3$ by the formula $\epsilon_i(d) := d_i$ for $i = 1, \ldots, 3$. We choose a system of positive roots for $(\mathfrak{g}, \mathfrak{h})$ as follows $\Phi^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\}$. Then $\Delta = \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \alpha_2 := \varepsilon_2 - \varepsilon_3\}$ is the system of simple roots for $(\mathfrak{g}, \mathfrak{h})$ and the choice of Φ^+. Let us put $\Sigma = \{\alpha_1\} \subseteq \Delta$. We construct the grading $(A_2, \{\alpha_1\})$. The highest root of A_2 is $\theta = \varepsilon_1 - \varepsilon_3$ (see Knapp [20], pp. 509). The expansion $\theta = \alpha_1 + \alpha_2$ shows that $k = 1$. In this case $\Phi_1^+ = \{\alpha_1, \alpha_1 + \alpha_2\}$, $\Phi_0^+ = \{\alpha_2\}$. Thus dim $\mathfrak{g}_0 = 2 + 1 + 1 = 4$, and dim $\mathfrak{g}_{-1} =$ dim $\mathfrak{g}_1 = 2$. In terms of matrices this grading is given as follows

$$A = \left(\begin{array}{c|c} \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right),$$

where the the diagonal boxes are of size one and two. The associated parabolic subalgebra consists of matrices of type A_2 where the $\mathfrak{g}_{-1}$ block is zero, i.e.,

$$\mathfrak{p} = \left\{ A = \left(\begin{array}{c|cc} * & * & * \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right) \in \mathfrak{sl}(3, \mathbb{C}) \right\}.$$

Gradings of a real form of a simple complex Lie algebra. In the real case, there is a similar result as in the complex one. We will mention it only briefly, for details see Yamaguchi [31]. Let $\mathfrak{g}$ be a real form of a complex simple Lie algebra of rank l. Let S_l be the Satake diagram of $\mathfrak{g}$. We call any subset Σ of the set of white nodes of the Satake diagram S_l *admissible subset,* if it is invariant under the symmetry $s\nu$ of the Satake diagram S_l. In Yamaguchi [31], a construction of a grading corresponding to an admissible subset Σ is written. Moreover, a bijective correspondence (in the similar sense as in the complex case) between the set of all admissible subsets Σ and gradings of the real form $\mathfrak{g}$ is established.

We will denote this grading or the corresponding standard parabolic subalgebra by (S_l, Σ) if the Satake diagram of $\mathfrak{g}$ is S_l. Sometimes we will write the symbol (S_l, Σ) as the so called *crossed Satake diagram* which is the Satake diagram S_l with nodes in Σ being crossed.

Contact gradings. In sections 4.4 and 5.1, we will need the notion of contact grading.

Definition 2.1.3. Let $\mathfrak{g}$ be a real or complex $|k|$-graded semisimple Lie algebra. We call this grading *contact* if it is a depth two grading, i.e.,

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

satisfying the following two properties

(1) $[,] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is nondegenerate and

(2) $\dim \mathfrak{g}_{-2} = 1$.

In Yamaguchi [31] there is a classification of contact gradings for each complex simple Lie algebra and each real form of a complex simple Lie algebra. We only mention that for each complex simple Lie algebra there is exactly one contact grading. For each real form there is exactly one contact grading, except of the following real forms: $AI(l=1)$, AII, BII, CII, DII, EIV or FII (in the list of table VI in Helgason [17]).

Groups associated to a $|k|$-grading. Now, let us define some groups associated to a $|k|$-grading of a semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k$ be a $|k|$-graded complex semisimple Lie algebra. Let G be a complex Lie group such that its Lie algebra is $\mathfrak{g}$. We denote the subgroup of G of elements $g \in G$ for which $Ad(g)\mathfrak{g}_i \subseteq \mathfrak{g}_i$ for $i = -k, \ldots, k$ by G_0. To the $|k|$-grading of $\mathfrak{g}$ we associate the filtration $\mathfrak{g} = \mathcal{F}_{-k}(\mathfrak{g}) \subseteq \ldots \subseteq \mathcal{F}_k(\mathfrak{g})$ of $\mathfrak{g}$ defined by $\mathcal{F}_i(\mathfrak{g}) = \mathfrak{g}_i \oplus \ldots \oplus \mathfrak{g}_k$, $i = -k, \ldots, k$, and call it the *associated filtration.* We denote the subgroup of G consisting of elements $g \in G$ such that their Ad-action preserves the associated filtration, i.e., $Ad(g)(\mathfrak{g}_i) \subseteq \mathcal{F}_i(\mathfrak{g})$ for $i = -k, \ldots, k$, by P. It can be shown that this group is a parabolic subgroup of G. We will call it an *associated parabolic group* to the $|k|$-graded Lie algebra $\mathfrak{g}$ and the group G.

Theorem 2.1.1. *Let $\mathfrak{g}$ be $|k|$-graded simple Lie algebra. Then the associated parabolic group P has Lie algebra $\mathfrak{p}$, and G_0 has Lie algebra $\mathfrak{g}_0$. Let $g \in P$ be any element. Then there exist unique elements $g_0 \in G_0$ and $X_i \in \mathfrak{g}_i$ for $i = 1, \ldots, k$, such that*

$$g = g_0 \exp X_1 \ldots \exp X_k.$$

Proof. See Čap, Schichl [6].□

Remark 2.1.1. If $\mathbb{V}, \mathbb{W}$ are P-modules and $\Phi : \mathbb{V} \to \mathbb{W}$ is a linear mapping which is G_0-equivariant and infinitesimally $\mathfrak{p}_+$-equivariant then it is a P-module homomorphism. This remark is an easy consequence of the previous theorem, for comments see Čap, Slovák, Souček [10].

2.2 Representation theory of semisimple Lie algebras

There is a well known theorem which holds for all complex simple Lie algebras $\mathfrak{g}$ about their complex representation. It says that the set of all complex irreducible finite dimensional representations of $\mathfrak{g}$ is in a bijective correspondence with certain semigroup in the dual $\mathfrak{h}^*$ of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. In this section we present a generalization of this theorem and its extension to the real case (i.e., the case in which $\mathfrak{g}$ is a real form of a complex simple Lie algebra) together with introducing some preparatory notions.

2.2.1 Standard cyclic modules

The main source of this subsection is the book of Humphrey [18]. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, B its Killing form, $\mathfrak{h}$ its Cartan subalgebra, Φ the system of roots with respect to $(\mathfrak{g}, \mathfrak{h})$, Φ^+ a system of positive roots and Δ the system of simple ones with respect to the previous choices of $\mathfrak{h}$ and Φ^+. Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and δ the half sum of all positive roots, i.e.,

$$\delta = \frac{1}{2} \sum_{\phi \in \Phi^+} \phi \in \mathfrak{h}^*.$$

Let $\mathbb{V}$ be a complex finite or infinite dimensional $\mathfrak{g}$-module; the action of $\mathfrak{g}$ on $\mathbb{V}$ is denoted by a dot. For an element $\lambda \in \mathfrak{h}^*$ let us denote by $\mathbb{V}(\lambda)$ the vector subspace

$$\mathbb{V}(\lambda) = \{v \in V; H.v = \lambda(H)v, H \in \mathfrak{h}\}.$$

(This subspace is called *weight space* if $\mathbb{V}(\lambda) \neq \{0\}$ and in such case λ is called *weight* of $\mathfrak{h}$ on $\mathbb{V}$.) Due to the notation introduced before, the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$ can be seen as weight spaces of the adjoint representation of $\mathfrak{g}$.

A *maximal vector (of weight λ)* is a nonzero vector $v^+ \in \mathbb{V}(\lambda)$ such that it is killed by $\mathfrak{g}_\alpha$ for all $\alpha \in \Delta$, i.e., $X.v^+ = 0$ for all $X \in \mathfrak{g}_\alpha$, $\alpha \in \Delta$. (In case in which $\dim \mathbb{V} = \infty$, the maximal vector need not to exist.) A $\mathfrak{g}$-module $\mathbb{V}$ is called *standard cyclic* if it is generated by its maximal vector, i.e., $\mathbb{V} = \mathfrak{U}(\mathfrak{g}).v^+$ for a maximal vector $v^+ \in \mathbb{V}(\lambda)$, and we call λ the *highest weight* of $\mathbb{V}$.

Similar theorems hold for standard cyclic modules as for the finite dimensional one. For example, the maximal vector is unique up to a scalar multiple and therefore the highest weight is uniquely defined.

We present the following existence theorem.

Theorem 2.2.1. *Let $\lambda \in \mathfrak{h}^*$. Then there exist an irreducible standard cyclic module of the highest weight λ, denoted by $L(\lambda)$.*

Proof. See Humphrey [18], pp. 110. □

Remark 2.2.1. This irreducible module of the highest weight λ is constructed as a quotient of a standard cyclic module of the same highest weight by its maximal submodule.

We shall also need the notion of Casimir element and some formula of its action on a standard cyclic module. Let $\{h_\alpha\}_{\alpha\in\Delta}$ be the standard basis of the Cartan subalgebra $\mathfrak{h}$ and $\{k_\alpha\}_{\alpha\in\Delta}$ the dual basis with respect to the restriction of the Killing form B to the Cartan subalgebra $\mathfrak{h}$. Chose a nonzero element x_α of $\mathfrak{g}_\alpha$ and an element $z_\alpha \in \mathfrak{g}_{-\alpha}$ such that $B(x_\alpha, z_\alpha) = 1$ for each $\alpha \in \Phi$. The element c of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ defined by the equation

$$c := \sum_{\alpha\in\Delta} h_\alpha k_\alpha + \sum_{\alpha\in\Phi} x_\alpha z_\alpha$$

is called the *universal Casimir element* of $\mathfrak{g}$.

Theorem 2.2.2. *The action of the universal Casimir element on a standard cyclic module of the highest weight λ is by a scalar*

$$(\lambda+\delta, \lambda+\delta) - (\delta, \delta) = (\lambda + 2\delta, \lambda).$$

Proof. See Humphrey [18], pp. 143. □

2.2.2 Real representations of real forms of simple Lie algebras

Let $\mathbb{V}$ be a real vector space. We call an automorphism J of $\mathbb{V}$ *complex structure* if $J^2 = -id$. For a complex vector space $\mathbb{W}$, a *real (quaternionic) structure* is an antiautomorphism of $\mathbb{W}$, such that $J^2 = id$ ($J^2 = -id$). A real vector space equipped with a complex structure can be understood as a complex vector space, the multiplication by the imaginary unit corresponds to the application of the complex structure J (the complex dimension of such complex vector space is half of the real dimension of the real space).

Denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}^{\mathbb{C}}$ and the complexification of $\mathbb{V}$ by $\mathbb{V}^{\mathbb{C}}$. The *complexification* $(\rho^{\mathbb{C}}, \mathbb{V}^{\mathbb{C}})$ *of the representation* $(\rho, \mathbb{V})$ is a complex linear $\mathfrak{g}$-module homomorphism of $\mathfrak{g}^{\mathbb{C}}$ into $\mathrm{End}(\mathbb{V}^{\mathbb{C}})$ defined by $\rho^{\mathbb{C}}(X+iY)(v+iw) = \rho(X)v - \rho(Y)w + i(\rho(Y)v + \rho(X)w)$ for $X, Y \in \mathfrak{g}$ and $v, w \in \mathbb{V}$.

Let $(\rho, \mathbb{V})$ be a real representation of a real Lie algebra $\mathfrak{g}$ on a real vector space $\mathbb{V}$. We define the so called representations of real, quaternionic and complex types.

(1) $(\rho, \mathbb{V})$ is called of *quaternionic type*, if there exists a complex structure and a quaternionic structure commuting with ρ (in this case $\mathbb{V}$ is viewed as a complex vector space, the multiplication by i corresponds to the application of the complex structure);

(2) $(\rho, \mathbb{V})$ is called of *complex type* if there exists a complex structure commuting with ρ and $(\rho, \mathbb{V})$ is not of quaternionic type;

(3) $(\rho, \mathbb{V})$ is called of *real type* if there is no complex structure commuting with ρ.

To a complex representation $(\rho, \mathbb{V})$ on a finite dimensional complex vector space $\mathbb{V}$ of a complex Lie algebra $\mathfrak{f}$, we can associate the so called *conjugated representation* $\overline{\rho} : \mathfrak{f} \to \mathrm{End}(\mathbb{V})$ given in the following way. If $[\rho(X)]$ is a matrix of the endomorphism $\rho(X)$ $(X \in \mathfrak{f})$ of the vector space $\mathbb{V}$ with respect to some basis of $\mathbb{V}$ then the matrix $[\overline{\rho}(X)]$ of the endomorphism $\overline{\rho}(X)$ equals $\overline{[\rho(X)]}$, i.e., the matrix entries of $[\overline{\rho}(X)]$ are complex conjugate of the entries of $[\rho(X)]$. We call a complex representation *self-dual* if it is isomorphic to its complex conjugate.

Let us suppose that $(\rho, \mathbb{V})$ is an irreducible real representation of a real Lie algebra $\mathfrak{g}$. The complexification of this representation depends on the type of this real representation in the following way. If $(\rho, \mathbb{V})$ is of complex (quaternionic) type then $(\rho^{\mathbb{C}}, \mathbb{V}^{\mathbb{C}})$ is reducible and $\rho^{\mathbb{C}} = \pi \oplus \overline{\pi}$ where π is a complex representations of $\mathfrak{g}^{\mathbb{C}}$ and $\pi \nsim \overline{\pi}$ $(\pi \sim \overline{\pi})$. If $(\rho, \mathbb{V})$ is of real type then its complexification is irreducible and $\rho^{\mathbb{C}} \simeq \overline{\rho^{\mathbb{C}}}$. See Zhida, Dagan [32].

If $\mathfrak{g}$ is a real form of a complex simple Lie algebra $\mathfrak{f} = \mathfrak{g} \otimes \mathbb{C}$ then we can find out whether a finite dimensional irreducible complex representation of $\mathfrak{f}$ with the highest weight λ is a complexification of a real representation of the given real form $\mathfrak{g}$ of $\mathfrak{f}$ (i.e., a complexification of a representation of real type) or it is a proper component of the complexification of a real representation of $\mathfrak{g}$ (i.e., necessary of complex or quaternionic type). Moreover, in the latter case we can distinguish whether it is a component of the complexification of a representation of quaternionic or of complex type. Our procedure will go in the opposite direction to the mentioned one as follows. At first we give a tool (symmetry of the Satake diagram) how to distinguish the representations of real or quaternionic type from the representations of complex type, at second we introduce a notion (Maltsev height) which enables to distinguish the representations of quaternionic type from the representations of real type.

Let $(\rho, \mathbb{V})$ be a finite dimensional irreducible real representation of a real form $\mathfrak{g}$ of a complex simple Lie algebra on a real vector space $\mathbb{V}$. Let λ be the highest weight of a component of the complexification $(\rho^{\mathbb{C}}, \mathbb{V}^{\mathbb{C}})$ of the representation $(\rho, \mathbb{V})$. Writing the weight λ above the Satake diagram of the real form $\mathfrak{g}$ we obtain the so called *weighted Satake diagram.* Then it is well known that this component is self-dual (i.e., it is the complexification or a direct summand of the complexification of a representation of real or quaternionic type, respectively) if and only if $s\nu(\lambda) = \lambda$, where $s\nu$ is the symmetry of the Satake diagram of $\mathfrak{g}$. Thus we know how to distinguish the representations of real and quaternionic type from the complex type ones; by the symmetry $s\nu$.

Now,we would like to know how to distinguish the representations of real type from the representations of quaternionic one. To each weighted Satake diagram we can associate the so called *Maltsev height* $m(\lambda)$ (we omit the dependence of the Maltsev height on the Satake diagram in its symbolic notation) which is a certain integer, see Goodman, Wallach [14], chapter 5.1.7, Zhida, Dagan [32] or Šilhan [28]. (In the work of Šilhan [28], the so called index is used instead of the Maltsev height; it equals $(-1)^{m(\lambda)}$.)

Summing-up the previous discussion, we get the following

Theorem 2.2.3. *Let $\mathfrak{g}$ be a real form of a complex simple Lie algebra $\mathfrak{f} = \mathfrak{g} \otimes \mathbb{C}$ and $(\rho, \mathbb{V})$ be a finite dimensional irreducible real representation of $\mathfrak{g}$. Let $(\rho^{\mathbb{C}}, \mathbb{V}^{\mathbb{C}})$ be a complexification of $(\rho, \mathbb{V})$. Denote by λ the highest weight of a component of this complexification. Then $(\rho, \mathbb{V})$ is of*

(1) real type, if and only if $s\nu(\lambda) = \lambda$ and $m(\lambda)$ is even;

(2) quaternionic type, if and only if $s\nu(\lambda) = \lambda$ and $m(\lambda)$ is odd;

(3) complex type, if and only if $s\nu(\lambda) \neq \lambda$.

Proof. See Zhida, Dagan [32]. □

2.3 Lie algebra cohomology, Hasse and Bernstein-Gelfand-Gelfand diagrams

In this section we introduce some basic algebraic and combinatorial structures related to $|k|$-graded Lie algebras and their representations, like Lie algebra cohomology, saturated sets, Hasse and Bernstein-Gelfand-Gelfand (BGG) diagrams.

2.3.1 Lie algebra cohomology

We begin with the algebraic notions. Let $\mathfrak{g}$ be a $|k|$-graded complex semisimple Lie algebra and $\mathfrak{g}_-$ its negative part. Suppose there is given a representation $(\rho, \mathbb{V})$ of $\mathfrak{g}$. Then one can define the so called n^{th} *chain group* $C^n(\mathfrak{g}_-, \mathbb{V}) := \bigwedge^n \mathfrak{g}_-^* \otimes \mathbb{V}$ for each $n \in \mathbb{N}_0$. For this chain groups we can define a *(Lie algebra) differential* $\partial : C^n(\mathfrak{g}_-, \mathbb{V}) \to C^{n+1}(\mathfrak{g}_-, \mathbb{V})$ acting between them by the formula

$$\partial\omega(X_1, \dots, X_{n+1}) := \sum_{j=1}^{n+1} \rho(X_j)\omega(X_1, \dots, \widehat{X_j}, \dots, X_{n+1}) +$$
$$+ \sum_{1 \leq r < s \leq n+1} (-1)^{r+s}\omega([X_r, X_s], X_1, \dots, \widehat{X_r}, \dots, \widehat{X_s}, \dots, X_{n+1}),$$

where $X_i \in \mathfrak{g}_-$, $i = 1, \dots, n+1$ and $\omega \in C^n(\mathfrak{g}_-, \mathbb{V})$. It can be checked that ∂ is a boundary operator, i.e., $\partial \circ \partial = 0$, thus $(C^\bullet(\mathfrak{g}_-, \mathbb{V}), \partial)$ is a chain complex for which one can define its cohomology, so called n^{th} *Lie algebra cohomology* with coefficients in $\mathbb{V}$

$$H^n(\mathfrak{g}_-,\mathbb{V}) := Z^n(\mathfrak{g}_-,\mathbb{V})/B^n(\mathfrak{g}_-,\mathbb{V})$$

where $Z^n(\mathfrak{g}_-,\mathbb{V}) := \mathrm{Ker}(\partial : C^n(\mathfrak{g}_-,\mathbb{V}) \to C^{n+1}(\mathfrak{g}_-,\mathbb{V}))$ and $B^n(\mathfrak{g}_-,\mathbb{V}) := \partial(C^{n-1}(\mathfrak{g}_-,\mathbb{V}))$.

Let G be a Lie group with Lie algebra is $\mathfrak{g}$. It is well known that the Lie algebra codifferential ∂^* is a P-module homomorphism where P is the associated parabolic subgroup to the $|k|$-graded Lie algebra $\mathfrak{g}$ and the group G. This result is proved in the paper of Čap, Schichl [6] for example.

2.3.2 Saturated sets, Hasse and BGG diagrams

In this subsection we define some combinatorial structure associated to $|k|$-graded simple Lie algebras and their finite dimensional representations.

Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra and $\mathfrak{p}$ the associated parabolic subalgebra, $\mathfrak{p}_+$ the positive part and $\mathfrak{g}_-$ the negative part of $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{h}$ and a system of positive roots Φ^+ of $(\mathfrak{g},\mathfrak{h})$ such that $\mathfrak{p}$ is a standard parabolic algebra with respect to $(\mathfrak{h},\Phi^+)$, for details see 2.1. For the pair $(\mathfrak{g},\mathfrak{h})$, the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ is defined. A subset Q of the system of positive roots of the Lie algebra $\mathfrak{g}$, $Q \subseteq \Phi^+$ is called *saturated set* if there is an element w of the Weyl group $\mathcal{W}$ such that $Q = Q(w)$, where $Q(w)$ is the set of all positive roots which are mapped into the set of negative roots by the element w^{-1}, i.e., $Q(w) := \{\alpha \in \Phi^+; w^{-1}\alpha \in -\Phi^+\}$.

Example: Let $\mathfrak{g} = A_2 = \mathfrak{sl}(3,\mathbb{C})$ and $\{\varepsilon_i\}_{i=1}^3$ be the canonical basis of $(\mathbb{C}^3)^*$. The set of positive roots looks like $\Phi^+ = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3\, \varepsilon_1-\varepsilon_3\}$. Let us denote $w_1 = \sigma_{\varepsilon_1-\varepsilon_2}$, $w_2 = \sigma_{\varepsilon_2-\varepsilon_3}$, $w_3 = \sigma_{\varepsilon_1-\varepsilon_3}$ the reflections in the corresponding roots (i.e., with respect to the planes perpendicular to the corresponding roots). Then $Q(id) = \emptyset$, $Q(w_1) = \{\varepsilon_1-\varepsilon_2\}$, $Q(w_3) = \{\varepsilon_1-\varepsilon_3\}$, $Q(w_1w_2) = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3\}$, $Q(w_1w_3) = \{\varepsilon_2-\varepsilon_3, \varepsilon_1-\varepsilon_3\}$, $Q(w_2) = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \varepsilon_1-\varepsilon_3\}$.

A subset A of the system of positive roots is called *saturated set for the pair* $(\mathfrak{g},\mathfrak{p})$, if and only if it is a saturated set and $A \subseteq \Phi_{\mathfrak{p}_+}$ where $\Phi_{\mathfrak{p}_+}$ is the set of all positive roots for which the corresponding root spaces belong to $\mathfrak{p}_+$. The following theorem is well known (see Goodman, Wallach [14], section 7.3.2).

Theorem 2.3.1. *A subset $A \subseteq \Phi_{\mathfrak{p}_+}$ is a saturated set for the pair $(\mathfrak{g}, \mathfrak{p})$ if and only if the following two conditions*

(R1) if $\alpha, \beta \in A$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in A$ and

(R2) if $\gamma \in A$ and $\gamma = \alpha + \beta$ where $\alpha, \beta \in \Phi$ then $\alpha \in A$ or $\beta \in A$

hold.

Proof. See Goodman, Wallach [14], pp. 328. □

Hasse diagram of the pair $(\mathfrak{g}, \mathfrak{p})$ is a labelled oriented graph defined as follows. Vertices are saturated sets for $(\mathfrak{g}, \mathfrak{p})$ and there is a labelled oriented arrow $Q(w_1) \xrightarrow{\alpha} Q(w_2)$ $(w_1, w_2 \in \mathcal{W})$ if and only if there is an element $\alpha \in \Phi_{\mathfrak{p}_+}$ such that $w_2 = \sigma_\alpha w_1$ and $|w_2| = |w_1| + 1$, where σ_α is the reflection in the root α and $|w|$ for a $w \in \mathcal{W}$ is the reduced length of the element w. (The *reduced length* of an element of the Weyl group $\mathcal{W}$ is the smallest number of members of the decomposition of this element into simple reflections.)

The following lemma gives an information about arrows of Hasse diagram.

Lemma 2.3.1 (Arrows in Hasse diagram). *There is an oriented labelled arrow $Q \xrightarrow{\alpha} Q'$ in the Hasse diagram for $(\mathfrak{g}, \mathfrak{p})$ if and only if there is a positive integer $k \in \mathbb{N}$ such that $|Q'| - |Q| = k\alpha$, where*

$$|Q| := \sum_{\beta \in Q} \beta$$

for an element $\alpha \in \Phi_{\mathfrak{p}_+}$ and saturated sets Q, Q' for the pair $(\mathfrak{g}, \mathfrak{p})$.

Proof. See Krump, Souček [21].□

There is a famous connection between the cohomology groups and Hasse diagrams described by the Kostant theorem. The explicit version of Kostant theorem is as follows. (It does not use the notion of Hasse diagram.) The weights of $\mathfrak{h}$ on $\mathfrak{g}_-^*$ are $-\alpha$ with multiplicity one, where $\alpha \in \Phi^+$. We choose a nonzero element $\omega_{-\alpha}$ in the weight space $\mathfrak{g}_-^*(-\alpha)$. For each subset $Q = \{\beta_1, \dots, \beta_p\} \subseteq \Phi^+$ we set $\omega_Q = \omega_{-\beta_1} \wedge \ldots \wedge \omega_{-\beta_p} \in \bigwedge^p \mathfrak{g}_-^*$. Let an irreducible finite dimensional $\mathfrak{g}$-module $\mathbb{V}$ be given. We denote by $v_\lambda \in \mathbb{V}$ a nonzero vector in the weight space $\mathbb{V}(\lambda)$.

Theorem 2.3.2 (Kostant). *If $\mathbb{V}$ is an irreducible finite-dimensional complex $\mathfrak{g}$-module with the highest weight λ then*

$$H^p(\mathfrak{g}_-, \mathbb{V}) = \bigoplus_{s\in\mathcal{W}, |s|=p} H^p_s, \quad p = 0, \dots, \dim \mathfrak{g}_-$$

is a decomposition into irreducible $\mathfrak{p}$-modules, where H^p_s is a module with the highest weight vector $\omega_{-Q(s^{-1})} \otimes v_{s(\lambda)}$ for $p = 0, \dots, \dim \mathfrak{g}_-$ and $s \in \mathcal{W}, |s| = p$.

Proof. See Goodman, Wallach [14], pp. 332. □

Using the Kostant theorem, we obtain an explicit description of the 1-1 correspondence between the vertices of the Hasse graph and Lie algebra cohomologies. Let $\mathbb{V}$ be the irreducible $\mathfrak{g}$-module with the highest weight λ. Denote by $\mathbb{H}^j(\lambda)$ the Lie algebra cohomology $H^j(\mathfrak{g}_-, \mathbb{V})$. Let us consider the following decomposition into irreducible $\mathfrak{p}$-modules:

$$\mathbb{H}^j(\lambda) = \oplus_{k=1}^{m_j} \mathbb{H}^j_k(\lambda), \quad j = 0, \dots, \dim \mathfrak{g}_-.$$

Then there is a 1-1 correspondence between the modules $\mathbb{H}^j_1(\lambda)$,..., $\mathbb{H}^j_{m_j}(\lambda)$ and the vertices $Q(w^j_1), \dots, Q(w^j_{m_j})$ of the Hasse diagram for $(\mathfrak{g}, \mathfrak{p})$ with $|w^j_r| = j$, $r = 1, \dots, m_j$, $j = 0, \dots, \dim \mathfrak{g}_-$.

Now, we are able to write down the definition of the Bernstein-Gelfand-Gelfand (BGG) diagram. Let $\mathfrak{g}$ and $\mathfrak{p}$ as defined above. Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight for $\mathfrak{g}$. [3] Let us denote the highest weight of $\mathbb{H}^j_k(\mathfrak{g}, \mathbb{V}^*)^*$ for $j = 0, \dots, \dim \mathfrak{g}_-$, $k = 1, \dots, m_j$ by λ^j_k. *Bernstein-Gelfand-Gelfand (BGG) diagram* for the triple $(\mathfrak{g}, \mathfrak{p}, \lambda)$ is an oriented labelled graph. Its vertices are the weights (not necessary dominant for $\mathfrak{g}$) λ^j_k mentioned above and there is a labelled oriented arrow $\lambda^j_r \xrightarrow{\alpha} \lambda^{j+1}_s$ if and only if $\alpha \in \Phi_{\mathfrak{p}_+}$ is a positive root in the parabolic part of the root system of $\mathfrak{g}$ and

$$\lambda^{j+1}_s = \lambda^j_r - 2\frac{(\lambda^j_r + \delta, \alpha)}{(\alpha, \alpha)}\alpha,$$

where δ is the half-sum of all positive roots of $\mathfrak{g}$.

Remark 2.3.1. From practical reasons, we will compute the BGG diagrams using the "non-shifted" version of the formula written above, i.e., we omit the weight

[3] with respect to a Cartan subalgebra $\mathfrak{h}$ and a choice of the set of positive roots Φ^+ for which $\mathfrak{p}$ is a standard parabolic with respect to $(\mathfrak{h}, ^+)$.

δ in our computations. In the fourth and fifth chapter, some prescription for vertices of BGG diagrams will be given. These computations use the non-shifted version. To get the shifted version, one can use the mentioned prescription for $\lambda + \delta$ and after this application the weight δ is to be subtracted.

Remark 2.3.2. The BGG diagram for $(\mathfrak{g}, \mathfrak{p}, \lambda)$ for any $\mathfrak{g}$-dominant integral weight λ and the Hasse diagram are isomorphic as labelled oriented graphs (see Krump, Souček [21]). For a more general setting, see Lepowsky [26].

Remark 2.3.3. The oriented arrow in the BGG diagram for $(\mathfrak{g}, \mathfrak{p}, \lambda)$ represents a unique differential operator up to a scalar multiple acting between sections of homogeneous vector bundles $H^j_k(\lambda)$ (associated to the principle P-bundle $G \to G/P$ via the P-modules $\mathbb{H}^j_k(\lambda)$), see the notion of BGG-sequence in Čap, Slovák, Souček [10] and section 2.6.

2.4 Symplectic Clifford algebra, metaplectic group and Segal-Shale-Weil representation

In this section we review some well known concepts like symplectic Clifford algebra, Weyl algebra, Heisenberg algebra, Heisenberg group and metaplectic group. We also introduce the Segal-Shale-Weil representation of the metaplectic group.

2.4.1 Symplectic Clifford algebra

Consider a *linear symplectic structure* $(\mathbb{V}, \omega)$, i.e., $\mathbb{V}$ is a finite dimensional vector space and ω is an antisymmetric nondegenerate bilinear form on $\mathbb{V}$. A linear map $F : \mathbb{V} \to \mathbb{V}'$ is called a *symplectomorphism* of $(\mathbb{V}, \omega)$ and $(\mathbb{V}', \omega')$ if $F^*\omega' = \omega$. Two linear symplectic structures are called *equivalent* if there is a symplectomorphism of these structures which is a linear isomorphism. It is well known that there is a unique linear symplectic structure up to an equivalence (Darboux theorem for vector spaces). The dimension of the linear symplectic structure is necessarily even. We call a basis $\{q^1, \ldots, q^k, p_1, \ldots, p_k\}$ *symplectic basis* if and only if $\omega(q^j, p_j) = -1$ for $j = 1, \ldots, k$ and other products are zero.

Definition 2.4.1 (Symplectic Clifford algebra). Let $(\mathbb{V}, \omega)$ be a linear symplectic structure. We call a pair (A, i) *symplectic Clifford algebra* if $i : \mathbb{V} \to A$ is a linear map and A is an associative algebra with unit, such that

(1) $i(v).i(w) - i(v).i(w) = -\omega(v, w)$ for all $v, w \in \mathbb{V}$,

(2) *(universal property)* if (A', i') is a pair of a linear map $i' : \mathbb{V} \to A'$ and A' is an associative algebra with unit satisfying the first item then there is a unique algebra homomorphism $\rho : A \to A'$ satisfying $i' = \rho i$.

Remark 2.4.1. Each linear map $i : \mathbb{V} \to A$ satisfying

$$i(v).i(w) - i(v).i(w) = -\omega(v, w)$$

(i.e., the first item of the definition of the symplectic Clifford algebra) is called *symplectic inclusion.* The condition $i' = \rho i$ described in the second item is called *compatibility property* and such a map is called *compatible.*

Theorem 2.4.1 (Uniqueness of symplectic Clifford algebra). *For each linear symplectic structure* $(\mathbb{V}, \omega)$, *there is at most one symplectic Clifford algebra up to an isomorphism. We denote it by* $sCliff(\mathbb{V}, \omega)$.

Proof. The uniqueness of such an algebra is a categorial fact (the second condition is the well known universal property which establishes the uniqueness), but we prove it explicitely. Consider an another symplectic Clifford algebra $(sCliff(\mathbb{V}, \omega)', i')$. Using the universal property for both symplectic Clifford algebras, we obtain an existence of a homomorphisms $\rho : sCliff(\mathbb{V}, \omega) \to sCLiff(\mathbb{V}, \omega)'$ and $\rho' : sCliff(\mathbb{V}, \omega)' \to sCliff(\mathbb{V}, \omega)$, such that $i' = \rho i$ and $i = \rho' i'$. Combining theses two equations we obtain $i' = \rho\rho' i'$ and $i = \rho'\rho i$. Thus $\rho\rho'$ and $\rho'\rho$ are endomorphisms of $sCliff(\mathbb{V}, \omega)'$ and $sCliff(\mathbb{V}, \omega)$ respectively and these endomorphisms satisfy the compatibility property. The mappings $id_{sCliff(\mathbb{V},\omega)}$ and $id_{sCliff(\mathbb{V},\omega)'}$ are compatible mapping too, thus using the universality property, we obtain $\rho'\rho = id_{sCliff(\mathbb{V},\omega)}$ and $\rho\rho' = id_{sCliff(\mathbb{V},\omega)'}$, i.e., $sCliff(\mathbb{V}, \omega)$ and $sCliff(\mathbb{V}, \omega)'$ are isomorphic.□

Theorem 2.4.2 (Existence of symplectic Clifford algebra). *For each linear symplectic structure* $(\mathbb{V}, \omega)$, *there is a symplectic Clifford algebra* $(sCliff(\mathbb{V}, \omega), i)$.

Proof. We define a two sided ideal $I(\mathbb{V},\omega)$ in the tensor algebra $\mathcal{T}(\mathbb{V})$ generated by $v\otimes w - w\otimes v + \omega(v,w)$, i.e

$$I(\mathbb{V},\omega) := < v\otimes w - w\otimes v + \omega(v,w); v,w\in\mathbb{V} > .$$

The symplectic Clifford algebra is defined to be the quotient $sCliff(\mathbb{V},\omega) := \mathcal{T}(\mathbb{V})/I(\mathbb{V},\omega)$. Denote by π the canonical projection onto this quotient space, $\pi : \mathcal{T}(\mathbb{V}) \to sCliff(\mathbb{V},\omega)$. The inclusion $j : \mathbb{V} \to \mathcal{T}(\mathbb{V})$ together with the projection define a mapping $i := \pi j$. This mapping satisfies $i(v).i(w) - i(w).i(v) = j(v)\otimes j(w) - j(w)\otimes j(v) \bmod I(\mathbb{V},\omega) = -\omega(v,w)$, for all $v,w\in\mathbb{V}$, i.e., this map is a symplectic inclusion. Consider an another associative algebra A' with unit together with the linear mapping $i' : \mathbb{V} \to A'$ satisfying the first item of the definition of symplectic Clifford algebra. We shall show that there is a unique compatible algebra homomorphism $\rho : sCliff(\mathbb{V},\omega) \to A'$, i.e., $\rho i = i'$. The homomorphism ρ is uniquely determined on the image of i, thus on $\mathbb{V} \subseteq sCliff(\mathbb{V},\omega)$. There is a unique linear extension of its definition on $\mathbb{V}$ to the tensor algebra $\mathcal{T}(\mathbb{V})$ and thus its definition on the factor $sCliff(\mathbb{V},\omega) = \mathcal{T}(\mathbb{V})/I(\mathbb{V},\omega)$ is determined and therefore ρ is unique too.□

Remark 2.4.2. The definition of the symplectic Clifford algebra is analogous to the standard modern definition of Clifford algebra for a linear symmetric (not necessarily nondegenerate) structure. In the work of Klein [19], an explicit definition is used.

2.4.2 Heisenberg algebra, metaplectic group and the Segal-Shale-Weil representation

Weyl and Heisenberg algebra, Heisenberg and metaplectic group. The (associative) symplectic Clifford algebra equipped with a bracket $[x,y] := x.y - y.x$ becomes a Lie algebra which is often called *Weyl algebra.*

Remark 2.4.3. The name Weyl algebra is sometimes reserved for the set of polynomial coefficients linear differential operators on the ring of polynomials or for some notions generalizing (in the realm od D-modules) this one. In physical literature, the name Weyl algebra is reserved for what is called Heisenberg algebra in mathematics. We will follow the mathematical convention.

Now, we can define the *Heisenberg algebra*. Let $(\mathbb{V}, \omega)$ be a linear symplectic structure. Choose a symplectic basis $\{q^1, \ldots, q^k, p_1, \ldots, p_k\}$ of $(\mathbb{V}, \omega)$. The set of polynomials of degree less or equal to one defines a Lie subalgebra of the Weyl algebra, isomorphic to $\mathbb{R} \oplus \mathbb{R}^{2k}$. To see that it is subalgebra of the Weyl algebra, consider two elements of Heisenberg algebra $x = T + \sum_{i=1}^{k} P^i p_i + \sum_{i=1}^{k} Q_i q^i$ and $x' = T' + \sum_{i=1}^{k} P'^i p_i + \sum_{i=1}^{k} Q'_i q^i$ for $T, T', P^i, P^{i'}, Q^i, Q^i$, $i = 1, \ldots, k$. The only nontrivial bracket is $[q^i, p_i] = -1$ for $i = 1, \ldots, k$. The bracket $[x, x'] = \sum_{i=1}^{k} Q_i P'^i - \sum_{i=1}^{k} P_i Q'^i \in \mathbb{R}$; this proves that the Heisenberg algebra is a subalgebra of the Weyl algebra. We shall denote it by H_k, if it is isomorphic to $\mathbb{R} \oplus \mathbb{R}^{2k}$.

There is also a group structure on the Heisenberg algebra the multiplication of which is defined in coordinates by

$$(t, v).(s, w) := (t + s + \frac{1}{2}\omega(v, w), v + w)$$

for all $(t, v), (s, w) \in H_k$. The neutral element of this group is $(0, 0)$. The inverse element to an element (t, v) is $(-t, -v)$. This group is called the *Heisenberg group*.

Let us introduce the so called metaplectic group. From the differential topology of Lie groups it is well known that the first homotopy group of a Lie group G is equal to the first homotopy group of the compact component of its Iwasawa decomposition $G = KAN$, see Knapp [20], pp. 72 or 317. It is known that the compact component of the symplectic group $Sp(2k, \mathbb{R})$ is diffeomorphic to the unitary group, $U(k)$ (see again Knapp [20], pp. 72), the first homotopy group of which is isomorphic to $\mathbb{Z}$, see Hatcher [16], pp. 416. Summing-up, $\pi_1(Sp(2k, \mathbb{R})) \simeq \pi_1(U(k)) \simeq \mathbb{Z}$. By the theory of covering spaces (see again Hatcher [16], chapter 1.3), it is well known that there is a uniquely determined two-fold covering of this symplectic group up to a diffeomorphism, which is called *metaplectic group* and denoted by $Mp(2k, \mathbb{R})$.

Stone-von Neumann theorem and Segal-Shale-Weil representation. Now, let us concentrate to some representations of the Heisenberg algebra and Heisenberg group. The most known representation is the so called Schrödinger quantization prescription used in Quantum Mechanics.

Definition 2.4.2 (Schrödinger quantization prescription). Let $(\mathbb{V},\omega)$ be a linear symplectic structure and $\{q^i,p_i\}_{i=1}^k$ some symplectic basis of $(\mathbb{V},\omega)$. The mapping $\sigma : H_k \to \mathrm{End}(L^2(\mathbb{R}^k))$ given by

(1) $1 \in H_k \mapsto i$,

(2) $q^i \in H_k \mapsto ix^i, i = 1,\dots,k$

(3) $p_i \in H_k \mapsto i\frac{\partial}{\partial x^i}, i = 1,\dots,k$.

is called *Schrödinger quantization prescription*[4].

To see that σ is a representation of a Lie algebra, it is sufficient to observe that multiplication commutes on the space $\mathcal{S}(\mathbb{R}^k)$ and the same is true for partial derivatives. The last nontrivial condition we shall check is

$$\sigma([q^i,p_i])f = [\sigma(q^i),\sigma(p_i)]f$$

for $i = 1,\dots,k$ and an element $f \in \mathcal{S}(\mathbb{R}^k)$ The R.H.S. equals

$$ix^i\frac{\partial f}{\partial x^i} - \frac{\partial(ix^i f)}{\partial x^i} = -if.$$

The L.H.S. equals

$$\sigma([q^i,p_i])f = \sigma(-1)f = -if.$$

Thus R.H.S.=L.H.S.

Let us define a representation of the Heisenberg group,
$\pi : H_k \to \mathrm{Aut}\ (L^2(\mathbb{R}^k))$ given by

$$\pi(t,(x,z))f = e^{i(t+<x,z-\frac{1}{2}y>)}f(z-y)$$

for a function $f \in L^2(\mathbb{R}^k)$ and an element $(t,(x,z)) \in H_k \simeq \mathbb{R}\oplus(\mathbb{R}^k\oplus\mathbb{R}^k)$ where $<\ \ ,>$ is the standard Euclidean product on $\mathbb{R}^k$. We can define an action of the symplectic group $Sp(2k,\mathbb{R})$ on the Heisenberg algebra H_k,

$$Sp(2k,\mathbb{R})\times H_k \to H_k$$

[4] In fact, the mappings $\sigma(X)$ for $X \in H_k$ are defined only on the Schwartz space $\mathcal{S}(\mathbb{R}^k)$ which is dense in $L^2(\mathbb{R}^k)$.

by

$$(g,(t,v)) \mapsto (t, gv),$$

for $g \in Sp(2k,\mathbb{R})$, $(t,v) \in H_k \simeq \mathbb{R} \oplus \mathbb{R}^{2k}$. This action determines a family representations $\{\pi^g; g \in Sp(2n,\mathbb{R})\}$ of H_k, $\pi^g(t,v) := \pi(t,gv)$ which satisfies the condition $\pi^g(0,t) = e^{it} id_{L^2(\mathbb{R}^k)}$ for all $t \in \mathbb{R}$.

Theorem 2.4.3 (Stone-von Neumann theorem). *There is exactly one irreducible representation of H_k on $L(\mathbb{R}^k)$, $\pi : H_k \rightarrow Aut(L^2(\mathbb{R}^k))$ up to a unitary equivalence satisfying the condition $\pi(0,t) = e^{it} id_{L^2(\mathbb{R}^k)}$ for all $t \in \mathbb{R}$.*

Proof. See Folland [13], chapter 1.5. □

Using this theorem for the family $\{\pi^g; g \in Sp(2k,\mathbb{R})\}$, there are unitary operators $U(g)$ such that

$$\pi^g = U(g)\pi U(g)^{-1},$$

for each $g \in Sp(2k,\mathbb{R})$. It can be checked, that this procedure defines a projective unitary representation of $Sp(2n,\mathbb{R})$, i.e.,

$$U(gh) = c(g,h)U(g)U(h)$$

for some $c(g,h) \in \mathbb{C}$. After Shale and Weil (see Kashiwara, Vergne [25]), this projective unitary representation lifts to a unitary representation of $Mp(2k,\mathbb{R})$ in a unique manner. This representation

$$L : Mp(2k,\mathbb{R}) \rightarrow U(L^2(\mathbb{R}^k)),$$

where $U(\mathcal{H})$ is the group of all unitary operators on a Hilbert vector space $\mathcal{H}$, is called the *Segal-Shale-Weil representation.*

2.5 Cartan geometries

In this part we introduce some basic concepts used in this dissertation thesis - the Cartan geometries, especially the parabolic ones. Élie Cartan's point of view, which generalizes the Klein's Erlangen program and unify it with the investigation of metric/tensor defined geometries, helps to see many well known geometries like Riemannian, conformal, CR, projective or contact as examples of a unique geometrical structure, namely the Cartan geometry. For a detailed approach see the book of Sharpe [27], in which foremost the Riemannian, Mӧbius and conformal geometries are treated.

Definition 2.5.1 (Cartan geometry). Let $H \subseteq G$ be a Lie subgroup of a Lie group G, and $\mathfrak{g}$ be the Lie algebra of G. *Cartan geometry of type* (G, H) on a manifold M is a principal H-bundle $p : \mathcal{P} \to M$ endowed with a $\mathfrak{g}$-valued differential one form $\omega \in \Omega^1(M, \mathfrak{g})$, which is called *Cartan connection,* such that

(1) ω is H-equivariant, i.e., $(r^h)^*\omega = Ad_{h^{-1}} \circ \omega$, for all $h \in H$,

(2) ω reproduces the fundamental vector fields, i.e., $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{h}, u \in \mathcal{P}$ and ζ_X being the fundamental vector filed corresponding to X,

(3) ω is an absolute parallelism, i.e., $\omega_{T_u\mathcal{P}} : T_u\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism for each $u \in \mathcal{P}$.

We denote the Cartan geometry of type (G, H) on a manifold M by
$(\mathcal{P}, M, G, H, \omega)$ or simply by $(\mathcal{P}, \omega)$.

To each Cartan geometry, we can associate the vector space of constant vector fields. Let $X \in \mathfrak{g}$ then we denote by $\omega^{-1}(X)$ the vector field on $\mathcal{P}$ defined by the formula $\omega_u(\omega^{-1}(X))_u = X$ for each $u \in \mathcal{P}$ and call it *constant vector field.*

Further, to any Cartan geometry $(\mathcal{P}, \omega)$, we can associate its *curvature* $K \in \Omega^2(\mathcal{P}, \mathfrak{g})$ defined by the formula

$$K(\xi, \eta) =: d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)],$$

for any vector fields $\xi, \eta \in \mathfrak{X}(\mathcal{P})$.

One can show (see Čap, Slovák [7]) that this tensor field is horizontal and H-equivariant and thus in particular it defines the *curvature function* $\kappa : \mathcal{P} \to \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$

$$\kappa(u)(X, Y) := K_u((\omega^{-1}(X))_u, (\omega^{-1}(Y))_u),$$

for $u \in \mathcal{P}$ and $X, Y \in \mathfrak{g}$.

Definition 2.5.2 (Flat Cartan geometry). We call a Cartan geometry $(\mathcal{P}, \omega)$ *flat* if $\kappa = 0$.

Definition 2.5.3 (Parabolic geometry). *Parabolic geometry of type* (G, P) on a manifold M is a Cartan geometry $(\mathcal{P}, M, G, P, \omega)$ with P being a parabolic subgroup of a semisimple Lie group G.

Let $\mathfrak{g}$ be a $|k|$-graded simple Lie algebra, $\mathfrak{g}_-$ its negative part and G a Lie group the Lie algebra of which is $\mathfrak{g}$ and P be the associated parabolic subgroup of G. Let $(\mathcal{P}, G, P, M, \omega)$ be a parabolic geometry of type (G, P) on a manifold M and $(\rho, \mathbb{V})$ be a representation of P on a vector space $\mathbb{V}$. Let $VM = \mathcal{P} \times_\rho \mathbb{V}$ be the associated vector bundle over M to the principle P-bundle $\mathcal{P} \to M$ via the representation $(\rho, \mathbb{V})$. We would like to derive a notion of differentiation of sections of VM. To do this, we can identify the smooth sections $\Gamma(M, VM)$ with P-equivariant maps from $\mathcal{P}$ to $\mathbb{V}$, $\mathcal{C}^\infty(\mathcal{P}, \mathbb{V})^P$. To any equivariant function $s : \mathcal{P} \to \mathbb{V}$ we associate a smooth function $\nabla^\omega s : \mathcal{P} \to \mathfrak{g}_-^* \otimes \mathbb{V}$ defined by

$$\nabla^\omega s(u)(X) = \mathcal{L}_{\omega_u^{-1}(X)} s,$$

for $X \in \mathfrak{g}_-, u \in \mathcal{P}$ and $\mathcal{L}$ denotes the Lie derivative. This operation is called the *absolutely invariant derivative*. It is associated to the $|k|$-graded Lie algebra $\mathfrak{g}$, parabolic geometry $(\mathcal{P}, G, P, M, \omega)$ and the representation $(\rho, \mathbb{V})$ of P.

2.6 Invariant differential operators and Bernstein-Gelfand-Gelfand resolutions

In this subsection, we will summarize some concepts related to invariant differential operators on parabolic geometries. Our definitions will be very pragmatic; the motivations and other details on this topic can be found in Čap, Slovák, Souček [10].

Let $\mathfrak{g}$ be a complex $|k|$-graded semisimple Lie algebra, $\mathfrak{g}_-$ its negative part, G a Lie group the Lie algebra of which is $\mathfrak{g}$ and P the associated parabolic subgroup of G. Let $(\mathcal{P}, G, P, M, \omega)$ be a parabolic geometry of type (G, P) on a manifold M. Suppose that $\mathbb{E}$ and $\mathbb{F}$ are two P-modules. Let us denote by $J^1\mathbb{E} = \mathbb{E} \oplus (\mathfrak{g}_-^* \otimes \mathbb{E})$ the so called *first jet prolongation* of the P-module $\mathbb{E}$.

Let us denote the action of $\mathfrak{p}$ on $\mathbb{E}$ by λ, i.e., $\lambda : \mathfrak{p} \to \mathrm{End}(\mathbb{E})$ is a Lie algebra homomorphism obtained from the P-action by derivation. The action of $\mathfrak{p}$ on the vector space $J^1\mathbb{E}$ is given by

$$Z.(v,\phi) = (\lambda(Z)v, \lambda(Z)\circ\phi - \phi\circ ad_-(Z) + \lambda(ad_{\mathfrak{p}}(Z)(_-)v))$$

for $Z \in \mathfrak{p}$, $v \in \mathbb{E}$ and $\phi \in \mathfrak{g}_-^* \otimes \mathbb{E}$. This action is called *induced action.* The action of G_0 is given by restriction, tensor product and direct sum. With these two actions $J^1\mathbb{E}$ becomes a P-module due to the theorem 2.1.1 and remark 2.1.1.

Remark 2.6.1. The importance of the first jet prolongation is that the absolute invariant derivative associated to a $|k|-$graded Lie algebra and a parabolic geometry $(\mathcal{P}, G, P, M, \omega)$ and a P-module $\mathbb{E}$ is invariant in the sense that given any $s \in \mathcal{C}^\infty(\mathcal{P}, \mathbb{E})^P$, the element $(s, \nabla^\omega s) \in \mathcal{C}^\infty(\mathcal{P}, J^1\mathbb{E})^P$, see Čap, Slovák, Souček [10].

Definition 2.6.1 (Invariant differential operator). Let $(\mathcal{P}, \omega)$ be a parabolic geometry of type (G, P) on a manifold M. Let EM and FM be the associated vector bundles to the principle P-bundle $\mathcal{P} \to M$ via the P-modules $\mathbb{E}$ and $\mathbb{F}$, respectively. We call a mapping $D_{(\mathcal{P},\omega)}$ from $\Gamma(M, EM) \to \Gamma(M, FM)$ *invariant differential operator* of the parabolic geometry $(\mathcal{P}, \omega)$ of degree 1 if and only if there is a P-module homomorphism $\Phi : J^1\mathbb{E} \to \mathbb{F}$ such that for a section $s \in \Gamma(M, EM)$ considered as an equivariant mapping $s \in \mathcal{C}^\infty(\mathcal{P}, \mathbb{E})^P$ we can write

$$D_{(\mathcal{P},\omega)}(s)(u) = \Phi(s(u), \nabla^\omega s(u))$$

for $u \in \mathcal{P}$.

Remark 2.6.2. It can be shown that $D_{(\mathcal{P},\omega)}$ is a natural operator of order ≤ 1; for details and definition of natural operator see Čap, Slovák, Souček [10].

Now, we can define the so called BGG sequence and standard invariant operators. Consider the homogeneous principle P-bundle $G \to G/P$. Let $\mathbb{V}$ be the representation of G with the highest weight λ. Let us denote by $\mathbb{H}^j(\lambda)$ the j^{th} cohomology $H^j(\mathfrak{g}_-, \mathbb{V})$ of $\mathfrak{g}_-$ with coefficients in $\mathbb{V}$. The homogenous vector bundle associated to the principle homogeneous bundle $G \to G/P$ via the P-module $\mathbb{H}^j(\lambda)$ will be denoted by $H^j(\lambda)$. The *Bernstein-Gelfand-Gelfand (BGG) sequence* for the triple $(\mathfrak{g}, \mathfrak{p}, \lambda)$ consists of invariant differential operators acting between the sections of the bundles $H^j(\lambda)$, $D : \Gamma(G/P, H^j(\lambda)) \to \Gamma(G/P, H^{j+1}(\lambda))$. These operators are called *standard invariant operators.*

Chapter 3

Harmonic representations

This chapter is devoted to a distinguished class of infinite-dimensional irreducible representations of the symplectic algebra C_n, called harmonic representations. We shall show a theorem on decomposition of the tensor product of such a representation and the defining representation of C_n (theorem 3.5.1). We shall also prove a theorem on complete reducibility of the tensor product of the tensor powers of the defining representation and a harmonic representation (theorem 3.5.2).

3.1 Tensor products of finite and infinite dimensional representations

In this section we shall review some basic facts on tensor products of finite and infinite dimensional representations, details can be found in Kostant [24]. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Φ the system of roots with respect to $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and $\mathfrak{U}(\mathfrak{h})$ the universal enveloping algebra of $\mathfrak{h}$. Let us choose a system Φ^+ of positive roots and the corresponding decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_\pm$ are nilpotent subalgebras, defined by $\mathfrak{n}_\pm := \bigoplus_{\alpha \in \pm\Phi^+} \mathfrak{g}_\alpha$. Denote by Z the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ and by Z^* the set of all infinitesimal characters $\chi : Z \to \mathbb{C}$, i.e., associative algebra homomorphisms from Z to $\mathbb{C}$. Consider a representation $\pi : \mathfrak{g} \to \mathrm{End}(\mathbb{V})$, where $\mathbb{V}$ is a finite or infinite dimensional complex vector space. Assume that π *admits a central* *character*[1] $\chi : Z \to \mathbb{C}$, i.e., $\pi(X)v = \chi(X)v$ for all $v \in \mathbb{V}$ and $X \in Z$. This is the case, e.g., if π is irreducible, see Kostant [24]. For any $u \in Z$ one knows that there is a unique element $f_u \in \mathfrak{U}(\mathfrak{h})$ such that

$$u - f_u \in \mathfrak{U}\mathfrak{n}_+,$$

where $\mathfrak{U}\mathfrak{n}_+$ is the ideal in $\mathfrak{g}$ generated by $\mathfrak{n}_+$. There is a map

$$\mathfrak{h}^* \to Z^*$$

given by $\lambda \mapsto \chi_\lambda$, where $\chi_\lambda(u) := f_u(\lambda)$, $u \in Z$ and $\lambda \in \mathfrak{h}^*$. Let $\mathcal{W}$ denote the Weyl group of the algebra $\mathfrak{g}$ and let $\tilde{\sigma}$ denote the *affine action* of a Weyl group element $\sigma \in \mathcal{W}$ on $\mathfrak{h}^*$, i.e.,

$$\tilde{\sigma}(\lambda) = \sigma(\lambda + \delta) - \delta,$$

where $\lambda \in \mathfrak{h}^*$ and

$$\delta = \frac{1}{2} \sum_{\phi \in \Phi^+} \phi \in \mathfrak{h}^*$$

as we have already defined.

Theorem 3.1.1 (Harish-Chandra). *The map $\mathfrak{h}^* \to Z^*$ sending $\lambda \to \chi_\lambda$ is an epimorphism and $\chi_\lambda = \chi_\nu$ if and only if λ and ν are conjugate with respect to the affine action $\tilde{\sigma}$.*

Proof. See Knapp [20], pp. 249. □

Let us consider a representation $\pi_\lambda : \mathfrak{g} \to \mathrm{End}(\mathbb{V}_\lambda)$ of the algebra $\mathfrak{g}$ on a finite dimensional complex vector space $\mathbb{V}_\lambda$ with the highest weight $\lambda \in \mathfrak{h}^*$. The main result needed from Kostant [24] is the following theorem.

Theorem 3.1.2. *Let $\Pi := \{\mu_1, \ldots, \mu_k\}$ denote the set of all weights of the representation π_λ and*

$$Y_i := \{y \in \mathbb{V} \otimes \mathbb{V}_\lambda; uy = \chi_{\nu+\mu_i}(u)y, u \in Z\}, i = 1, \ldots, k.$$

Assume that the characters $\chi_{\nu+\mu_i}$ are all distinct. Then

$$\mathbb{V} \otimes \mathbb{V}_\lambda = \bigoplus_{i=1}^{k} Y_i.$$

[1]The central character is often called infinitesimal character.

Moreover, if Y_i is not zero, then Y_i is the maximal submodule of $\mathbb{V}\otimes\mathbb{V}_\lambda$ admitting $\chi_{\nu+\mu_i}$.

Proof. See Kostant [24].□

3.2 Complete reducibility

Let us prove some theorems on complete reducibility for infinite dimensional modules, which are well known in the finite dimensional case. We shall begin with a definition of complete reducibility.

Definition 3.2.1. Let $\mathfrak{g}$ be a complex Lie algebra and $\mathbb{V}$ a $\mathfrak{g}$-module. We say that $\mathbb{V}$ is *completely reducible* if for every submodule $\mathbb{U}$ there is a $\mathfrak{g}$-module $\mathbb{W}$ such that $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$ where the sum is supposed to be direct sum of $\mathfrak{g}$-modules.

Lemma 3.2.1. *Let $\mathfrak{g}$ be a complex Lie algebra. Let $\mathbb{V}$ be a $\mathfrak{g}$-module and let $\mathbb{V} = \mathbb{V}_1 \oplus \ldots \oplus \mathbb{V}_d$ for some $d \in \mathbb{N}$ be a direct sum decomposition into irreducible $\mathfrak{g}$-modules. Then $\mathbb{V}$ is completely reducible.*

Proof. We shall prove this theorem using the induction on d. I. If $d = 1$, the proof is trivial. II. Let $d > 1$ and $\mathbb{W}$ be a $\mathfrak{g}$-invariant subspace of $\mathbb{V}$. Let us denote by $\pi_i : \mathbb{V} \to \mathbb{V}_i$ for $i = 1, \ldots, d$ the projections of $\mathbb{V}$ onto the submodules $\mathbb{V}_i$. These projections are $\mathfrak{g}$-module homomorphisms. We shall distinguish two cases: first, if $\pi_1(\mathbb{W}) = 0$ then

$$\mathbb{W} \subseteq \mathbb{V}_2 \oplus \ldots \oplus \mathbb{V}_d.$$

By induction hypothesis, we know that there is a $\mathfrak{g}$-module $\mathbb{U}$ in $\mathbb{V}_2 \oplus \ldots \oplus \mathbb{V}_d$ such that $\mathbb{V}_2 \oplus \ldots \oplus \mathbb{V}_d = \mathbb{W} \oplus \mathbb{U}$. Then it follows that $\mathbb{V} = \mathbb{W} \oplus (\mathbb{U} \oplus \mathbb{V}_1)$. Second, suppose that $\pi_1(\mathbb{W}) \neq 0$. Because of the irreducibility of $\mathbb{V}_1$ we know that $\pi_1(\mathbb{W}) = \mathbb{V}_1$. Set $\mathbb{W}' := \mathrm{Ker}\ (\pi_{1|\mathbb{W}})$, which is clearly a $\mathfrak{g}$-invariant subspace as a kernel of a $\mathfrak{g}$-module homomorphism. We have

$$\mathbb{W}' \subseteq \mathbb{V}_2 \oplus \ldots \oplus \mathbb{V}_d.$$

Using the induction hypothesis, we obtain that there is a $\mathfrak{g}$-module $\mathbb{U}$ such that

$$\mathbb{W}' \oplus \mathbb{U} = \mathbb{V}_2 \oplus \ldots \oplus \mathbb{V}_d. \tag{3.1}$$

Take $x \in \mathbb{W} \cap \mathbb{U}$, then we know that $\pi_1(x) = 0$, because $x \in \mathbb{U}$. Since also $x \in \mathbb{W}$, then $x \in \mathbb{W}'$, i.e., $x \in \mathbb{W}' \cap \mathbb{U}$ which equals $\{0\}$ and thus $\mathbb{U} \oplus \mathbb{W}$ is a direct sum (of two $\mathfrak{g}$-modules). Further, we would like to show that $\mathbb{W} \oplus \mathbb{U} = \mathbb{V}$. It is sufficient to show that each element of $\mathbb{V}$ can be expressed as a sum of two elements $U, W \in \mathbb{U}, \mathbb{W}$, respectively. Take $x \in \mathbb{V}$. Then there are $x_i \in \mathbb{V}_i$, $i = 1, \ldots, d$ such that $x = \sum_{i=1}^d x_i$. Thus we may write $x = x_1 + \sum_{i=1}^d x_i$. Because of the decomposition (3.1) we know that there are $u \in \mathbb{U}, w' \in \mathbb{W}'$ such that $x = x_1 + u + w'$. Now, we need to decompose x_1. Take some $y \in \pi_{1|\mathbb{W}}^{-1}(\{x_1\})$; the set $\pi_{1|\mathbb{W}}^{-1}(\{x_1\})$ is nonempty because of $\pi_1(\mathbb{W}) = \mathbb{V}_1$. Define $\widetilde{u} := x_1 - y$. To show that $\tilde{u} \in \mathbb{V}_2 \oplus \ldots \oplus \mathbb{V}_d$, let us compute $\pi_1(\widetilde{u}) = \pi_1(x_1) - \pi_1(y) = x_1 - \pi_{1|\mathbb{W}}(y) = x_1 - x_1 = 0$ and thus $\widetilde{u} \in \mathbb{V}_2 \oplus \ldots \oplus \mathbb{V}_d = \mathbb{U} \oplus \mathbb{W}'$. Hence, we can write $\tilde{u} = u' + w''$ for $u' \in \mathbb{U}$ and $w'' \in \mathbb{W}'$ Summing up, $x = x_1 + u + w' = \widetilde{u} + y + u + w' = u' + w'' + y + u + w' = (u + u') + (y + w' + w'')$. Set $U := u + u'$ and $V := y + w' + w''$. We have $x = U + V$ for some $U \in \mathbb{U}$ and $\mathbb{V}$, what we have had to prove. $\square$

Remark 3.2.1. It is easy to see that the preceding lemma holds not only for complex Lie algebra modules but also for any G-modules and modules over associative algebras. This theorem is a generalization of a theorem in the book Goodman, Wallach [14], pp. 119. This cited theorem needs this generalization because it uses the implication: if $\mathbb{W}$ is a submodule of $\mathbb{V}$ which is isomorphic to $\mathbb{V}$, then $\mathbb{W} = \mathbb{V}$, which does not hold in the infinite dimension.

Remark 3.2.2. Let us remark that any submodule $\mathbb{W}$ of a finite direct sum of irreducible submodules $\mathbb{V}$ is equipped with a projection $\pi_{\mathbb{W}} : \mathbb{V} \to \mathbb{W}$ defined in the following manner. Let $\mathbb{V} = \mathbb{W} \oplus \mathbb{U}$ for some submodule $\mathbb{U}$, the existence of which establishes the preceding lemma. Then we can define $\pi_{\mathbb{W}}(w \oplus u) := w$ for $w \in \mathbb{W}$ and $u \in \mathbb{U}$. This projection is clearly a module homomorphism.

Next, we show a lemma which characterizes all submodules of a module which is a direct sum of irreducible submodules.

Lemma 3.2.2. *Let $\mathfrak{g}$ be a complex Lie algebra. Let $\mathbb{V} = \mathbb{V}_1 \oplus \ldots \oplus \mathbb{V}_d$ for some $d \in \mathbb{N}$ be a direct sum decomposition into irreducible $\mathfrak{g}$-modules and $\mathbb{W}$ be a submodule of $\mathbb{V}$. Then $\mathbb{W} = \mathbb{W}_1 \oplus \ldots \oplus \mathbb{W}_c$ for some $c \leq d$ such that for* *each $j \in \{1, \ldots, c\}$, there is an element $i(j) \in \{1, \ldots, d\}$ for which the relation $\mathbb{W}_j \simeq \mathbb{V}_{i(j)}$ holds and if $j \neq j'$, then $i(j) \neq i(j')$.*

Proof. Because of the previous lemma 3.2.1, we know that there is an invariant vector space $\mathbb{U} \subseteq \mathbb{V}$ such that $\mathbb{V} = \mathbb{W} \oplus \mathbb{U}$, i.e., there is a projection $\pi = \pi_{\mathbb{W}} : \mathbb{V} \to \mathbb{W}$, which is a $\mathfrak{g}$-module homomorphism. It is easy to see that $\pi(\mathbb{V}_r)$ for $r = 1, \ldots, d$ is an irreducible $\mathfrak{g}$-module. Indeed, suppose there is a $\mathfrak{g}$-invariant subspace $\{0\} \subsetneq Q \subsetneq \pi(\mathbb{V}_r)$. Take $\pi^{-1}(Q) \subseteq \mathbb{V}_r$. This subspace is $\mathfrak{g}$-invariant, because for $q \in \pi^{-1}(Q)$, there is a $w \in Q$ such that $w = \pi(q)$. For any $X \in \mathfrak{g}$ we have $Q \ni Xw = X\pi(q) = \pi(Xq)$. Therefore $Xq \in \pi^{-1}(Q)$, i.e., $\pi^{-1}(Q)$ is a $\mathfrak{g}$-invariant subspace. Suppose that $\pi^{-1}(Q) = \{0\}$ and take $y \in Q$. Then $\pi^{-1}(\{y\}) = \{0\}$ and therefore $y = \pi(0) = 0$, i.e., $Q = \{0\}$. If $\pi^{-1}(Q) = \mathbb{V}_r$ then it follows that $\pi(\mathbb{V}_r) \subseteq Q$, which is impossible. Indeed, take $y \in \pi(\mathbb{V}_r)$, then there is an element $x \in \mathbb{V}_r$ such that $\pi(x) = y$. Because of the equality $\pi^{-1}(Q) = \mathbb{V}_r$ we know that $x \in \pi^{-1}(Q)$ and therefore $y = \pi(x) \in Q$, i.e., $\pi(\mathbb{V}_r) \subseteq Q$. Summing up, we have proved that $\pi(\mathbb{V}_r)$ is an irreducible $\mathfrak{g}$-module.

We define a set $X \subseteq \{1, \ldots, d\}$ inductively. I. Denote by k the smallest element i of $\{1, \ldots, d\}$ for which $\pi(\mathbb{V}_i)$ is a nonzero module. II. For $k < i \leq d$, let $i \in X$ if and only if $\pi(\mathbb{V}_i) \neq \{0\}$ and $\pi(\mathbb{V}_j) \neq \pi(\mathbb{V}_i)$ for all $j \in X$ such that $j < i$.

Now, prove that $\bigoplus_{i \in X} \pi(\mathbb{V}_i)$ is a direct sum. Take two distinct $r, s \in X$ and suppose that $r < s$ without lost of generality. We know that $\pi(\mathbb{V}_r) \cap \pi(\mathbb{V}_s)$ is a submodule of $\pi(\mathbb{V}_r)$ and of $\pi(\mathbb{V}_s)$. Since these modules are irreducible $\pi(\mathbb{V}_r) \cap \pi(\mathbb{V}_s)$ is either $\{0\}$ or $\pi(\mathbb{V}_r) \cap \pi(\mathbb{V}_s) = \pi(\mathbb{V}_r)$ and $\pi(\mathbb{V}_s) \cap \pi(\mathbb{V}_r) = \pi(\mathbb{V}_s)$. But the second possibility implies that $\pi(\mathbb{V}_r) = \pi(\mathbb{V}_s)$ which is impossible due to the construction of the set X, i.e., $\pi(\mathbb{V}_r) \cap \pi(\mathbb{V}_s) = \{0\}$ and the considered sum is direct.

Further, we prove that $\mathbb{W} = \bigoplus_{i \in X} \pi(\mathbb{V}_i)$. Let $x = \sum_{i \in X} x_i \in \bigoplus_{i \in X} \pi(\mathbb{V}_i)$. There are y_i for $i \in X$ such that $\pi(y_i) = x_i$. Thus we may write $x = \sum_{i \in X} x_i = \sum_{i \in X} \pi(y_i) = \pi(\sum_{i \in X} y_i)$, i.e., $x \in \mathrm{Im}\ (\pi) = \mathbb{W}$. Suppose that $x \in \mathbb{W}$. Then there is an element $y \in \mathbb{V}$ such that $x = \pi(y)$. We can decompose y as $y = \sum_{i=1}^{d} y_i$ for $y_i \in \mathbb{V}_i$, $i = 1, \ldots, d$. Thus we obtain the decomposition

$x = \sum_{i=1}^{d} \pi(y_i)$. For any $i = 1, \dots, d$ we have either $\pi(y_i) = 0$ or $\pi(y_i) = \pi(z_j)$, where $z_j \in \mathbb{V}_j$ and $j \in X$. Therefor, we can write $x = \sum_{i=1}^{d} \pi(y_i) = \sum_{j \in X} \pi(z_j)$ what we have had to prove.

It is easy to see that $\pi_{|\mathbb{V}_j}$ is an isomorphism for all $j \in X$. The mapping $\pi_j := \pi_{|\mathbb{V}_j} : \mathbb{V}_j \to \pi(\mathbb{V}_j)$ for $j \in X$ is a map between irreducible modules. Thus either $\mathrm{Ker}(\pi_j) = \mathbb{V}_j$ or $\mathrm{Ker}(\pi_j) = \{0\}$. In the first case, we obtain that π_j is a zero homomorphism, which is impossible due to the construction of X. In the second case, π_j is a monomorphism. The surjectivity of π_j follows from the fact that $\pi(\mathbb{V}_j)$ is irreducible. Because π is a $\mathfrak{g}$-module homomorphism it follows that π_i for $i \in X$ are intertwining isomorphism and thus for $i \in X$ we have $\mathbb{V}_i \simeq \pi(\mathbb{V}_i)$ as $\mathfrak{g}$-modules. The statement follows (the last duty is only a renumbering of the elements of the set X). $\Box$

3.3 Spinor representations of $\mathfrak{sp}(2n, \mathbb{C})$

To fix a notation, we introduce some standard facts on the symplectic algebra C_n in this section. Further, we mention an analogy between the tensor representations of the algebra D_n and representation of the symplectic algebra C_n. We introduce the half-spinor representations of the algebra C_n and the notion of a harmonic module.

3.3.1 Some basic facts on C_n

First, let us recall some basic facts on the symplectic algebra $\mathfrak{g} = C_n = \mathfrak{sp}(2n, \mathbb{C})$. This algebra consists of $2n \times 2n$ matrices over complex numbers of the form

$$A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right),$$

where $A_1 = -A_4^T$, $A_2 = A_2^T$ and $A_3 = A_3^T$. We choose a Cartan subalgebra $\mathfrak{h}$ of C_n consisting of all diagonal $2n \times 2n$ matrices in $\mathfrak{g}$. If ϵ_i denotes the projection onto the (i, i) element of the matrix, then the set of all roots Φ equals

$$\Phi = \{\pm(\epsilon_i \pm \epsilon_j); 1 \leq i < j \leq n\} \cup \{\pm 2\epsilon_i; i = 1, \dots, n\}.$$

We chose a system Δ of simple roots as follows

$$\Delta = \{\alpha_1 := \epsilon_1 - \epsilon_2, \dots, \alpha_{n-1} := \epsilon_{n-1} - \epsilon_n, \alpha_n := 2\epsilon_n\}.$$

The Chevalley basis of C_n is given by

$$X_{\epsilon_i - \epsilon_j} := E_{i,j} - E_{n+j,n+i}, 1 \leq i < j \leq n,$$

$$X_{2\epsilon_i} := E_{i,n+i}, i = 1, \ldots, n$$

$$X_{\epsilon_i + \epsilon_j} := E_{i,n+j} - E_{j,n+i}, 1 \leq i < j \leq n,$$

$$Y_\mu := X_\mu^T, \mu \in \Phi,$$

$$H_i := E_{i,i} - E_{j,j} + E_{n+j,n+j} - E_{n+i,n+i}, i = 1, \ldots, n-1,$$

$$H_n := E_{n,n} - E_{2n,2n},$$

where $E_{i,j}$ is a matrix having 1 at the place (i,j) and 0 at the other places.

The algebra C_n has a very useful realization consisting of differential operators on $\mathbb{C}[x^1, \ldots, x^n]$. It is shown in Dixmier [12] that the Lie algebra generated by $\{x^i\partial_{i+1}, x^{i+1}\partial_i; i = 1, \ldots, n-1\} \cup \{\partial_1^2, (x^1)^2\}$ (where ∂_i is the partial differentiation in x^i, $i = 1, \ldots, n$) is isomorphic to the algebra C_n via the isomorphism $\psi : C_n \to \mathrm{End}(\mathbb{C}[x^1, \ldots, x^n])$, defined by

$$\psi(X_{\epsilon_i - \epsilon_{i+1}}) := x_{n-i}\partial_{n-i+1}, i = 1, \ldots, n-1,$$

$$\psi(X_{-(\epsilon_i - \epsilon_{i+1})}) := x_{n-i+1}\partial_{n-i}, i = 1, \ldots, n-1,$$

$$\psi(X_{2\epsilon_n}) := -\frac{1}{2}\partial_1^2,$$

$$\psi(X_{-2\epsilon_n}) := \frac{1}{2}(x^1)^2.$$

The requirement that the basis $\{\epsilon_i\}_{i=1}^n$ is an orthonormal basis defines an inner product $(\ ,\ )$ on $\mathfrak{h}^*$. Using the standard notation

$$\check{\alpha} := \frac{2\alpha}{(\alpha, \alpha)},$$

for $\alpha \in \mathfrak{h}^* - \{0\}$, the set of fundamental weights $\{\varpi_i\}_{i=1}^n$ is defined as the dual basis to the basis $\{\check{\alpha}_i\}_{i=1}^n$, i.e., $(\varpi_i, \check{\alpha}_j) = \delta_{ij}$. In our case, $\varpi_j = \sum_{i=1}^j \epsilon_i, j = 1, \ldots, n$.

3.3.2 Half-Spinor representations of C_n

Now, we can come to the representation part of this section. There is a very close analogy between representations of $C_n = \mathfrak{sp}(2n, \mathbb{C})$ and $D_n = \mathfrak{so}(2n, \mathbb{C})$. Finite dimensional representations of C_n have their counterpart in tensor representations of D_n (i.e., representations of D_n with highest weights consisting from integers). On the other hand, there is no finite dimensional representation of C_n similar to spinor representations of D_n.

It was suggested by Kostant ([23]) that a proper analogy of spinor representations of orthogonal algebras are certain infinite-dimensional representations of the symplectic algebra called the Segal-Shale-Weil representations. (One can show that the complexification of the derivation of the Segal-Shale-Weil representation of the metaplectic group considered acting on the ring of polynomials, introduced in section 2.4.1, decomposes into this Segal-Shale-Weil representations of the symplectic algebra.) The mentioned analogy can be nicely seen using the following realization of these representations.

Consider first the orthogonal algebras $D_n = \mathfrak{so}(2n, \mathbb{C})$ and choose a maximal isotropic subspace $\mathbb{V}$ of $\mathbb{C}^{2n}$, it has dimension n. Spinor representations of D_n can be realized on the Grassmann algebra $\mathbb{S} = \bigwedge^{\bullet}(\mathbb{V}) = \oplus_{i=1}^{n} \bigwedge^{i}(\mathbb{V})$. It decomposes into two irreducible subspaces $\mathbb{S} = \mathbb{S}_+ \oplus \mathbb{S}_-$, where
$\mathbb{S}_+ = \oplus_{j \in 2\mathbb{Z}} \bigwedge^{j}(\mathbb{V})$ and $\mathbb{S}_- = \oplus_{j \in 2\mathbb{Z}+1} \bigwedge^{j}(\mathbb{V})$ are the so called half-spinor representations.

In the case of symplectic algebra, there is a similar construction. Consider the defining representation $\mathbb{C}^{2n}$ of $\mathfrak{sp}(2n, \mathbb{C})$ with the corresponding standard symplectic form and choose again a maximal isotropic subspace $\mathbb{V} \simeq \mathbb{C}^n$ (as a vector space). The infinite dimensional space $\mathbb{C}[x^1, \ldots, x^n] \simeq \oplus_{i=1}^{\infty} \odot^i (\mathbb{V})$ is a representation of C_n as described above (using the isomorphism ψ). It also decomposes as $\oplus_{i=1}^{\infty} \odot^i (\mathbb{V}) = \mathbb{S}_+ \oplus \mathbb{S}_-$ as a C_n-module. As in the orthogonal case, the first representation is the direct sum of even dimensional symmetric powers and the second one of the odd dimensional ones. We will call this representations *half-spinor representations* of C_n.

Remark 3.3.1. This is a nice example of a supersymmetry, where the space of polynomials in n commuting variables (the symplectic case) has as an analogy the space of polynomials in n anticommuting variables (the orthogonal case). This analogy explains why spinor representations for C_n are infinite dimensional.

Finite dimensional representations of D_n can be all realized as spinor-tensors, i.e., as submodules of tensor products of one of the two half-spinor representations with a tensor representation. Consequently, an analogue of these finite-dimensional representations of D_n is a class of infinite dimensional representations of C_n consisting of submodules of tensor products of one of the two infinite dimensional half-spinor representations of C_n with a finite dimensional representation of C_n. This is a class of representations we are going to study in this chapter.

3.4 Completely pointed modules

In this section we review some basic facts on bounded and completely pointed modules from Britten, Lemire [2]. More details can be found there.

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{h}$ its Cartan subalgebra. Let us consider an $\mathfrak{h}$-*diagonalizable* $\mathfrak{g}$-module $\mathbb{V}$, i.e., $\mathbb{V} = \bigoplus_{\nu \in wt(\mathbb{V})} \mathbb{V}(\nu)$, where $wt(\mathbb{V}) \subseteq \mathfrak{h}^*$ is the space of all weights of the module $\mathbb{V}$.[2] We say that it is a module with *bounded multiplicities* if and only if there is a $k \in \mathbb{N}$ such that $\dim \mathbb{V}(\nu) \leq k$ for all $\nu \in wt(\mathfrak{h})$. The minimal k is called the *order* of the module. The module is called *completely pointed* provided the order of this module is 1. The modules with bounded multiplicities have some nice properties. For example, it is known that a simple complex Lie algebra has an infinite dimensional irreducible module with bounded multiplicities if and only if it is a either a special linear algebra or a symplectic algebra.

In this chapter, we shall consider irreducible standard cyclic modules. For any weight $\nu \in \mathfrak{h}^*$, we shall denote by $L(\nu)$ the unique irreducible standard cyclic module with highest weight ν, see 2.2.1. The half-spinor representations (or the Segal-Shale-Weil representations) of C_n belong to this class. It is easy to compute that the weight of a constant polynomial is $\nu_+ = -\frac{1}{2}\varpi_n$ and the

[2] Any standard cyclic module is $\mathfrak{h}$-diagonalizable, for instance.

weight of the monomial x^1 is $\nu_- = \varpi_{n-1} - \frac{3}{2}\varpi_n$. One can easily check that these monomials are highest weight vectors of the half-spinor representations. Hence $\mathbb{S}_+ \simeq L(\nu_+)$ and $\mathbb{S}_- \simeq L(\nu_-)$. Both these representations are completely pointed (different monomials have different weights), see Britten, Hooper, Lemire [3]. It can be shown that the opposite claim is also true. If an irreducible standard cyclic module $L(\nu)$ is a completely pointed C_n-module, then $\nu = \nu_+$ or $\nu = \nu_-$, see again Britten, Hooper, Lemire [3]).

The following key facts describe the structure of the tensor product of a spinor representations with a finite dimensional module (for details see Britten, Lemire [2] and Britten, Hooper, Lemire [3]).

Theorem 3.4.1. *Let $\nu = \sum_{i=1}^{n} \nu_i \varpi_i$ be a dominant integral weight of C_n and let $L(\nu)$ be the corresponding irreducible finite dimensional highest weight module. Let*

$$\begin{aligned} T_\nu^+ \quad &:= \quad \{\nu - \sum_{i=1}^{n} d_i \epsilon_i ; d_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} d_i \in 2\mathbb{Z}, 0 \leq d_i \leq \nu_i, \\ &\qquad i = 1, \ldots, n-1, 0 \leq d_n \leq 2\nu_n + 1\} \end{aligned}$$

$$\begin{aligned} T_\nu^- \quad &:= \quad \{\nu - \sum_{i=1}^{n} d_i \epsilon_i ; d_j + \delta_{n,j} \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} d_i \in 2\mathbb{Z}, 0 \leq d_i \leq \nu_i, \\ &\qquad i = 1, \ldots, n-1, 0 \leq d_n \leq 2\nu_n + 1\}. \end{aligned}$$

Then

$$L(\nu_\pm) \otimes L(\nu) = \bigoplus_{\kappa \in T_\nu^\pm} L(\nu_\pm + \kappa)$$

Proof. See Britten, Hooper, Lemire [3].□

Definition 3.4.1. Let us denote by $\mathbb{A}$ the following set of weights.

$$\mathbb{A} := \{\sum_{i=1}^{n} \lambda_i \varpi_i ; \lambda_i \geq 0, \lambda_i \in \mathbb{Z},\ i = 1, \ldots, n-1;\ \lambda_n \in \mathbb{Z} + \frac{1}{2}, \lambda_{n-1} + 2\lambda_n + 3 > 0\}.$$

Denote by $\mathcal{A}$ the following set of representations

$$\mathcal{A} := \{L(\nu); \nu \in \mathbb{A}\}$$

and call each member of $\mathcal{A}$ a *harmonic module* .

Theorem 3.4.2. *The following conditions are equivalent*

(1) L is a harmonic module

(2) L is a direct summand in the decomposition of $L(\lambda) \otimes L(-\frac{1}{2}\varpi_n)$ for some dominant integral weight λ

(3) L is an infinite dimensional C_n-module with bounded multiplicities.

Proof. See Britten, Lemire [2].□

Let us define two subsets of $\mathfrak{h}^*$.

$$\mathbb{Z}_+ := \{\sum_{i=1}^{n} \lambda_i \epsilon_i; \lambda_i \in \mathbb{Z}, i = 1, \ldots, n, \sum_{i=1}^{n} \lambda_i \in 2\mathbb{Z}\}$$

and

$$\mathbb{Z}_- := \{\sum_{i=1}^{n} \lambda_i \epsilon_i; \lambda_i \in \mathbb{Z}, i = 1, \ldots, n, \sum_{i=1}^{n} \lambda_i \in 2\mathbb{Z}+1\}.$$

(The first defined set is a lattice in $\mathfrak{h}^*$.) Let us write the set $\mathbb{A}$ as a disjoint union of two subsets $\mathbb{A} = \mathbb{A}_+ \cup \mathbb{A}_-$, where

$$\mathbb{A}_+ := \{-\frac{1}{2}\varpi_n + \kappa; \kappa \in \mathbb{Z}_+\} \cap \mathbb{A},$$

$$\mathbb{A}_- := \{-\frac{1}{2}\varpi_n + \kappa; \kappa \in \mathbb{Z}_-\} \cap \mathbb{A}.$$

For an integral number $k \in \mathbb{Z}$, we define $sn(k) := +$ if and only if $k \in 2\mathbb{Z}$ and $sn(k) := -$ if and only if $k \in 2\mathbb{Z}+1$.

3.5 Tensor products with the defining representation

In this section, we are going to study tensor products of any harmonic module with the defining representation $L(\varpi_1)$. We show that these products are completely reducible and characterize the constituents of the decomposition. By induction, we get complete reducibility also for a product with powers of $L(\varpi_1)$. These are exactly facts needed for future applications in a study of invariant differential operators on projective contact structures (see Čap, Slovák, Souček [10]).

Theorem 3.5.1. *Let $\lambda \in \mathbb{A}$ and let $\Pi(\varpi_1) := \{\pm\epsilon_i, i = 1, \ldots, n\}$ denote the set of all weights*[3] *of the defining representation $L(\varpi_1)$. Then $L(\lambda) \otimes L(\varpi_1)$ is completely reducible and*

$$L(\lambda) \otimes L(\varpi_1) = \bigoplus_{\kappa \in \mathbb{A}_\lambda} L(\kappa),$$

where $\mathbb{A}_\lambda \subseteq \{\kappa = \lambda + \mu; \kappa \in \mathbb{A}, \mu \in \Pi(\varpi_1)\}$.

Remark 3.5.1. For better understanding of the following proof, see the picture below the proof together with the remark 3.5.2.

Proof. **Part I.** Due to the theorem 3.4.2 we know that $L(\lambda)$ is a direct summand in the direct sum decomposition of

$$L(-\frac{1}{2}\varpi_n) \otimes L(\nu)$$

for some integral dominant weight ν. Thus $L(\lambda) \otimes L(\varpi_1) \subseteq (L(-\frac{1}{2}\varpi_n) \otimes L(\nu)) \otimes L(\varpi_1) = L(-\frac{1}{2}\varpi_n) \otimes (L(\nu) \otimes L(\varpi_1))$. The representation $L(\nu) \otimes L(\varpi_1)$ decomposes into (finitely many) irreducible finite dimensional representations directly, because both $L(\nu)$ and $L(\varpi_1)$ are finite dimensional representations of the simple Lie algebra C_n. Due to the theorem 3.4.1, we know that $L(-\frac{1}{2}\varpi_1) \otimes (L(\nu) \otimes L(\varpi_1))$ also decomposes into (finitely many) irreducible representation directly. The lemma 3.2.2 enables the conclusion that $L(\lambda) \otimes L(\varpi_1)$ is a direct sum of irreducible representations and due to the theorem 3.4.2 these direct summands are in $\mathcal{A}$. Further, because of the lemma 3.2.1, we know that $L(\lambda) \otimes L(\varpi_1)$ is completely reducible, which was to prove.

Part II. Now, we would like to use the Kostant theorem 3.1.2. We shall prove that the characters $\chi_{\lambda+\nu}$ and $\chi_{\lambda+\mu}$ are distinct for any distinct $\mu, \nu \in \Pi(\varpi_1)$. We know from theorem 3.1.1 that this is equivalent to the fact that $\lambda + \mu$ and $\lambda + \nu$ are not conjugated by the affine action of an element of the Weyl group $\mathcal{W}$ of the algebra C_n. Two elements $\phi, \psi \in \mathfrak{h}^*$ are conjugated by the affine action of an element of the Weyl group if and only if $\phi + \delta$ and $\psi + \delta$ are conjugated by an element of the Weyl group, i.e., if and only if $\sigma(\phi+\delta) = \psi+\delta$,

[3] One easily verifies that $\Pi(\varpi_1)$ is really the set of all weight of $L(\varpi_1)$, i.e., the smallest Φ-saturated set containing ϖ_1.

for some $\sigma \in \mathcal{W}$. We prove that $\{\lambda+\nu+\rho, \lambda+\mu+\rho\} \subseteq \overline{W_1} \cup \overline{W_2}$, where

$$W_1 := \{\sum_{i=1}^{n} \beta_i \epsilon_i; \beta_1 > \ldots > \beta_n > 0\},$$

$$W_2 := \{\sum_{i=1}^{n} \beta_i \epsilon_i; \beta_1 > \ldots > \beta_{n-1} > -\beta_n > 0\}$$

are two open neighbor Weyl chambers of C_n. Let us denote $\mu = s\epsilon_p$ for $s \in \{-1, 1\}$ and some $p = 1, \ldots, n$. In the case of C_n the element $\delta = n\epsilon_1 + (n-1)\epsilon_2 + \ldots + \epsilon_1$. Using the relation $\varpi_j = \epsilon_1 + \ldots + \epsilon_j$ for $j = 1, \ldots, n$ one easily computes that

$$\lambda + \mu + \delta =: \sum_{i=1}^{n} \beta_i \epsilon_i = \sum_{i=1}^{n} [(\sum_{j=i}^{n} \lambda_j) + n - i + 1 + s\delta_{ip}]\epsilon_i$$

for $\lambda = \sum_{i=1}^{n} \lambda_i \varpi_i$. Thus the requirement $\lambda + \mu + \delta \in \overline{W_1}$ reduces to $\lambda_i + 1 \geq s(\delta_{i+1,p} - \delta_{ip})$ which is evidently satisfied for all $i = 1, \ldots, n-1$, see the definition 3.4.1. For $i = n$, the condition we need to check is $\beta_{n-1} \geq \beta_n \geq 0$ or $\beta_{n-1} \geq -\beta_n \geq 0$. If $\lambda_n \geq 0$ then $\lambda + \mu + \delta \in \overline{W_1}$. Suppose that $\lambda_n < 0$, then the condition $\lambda + \mu + \delta \in \overline{W_2}$ reduces to $\lambda_{n-1} + 2\lambda_n + 3 + s(\delta_{n-1,p} - \delta_{np}) \geq 0$. This condition is satisfied because of the last inequality in the definition 3.4.1. (These conditions are nearly "optically" equivalent.)

Suppose that $\lambda + \mu + \delta$ and $\lambda + \nu + \delta$ are conjugated by an element σ of the Weyl group of C_n, i.e., $\sigma(\lambda + \mu + \delta) = \lambda + \nu + \delta$.

(1) Suppose that $\lambda + \mu + \delta \in W_1$ and $\lambda + \nu + \delta \in W_2$ (or $\lambda + \mu + \delta \in W_2$ and $\lambda + \nu + \delta \in W_1$, which is analogous). The condition $\sigma(\lambda+\mu+\delta) = \lambda+\nu+\delta$ implies $\sigma W_1 = W_2$. It is evident that $\sigma_{\epsilon_n} W_1 = W_2$. The Weyl group acts simply transitively on the set of open (or closed) Weyl chambers. Hence $\sigma = \sigma_{\epsilon_n}$.[4] Now, $\sigma_{\epsilon_n}(\lambda + \mu + \delta) = \lambda + \mu + \delta - 2(\epsilon_n, \lambda + \mu + \delta)\epsilon_n = \lambda + \mu + \delta - 2(\lambda_n + s\delta_{np} + 1)\epsilon_n$. This element equals $\lambda + \nu + \delta$ if and only if $\mu - \nu = 2(\lambda_n + s\delta_{np} + 1)\epsilon_n$ which is impossible due to the structure of the set $\Pi(\varpi_1)$ and the condition $\lambda_n \in \mathbb{Z} + \frac{1}{2}$.

(2) The case $\lambda + \mu + \delta, \lambda + \nu + \delta \in W_i$ and $\sigma(\lambda + \mu + \delta) = \lambda + \nu + \delta$ for $i = 1, 2$ leads to the condition $\sigma = id$, i.e., $\nu = \mu$ - a contradiction.

[4] Although ϵ_n does not belong to the system of simple roots, it is evident that we could have written $\sigma_{2\epsilon_n}$ instead of σ_{ϵ_n}.

(3) The remaining case is $\lambda+\mu+\delta, \lambda+\nu+\delta \in \overline{W_1} \cup \overline{W_2} - (W_1 \cup W_2)$, i.e., the considered elements lie on the walls of the two Weyl chambers. (The other cases are impossible: if there is an element lying on a wall of a closed Weyl chamber and an another one is lying in the open Weyl chamber, then they cannot be conjugated.) The inspection of the fact $\lambda+\mu+\delta, \lambda+\nu+\delta \in \overline{W_1} \cup \overline{W_2}$ showed that if these elements lie on the walls of $\overline{W_1}$ and $\overline{W_2}$, then they lie in their interior (i.e., they do not lie on the walls of codimension 2): the inequations $\beta_j \geq \beta_{j+1}$, for $j = 1, \dots, n-1$ happen equations exactly for one $j \in \{1, \dots, n-1\}$. Let us define two families of open Weyl chambers

$$W_r := \{\sum_{i=1}^{n} \beta_i \epsilon_i; \beta_1 > \dots > \beta_{r-1} > -\beta_r > \beta_{r+1} > \dots > \beta_n > 0\},$$

$r = 1, \dots, n-1$ and

$$W'_t := \{\sum_{i=1}^{n} \beta_i \epsilon_i; \beta_1 > \dots > \beta_{t-1} > -\beta_t > \beta_{t+1} > \dots > -\beta_n > 0\},$$

$t = 1, \dots, n-1$.

(3.1) Suppose that $\lambda+\mu+\delta \in \overline{W_1} \cap \overline{W_r}$ and $\lambda+\nu+\delta \in \overline{W_2} \cap \overline{W'_t}$ for some $r, t = 1, \dots, n-1$. If we suppose that $\sigma(\lambda+\mu+\delta) = \lambda+\nu+\delta$, then the fact that these elements lie in the interior of the walls implies that $\sigma W_1 = W_2$ or $\sigma W_1 = W'_t$. The first case leads to a contradiction as we have shown. Using the fact that the Weyl group acts simply transitively, we easily find that $\sigma = \sigma_{\epsilon_t}\sigma_{\epsilon_n}$ in the second case. Let us compute $\sigma_{\epsilon_t}\sigma_{\epsilon_n}(\lambda+\mu+\delta) = \lambda+\mu+\delta-2(\epsilon_t, \lambda+\mu+\delta)\epsilon_t - 2(\epsilon_n, \lambda+\mu+\delta)\epsilon_n = \lambda+\mu+\delta-2(\lambda_t+s\delta_{pt}+n-t+1)\epsilon_t - 2(\lambda_n+s\delta_{pn}+1)\epsilon_n$. This element equals $\lambda+\nu+\delta$ if and only if $\mu-\nu = 2(\lambda_t+s\delta_{pt}+n-t+1)\epsilon_t + 2(\lambda_n+\delta_{pt}+1)\epsilon_n$. Because of the structure of $\Pi(\varpi_1)$, we obtain: $\mu-\nu \in \{\pm 2\epsilon_t, \pm 2\epsilon_n, \pm\epsilon_t \pm \epsilon_t, \pm\epsilon_t \mp \epsilon_n\}$. The first possibility leads to $0 = \lambda_n + s\delta_{np} + 1$, which is impossible because λ_n is half-integral. The second possibility implies $0 = \lambda_t + s\delta_{tp} + n - t + 1 \geq \lambda_t + n - t > 0$ - a contradiction. The third and fourth possibilities force $\pm 1 = 2(\lambda_t + s\delta_{tp} + n - t + 1)$ - an odd number equals an even one, i.e., a contradiction.

(3.2) Suppose that $\lambda + \mu + \delta \in \overline{W_1} \cap \overline{W_r}$ and $\lambda + \nu + \delta \in \overline{W_1} \cap \overline{W_t}$. In this case, $\sigma W_1 = W_1$ or $\sigma W_1 = W_t$. The first case leads to a contradiction as we already know. In the second case, one easily finds that $\sigma_{\epsilon_t} W_1 = W_t$, i.e., using the simplicity of the Weyl group action, this implies $\sigma = \sigma_{\epsilon_t}$. Let us compute $\sigma_{\epsilon_t}(\lambda + \mu + \delta) = \lambda + \mu + \delta - 2(\lambda_t + s\delta_{pt} + n - t + 1)\epsilon_t$. This element equals $\lambda + \nu + \delta$ if and only if $\{\mu, \nu\} = \{\epsilon_t, -\epsilon_t\}$, i.e., $\mu - \nu = \pm 2\epsilon_t$. That means that $1 = \lambda_t + 1 + n - t + 1$ or $-1 = \lambda_t - 1 + n - t + 1$ which are impossible because $\lambda_t \geq 0$ and $t < n$ for $t = 1, \ldots, n-1$.

(3.3) The remaining cases are analogous to the previous ones and actually have been done.

Summarizing the part II, we have proved that the condition of the Kostant theorem 3.1.2 is satisfied.

Part III. Thus we know that $L(\lambda) \otimes L(\varpi_1) = \bigoplus_{i=1}^{k} Y_i$, where Y_i are submodules of $L(\lambda) \otimes L(\varpi_1)$ and if not zero, they admit the infinitesimal character $\chi_{\lambda+\nu_i}$, where $\nu_i \in \Pi(\varpi_1)$. (Moreover, $\nu_i \neq \nu_j$ for $i \neq j$ and for each $\mu \in \Pi(\varpi_1)$ there is an $i \in \{1, \ldots, k\}$ such that $\mu = \nu_i$.) Because Y_i is a submodule of a direct sum of irreducible modules (as we have proved in part I), we know that for each $i = 1, \ldots, k$ there are $\kappa_1^{(i)}, \ldots \kappa_{d_i}^{(i)} \in \mathbb{A}$ such that $Y_i = L(\kappa_1^{(i)}) \oplus \ldots \oplus L(\kappa_{d_i}^{(i)})$. Denote the infinitesimal character of $L(\kappa_j^{(i)})$ for $j = 1, \ldots d_i$ by $\chi_j^{(i)}$. For an element X in the center Z of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g} = C_n$ and $v = v_1 + \ldots + v_{d_i} \in L(\kappa_1^{(i)}) \oplus \ldots \oplus L(\kappa_{d_i}^{(i)})$ we can write $\pi(X)v = \pi(X)(v_1 + \ldots + v_{d_i}) = \pi_1(X)v_1 + \ldots + \pi_{d_i}(X)v_{d_i}$, where π denotes the considered representation on the space Y_i and π_j denote the representation on $L(\kappa_j^{(i)})$ for $j = 1, \ldots, d_i$. The last equation simplifies into

$$\chi_{\lambda+\nu_i}(X)v = \chi_1^{(i)}(X)v_1 + \ldots + \chi_{d_i}^{(i)}(X)v_{d_i}. \tag{3.2}$$

Inserting $v_1 = \ldots = v_{j-1} = v_j = \ldots = v_{d_i} = 0$ into 3.2 we get $\chi_{\lambda+\nu_i} = \chi_j^{(i)}$ for each $j = 1, \ldots, d_i$. But $\chi_j^{(i)} = \chi_{\kappa_j^{(i)}}$, i.e., $\chi_{\lambda+\nu_i} = \chi_{\kappa_j^{(i)}}$. According to the theorem 3.1.1, there is an element σ of the Weyl group $\mathcal{W}$ such that $\lambda + \nu_i = \sigma(\kappa_j^{(i)} + \delta) - \delta$, i.e., $\kappa_j^{(i)} + \delta$ is conjugated with $\lambda + \nu_i + \delta$ by the element $\sigma : \sigma(\kappa_j^{(i)} + \delta) = \lambda + \nu_i + \delta$.

Using the inspection of part II of this proof, we know that $\kappa_j^{(i)} + \delta \in W_1 \cup W_2$ and $\lambda + \nu_i + \delta \in \overline{W_1} \cup \overline{W_2}$. We may suppose that $\lambda + \nu_i + \delta \in W_1 \cup W_2$, because this element is supposed to be conjugated with element $\kappa_j^{(i)} + \delta$ lying in the union of two open Weyl chambers $W_1 \cup W_2$.

(1) Suppose that $\lambda + \nu_i + \delta, \kappa_j^{(i)} + \delta \in W_r$ for $r = 1, 2$. Due to the simplicity of the action of an Weyl group element on the set of open (closed) Weyl chambers, we get that $\sigma = id$ and thus $\kappa_j^{(i)} = \lambda + \nu_i$, which was to prove.

(2) Suppose that $\lambda + \nu_i + \delta \in W_1$ and $\kappa_j^{(i)} + \delta \in W_2$ (the remaining case can be treated in a similar way). We have already mentioned that in this case, $\sigma = \sigma_{\epsilon_n}$. To conclude the proof let us show the following

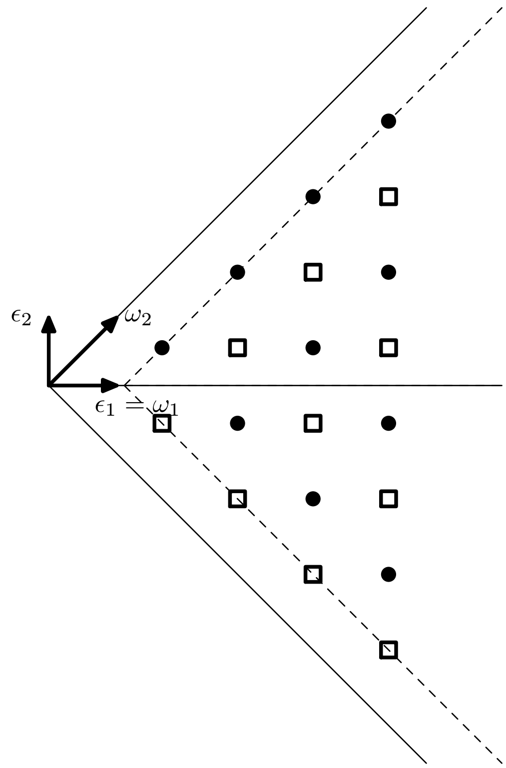
Lemma 3.5.1. *Let $L(\kappa) \subseteq L(\varpi_1)^{\otimes^k}$ be an irreducible summand with the highest weight κ, then $\kappa \in \mathbb{Z}_{sn(k)}$.*

Proof. I. For $k = 0$ the fact is obvious. II. Let us suppose that the statement holds for k. The tensor product $L(\varpi_1)^{\otimes^{k+1}}$ decomposes into a direct sum of irreducible modules (as a tensor power of finite dimensional modules over simple Lie algebras). Suppose that κ is the highest weight of one of the irreducible summands in $L(\varpi_1)^{\otimes^{k+1}}$. We can decompose $L(\varpi_1)^{\otimes^{k+1}} = L(\varpi_1)^{\otimes^k} \otimes L(\varpi_1) = [L(\kappa_1) \oplus \ldots \oplus L(\kappa_m)] \otimes L(\varpi_1) = (L(\kappa_1) \otimes L(\varpi_1)) \oplus \ldots \oplus (L(\kappa_m) \otimes L(\varpi_1))$ for some $m \in \mathbb{N}$ and $\kappa_j \in \mathbb{Z}_{sn(k)}$ (it is the induction hypothesis). Let $L(\kappa) \subseteq L(\kappa_j) \otimes L(\varpi_1)$ for some $j = 1, \ldots, m$. It is a well known fact in representation theory (a consequence of the Weyl character formula, see Knapp [20], pp. 285) that $\kappa = \kappa_j + \tau$ where τ is a weight of $L(\varpi_1)$. From the structure of the set $\Pi(\varpi_1)$, it follows that $\kappa - \kappa_j = \tau \in \mathbb{Z}_-$. Using the induction assumption, we get that $\kappa \in \mathbb{Z}_\pm$ if and only if $\kappa_j \in \mathbb{Z}_\mp$. □

Because $L(\nu)$ is a finite dimensional representation of C_n, there is a $k \in \mathbb{N}_0$ such that $L(\nu) \subseteq L(\varpi_1)^{\otimes k}$, i.e., it is a tensor representation. We know that $L(\lambda) \subseteq L(-\frac{1}{2}\varpi_n) \otimes L(\nu) \subseteq L(-\frac{1}{2}\varpi_n) \otimes L(\varpi_1)^{\otimes^k}$. From the theorem 3.4.1, it follows that $\lambda = -\frac{1}{2}\varpi_n + \kappa$, i.e., $\lambda \in \mathbb{A}_\pm$ if and only if $\kappa \in \mathbb{Z}_\pm$. Due the theorem 3.4.1, we know that $\kappa = \nu - \sum_{i=1}^n m_i\epsilon_i$ for $\sum_{i=1}^n m_i\epsilon_i \in 2\mathbb{Z}_+$, i.e., $\kappa \in \mathbb{Z}_{\pm}$ if and only if $\nu \in \mathbb{Z}_{\pm}$. Due to the previous lemma 3.5.1, we know that $\nu \in \mathbb{Z}_{sn(k)}$. Summing up, $\lambda \in \mathbb{A}_{sn(k)}$. This implies that $\lambda+\delta+\nu_i \in \mathbb{A}_{sn[k+\frac{1}{2}n(n+1)+1]}$. On the other hand, we know that $L(\kappa_j^{(i)}) \subseteq L(\lambda)\otimes L(\varpi_1) \subseteq L(-\frac{1}{2}\varpi_n)\otimes L(\varpi_1)^{\otimes^k}\otimes L(\varpi_1) \subseteq L(-\frac{1}{2}\varpi_n)\otimes L(\varpi_1)^{\otimes^{k+1}}$. Let us write the decomposition $L(\varpi_1)^{\otimes^{k+1}} = L(\xi_1)\oplus\ldots\oplus L(\xi_r)$ for some $r \in \mathbb{N}$ and $\xi_j \in \mathbb{Z}_{sn(k+1)}$ for $j = 1,\ldots,r$. Let us suppose that $L(\kappa_j^{(i)}) \subseteq L(-\frac{1}{2}\varpi_n)\otimes L(\xi_s)$ for some $s = 1,\ldots,r$. Then due to the theorem 3.4.1, it follows that $\kappa_j^{(i)} = -\frac{1}{2}\varpi_n + \kappa$, where $\kappa = \xi_s - \sum_{i=1}^n m_i\epsilon_i$ for some $\sum_{i=1}^n m_i\epsilon_i \in \mathbb{Z}_+$, i.e., $\kappa_j^{(i)} \in \mathbb{A}_{\pm}$ if and only if $\xi_s \in \mathbb{Z}_{\pm}$. Due to the previous lemma 3.5.1, we know that $\xi_s \in \mathbb{Z}_{sn(k+1)}$. Thus $\kappa_j^{(i)} \in \mathbb{A}_{sn(k+1)}$. This implies, $\kappa_j^{(i)}+\delta \in \mathbb{A}_{sn[k+1+\frac{1}{2}n(n+1)]}$. It is easy to see that $\sigma_{\epsilon_n} : \mathbb{A}_{\pm} \to \mathbb{A}_{\mp}$. Indeed, take $\psi = -\frac{1}{2}\varpi_n + \sum_{i=1}^n m_i\epsilon_n$ and compute $\sigma_{\epsilon_n}\psi - \psi = -2(\psi,\epsilon_n) = -2(-\frac{1}{2}+m_n) = -2m_n+1 \in 2\mathbb{Z}+1$ and the statement follows. Thus it is impossible that $\sigma_{\epsilon_n}(\lambda+\delta+\nu_i) = \kappa_j^{(i)}+\delta$, i.e., the remaining possibility $\sigma = id$ holds. This possibility implies $\kappa_j^{(i)} = \lambda+\nu_i$.

Part IV. Thus we know that if $Y_i \neq \{0\}$ then it is a direct sum of copies of $L(\lambda+\nu_i)$. For irreducible finite dimensional modules over simple Lie algebras, one can prove that the multiplicity of $L(\lambda+\nu_i)$ in the tensor product $L(\lambda)\otimes L(\nu)$ (where $\nu_i \in \Pi(\nu)$) is smaller or equal to the multiplicity of the weight ν_i in the representation $L(\nu)$, see the book Čap, Slovák [7]. It is easy to prove that the same is true for irreducible standard cyclic modules. Because each weight of $L(\varpi_1)$ has multiplicity one, we deduce that the multiplicity of representation with the highest weight $\lambda+\nu_i$ in the tensor product is at most one.$\square$

Remark 3.5.2. Weights from the set $\mathbb{A}+\delta$ (i.e., we consider weights from $\mathbb{A}$ shifted by δ) are all included in two Weyl chambers only - the union of the dominant Weyl chamber and its image under the reflection with respect to ϵ_n. All that can be nicely illustrated in the case of C_2. At the next picture, we can see the corresponding two Weyl chambers in the Cartan-Stiefel diagram of C_2 below. Elements of $\mathbb{A}$ are shifted by δ, elements of $\mathbb{A}_+$ are denoted by dots and elements of $\mathbb{A}_-$ by squares. At the picture of the Cartan-Stiefel diagram for C_2, we can see that if λ and κ are conjugated by the reflection in the plane orthogonal to ϵ_2 then one of them is represented by a dot and the second one by a square. (But $\lambda+\nu_i$ and $\kappa_j^{(i)}$ are both represented either by dots or by squares.)



Consider the representation $L(\varpi_1)^{\otimes^k} \otimes \mathbb{V}$ for some $\mathbb{V} \in \mathcal{A}$. We know that $L(\varpi_1)\otimes\mathbb{V}$ is completely reducible and its direct summands are in $\mathcal{A}$. Let us label these summands by integers, denoting their chosen position in the direct sum. Tensoring $L(\varpi_1)\otimes\mathbb{V} = \oplus_{b_1=1}^{n_1}\mathbb{V}_{b_1}$ by $L(\varpi_1)$, we obtain a direct sum again since each $\mathbb{V}_{b_1}$, $b_1 = 1,\ldots,n_1$ is in $\mathcal{A}$ and therefore decomposes when tensored by $L(\varpi_1)$ due to the previous theorem 3.5.1. We denote the b_2 therm of the direct sum of $\mathbb{V}_{b_1}\otimes L(\varpi_1)$ by $\mathbb{V}_{(b_1,b_2)}$. Continuing in this process (or by induction) we obtain $L(\varpi_1)^{\otimes^k} \otimes \mathbb{V} = \oplus_{b_1,\ldots,b_k}\mathbb{V}_{(b_1,\ldots,b_k)}$. Thus we have proved

Theorem 3.5.2. *The representation of $L(\varpi_1)^{\otimes^k} \otimes \mathbb{V}$ is completely reducible and decomposes as*

$$L(\varpi_1)^{\otimes^k} \otimes \mathbb{V} = \oplus_{b_1,\ldots,b_k}\mathbb{V}_{(b_1,\ldots,b_k)}.$$

Example: In this example we shall denote a module $L(\nu)$ with highest weight ν

simply by (ν) written in the basis of fundamental weights and we shall describe the set $\mathbb{A}_\lambda$ for $\lambda = (10\ldots01 - \frac{3}{2})$.

We know that

$$(0\ldots01 - \frac{3}{2}) \otimes (10\ldots0) = (0\ldots0 - \frac{1}{2}) \oplus (10\ldots01 - \frac{3}{2}).$$

We also know that

$$(10\ldots0) \otimes (10\ldots0) = (20\ldots0) \oplus (010\ldots0) \oplus (0).$$

We can decompose the following tensor products using the prescription of the theorem 3.4.1 to obtain that

$$(0\ldots01 - \frac{3}{2}) \otimes (20\ldots0) = (20\ldots01 - \frac{3}{2}) \oplus (10\ldots01 - \frac{3}{2}) \oplus (0\ldots01 - \frac{3}{2})$$
$$(0\ldots01 - \frac{3}{2}) \otimes (010\ldots0) = (010\ldots01 - \frac{3}{2}) \oplus (10\ldots0 - \frac{1}{2})$$
$$(0\ldots01 - \frac{3}{2}) \otimes (0) = (0\ldots01 - \frac{3}{2})$$

We also know that

$$(0\ldots0 - \frac{1}{2}) \otimes (10\ldots0) = (0\ldots01 - \frac{3}{2}) \oplus (1\ldots0 - \frac{1}{2})$$

From this we can deduce that:

$$\begin{aligned}(10\ldots01 - \frac{3}{2}) \otimes (10\ldots0) \quad &= \quad (20\ldots01 - \frac{3}{2}) \oplus (0\ldots01 - \frac{3}{2}) \oplus \\ &\oplus (010\ldots01 - \frac{3}{2}) \oplus (10\ldots0 - \frac{1}{2}).\end{aligned}$$

Hence we can see here that in this case, the set $\mathbb{A}_\lambda$ is given by

$$\mathbb{A}_\lambda = \{\kappa = \lambda + \mu; \kappa \in \mathbb{A}, \mu \in \Pi(\varpi_1)\}.$$

Many other examples lead to the same result, hence we conjecture that the same fact will be true in general.

Chapter 4

Projective Contact Geometry

4.1 Contact bundle and Levi form

Let M be a smooth n dimensional manifold.

Let us assume a smooth subbundle HM of corank one of the tangent bundle TM. Denote $QM := TM/HM$ the *quotient bundle* of rank one. This subbundle comes up with the *canonical quotient projection* $q : TM \to QM$. Recall the standard definition of *Levi form* $L : \Gamma(M, HM) \times \Gamma(M, HM) \to \Gamma(M, QM)$,

$$L(\xi, \eta)(m) := q([\xi, \eta]_m)$$

for each $\xi, \eta \in \Gamma(M, HM)$ and $m \in M$. The Levi form is antisymmetric and bilinear over $\mathcal{C}^\infty(M, \mathbb{R})$. The antisymmetry is evident. To prove the bilinearity, let us take $f \in \mathcal{C}^\infty(M, \mathbb{R})$, $\xi, \eta \in \Gamma(M, HM)$ and compute

$$\begin{aligned}
L(f\xi, \eta)(m) &= q([f\xi, \eta]_m) \\
&= q(f(m)\xi_m\eta - \eta_m(f\xi)) \\
&= q(f(m)\xi_m\eta - f(m)\eta_m\xi - (\eta_m.f)\xi_m) \\
&= q(f(m)[\xi, \eta]_m - (\eta_m.f)\xi_m) \\
&= f(m)q([\xi, \eta]_m) = f(m)L(\xi, \eta)(m)
\end{aligned}$$

which proves the bilinearity. So the Levi form factors to a tensor field, i.e., there is an element $\mathcal{L} \in \bigwedge^2 H^*M \otimes QM$ such that $\mathcal{L}_m(\xi_m, \eta_m) = L(\xi, \eta)(m)$.

We shall not distinguish between the Levi form L and the corresponding tensor field $\mathcal{L} \in \bigwedge^2 H^*M \otimes QM$ and denote them both by the calligraphic $\mathcal{L}$.

Definition 4.1.1. We call the subbundle HM *contact bundle* and the triple $(M, HM, \mathcal{L})$ *contact structure* if the associated Levi form $\mathcal{L}$ is nondegenerate, i.e for each $m \in M$ and $0 \neq \xi \in H_mM$ there is $\eta_m \in H_mM$ such that $\mathcal{L}_m(\xi_m, \eta_m) \neq 0$.

The nondegeneracy of the Levi form immediately implies that the dimension n of the manifold M is odd, i.e., $n = 2k+1$, if $(M, HM, \mathcal{L})$ is a contact structure.

Remark 4.1.1. According to the definition of the Levi form, the nondegeneracy means that to each $0 \neq \xi_m \in H_mM$ there is a $\eta_m \in H_mM$ such that $[\xi_m, \eta_m] \notin H_mM$. More classically, this condition can be expressed by the equation

$$\theta_m \wedge (d\theta_m)^k \neq 0,$$

where θ is a differential 1-form defining the contact subbundle HM, i.e.,

$$HM = \{t \in TM; \theta(t) = 0\}.$$

Thus the nondegeneracy of the Levi form represents the notion of maximal nonintegrability in Frobenius sense which is classically used to define a contact subbundle. Sometimes a contact structure will be denoted by the pair (M, θ). The differential 1-form θ defining the contact structure is called a *contact form.*

Example: There is a well known example of a contact structure, which is a model for all contact structures, namely

$$(\mathbb{R}^{2k+1}[q^1, \dots, q^k, p_1, \dots, p_k, t], \theta_0 = dt + \sum_{i=1}^{k} p_i dq^i).$$

A short computation gives that

$$HM = \{\sum_{i=1}^{k}(p_iQ^i)\frac{\partial}{\partial t} + \sum_{i=1}^{k} P_i \frac{\partial}{\partial p_i} + \sum_{i=1}^{k} Q^i \frac{\partial}{\partial q^i}; Q^i, P_i \in \mathcal{C}^\infty(\mathbb{R}^{2k+1}, \mathbb{R}), i = 1, \dots k\}.$$

In the section 4.3, we meet another example of contact structure, the so called projective contact sphere.

4.2 Connections on the contact and quotient bundle

Denote the kernel $\mathrm{Ker}(\mathcal{L}) \subseteq \bigwedge^2 HM$ of the Levi form $\mathcal{L}$ by $\bigwedge_0^2 HM$ and suppose that the above given contact structure $(M, HM, \mathcal{L})$ is equipped with a partial linear connection ∇, i.e., there is a mapping $\nabla : \Gamma(M, HM) \times \Gamma(M, HM) \to \Gamma(M, HM)$ satisfying the obvious relations

(1) $\nabla_{f\xi}\eta = f\nabla_\xi\eta$

(2) $\nabla_\xi f\eta = (\xi.f)\eta + f\nabla_\xi\eta,$

for all $f \in \mathcal{C}^\infty(M, \mathbb{R})$, $\xi, \eta \in \Gamma(M, HM)$.

Assume that the connection is *compatible* with the Levi form, i.e., that the induced connection satisfies $\nabla_\xi(\bigwedge_0^2 HM) \subseteq \bigwedge_0^2 HM$ for all $\xi \in \Gamma(M, HM)$. We call such a partial linear connection *contact connection.*

The contact connection ∇ on HM induces a partial linear connection ∇^Q on the one dimensional quotient bundle QM (the so called *induced quotient connection*) in the following way

$$\nabla^Q_\xi(\mathcal{L}(\eta, \zeta)) = \mathcal{L}(\nabla_\xi\eta, \zeta) + \mathcal{L}(\eta, \nabla_\xi\zeta), \xi, \eta, \zeta \in \Gamma(M, HM).$$

At first, let us rewrite its righthand side using the antisymmetry of the Levi form, $\mathcal{L}(\xi, \eta) = \mathcal{L}(\xi \wedge \eta)$ which holds for each $\xi, \eta \in \Gamma(M, HM)$, and the Leibniz like behavior of the contact connection induced to differential 2-forms: $R.H.S. = \mathcal{L}(\nabla_\xi\eta, \zeta) + \mathcal{L}(\eta, \nabla_\xi\zeta) = \mathcal{L}(\nabla_\xi\eta \wedge \zeta) + \mathcal{L}(\eta \wedge \nabla_\xi\zeta) = \mathcal{L}(\nabla_\xi(\eta \wedge \zeta))$. As a result we have

$$\nabla^Q_\xi\mathcal{L}(\eta, \zeta) = \mathcal{L}(\nabla_\xi(\eta \wedge \zeta))$$

for each $\eta, \zeta, \xi \in \Gamma(M, HM)$. To see that the definition of ∇^Q is independent on the choice of $\eta, \zeta \in \Gamma(M, HM)$, (i.e., the correctness of the definition) take $\omega, \omega' \in \Gamma(M, \bigwedge^2 HM)$ such that $\mathcal{L}(\omega) = \mathcal{L}(\omega')$. That means that there is an element $\tau \in \bigwedge_0^2 HM$ such that $\omega' = \omega + \tau$. Thus we may write $\nabla^Q_\xi\mathcal{L}(\omega') = \mathcal{L}(\nabla_\xi\omega') = \mathcal{L}(\nabla_\xi(\omega + \tau)) = \mathcal{L}(\nabla_\xi\omega) + \mathcal{L}(\nabla_\xi\tau) = \nabla^Q_\xi\mathcal{L}(\omega)$ which proves the correctness of the definition. In the last step we have used the compatibility property, $\nabla(\bigwedge_0^2 HM) \subseteq \bigwedge_0^2 HM$[1]. The nondegeneracy of the Levi form implies that it is surjective, i.e., to each vector filed $R \in \Gamma(M, QM)$ there are (not uniquely determined) vector fields $\eta, \zeta \in \Gamma(M, HM)$ such that $\mathcal{L}(\eta, \zeta) = R$.

Now, we shall show that this formula defines a partial connection. First, let us check the tensorial property in the ξ-argument. $\nabla^Q_{f\xi}(\mathcal{L}(\eta,\zeta)) = \mathcal{L}(\nabla_{f\xi}\eta, \zeta) + \mathcal{L}(\eta, \nabla_{f\xi}\zeta) = f\nabla^Q_\xi \mathcal{L}(\eta,\zeta)$, which follows from the definition of the contact connection on HM and the tensorial property of the Levi form $\mathcal{L}$. Second, to check the Leibniz rule, consider

$$\begin{aligned}
\nabla^Q_\xi(f\mathcal{L}(\eta,\zeta)) &= \nabla^Q_\xi(\mathcal{L}(f\eta,\zeta)) \\
&= \mathcal{L}(\nabla_\xi f\eta, \zeta) + \mathcal{L}(f\eta, \nabla_\xi \zeta) \\
&= \mathcal{L}(\xi.f\eta, \zeta) + f\mathcal{L}(\nabla_\xi \eta, \zeta) + f\mathcal{L}(\eta, \nabla_\xi \zeta) \\
&= f\nabla^Q_\xi \mathcal{L}(\eta,\zeta) + \xi.f\mathcal{L}(\eta,\zeta),
\end{aligned}$$

which proves the correctness of the definition of the quotient connection ∇^Q.

4.2.1 Contact bundle projections and quotient connections

In this paragraph we shall show that the choice of a partial linear connection ∇^Q on the quotient bundle QM is in 1-1 correspondence to a choice of a bundle projection $p: TM \to HM$.

First, we shall use the partial linear connection ∇^Q to define a bundle surjection $p: TM \to HM$. Consider the map $P: \Gamma(M, HM) \times \Gamma(M, TM) \to \Gamma(M, QM)$ given by the formula

$$P(\xi, \eta) := q([\xi, \eta]) - \nabla^Q_\xi q(\eta)$$

for each $\xi \in \Gamma(M, HM)$ and $\eta \in \Gamma(M, TM)$. Check that this expression is $\mathcal{C}^\infty(M, \mathbb{R})$-linear in η.

$$\begin{aligned}
P(\xi, f\eta) &= q([\xi, f\eta]) - \nabla^Q_\xi q(f\eta) \\
&= fq([\xi,\eta]) + \xi.fq(\eta) - \xi.fq(\eta) - f\nabla^Q_\xi q(\eta) \\
&= fP(\xi,\eta).
\end{aligned}$$

[1]Informally, the contact connection factors through the kernel of the Levi form to define the quotient connection.

This means that there is an element $p^{\nabla,Q}(\eta) \in \Gamma(M, HM)$ such that

$$P(\xi, \eta) = q([\xi, \eta]) - \nabla^Q_\xi q(\eta) = \mathcal{L}(\xi, p^{\nabla,Q}(\eta)),$$

for each $\xi \in \Gamma(M, TM)$. The element is uniquely defined due to the nondegeneracy of $\mathcal{L}$. We shall prove that the map $p^{\nabla,Q} : \eta \mapsto p^{\nabla,Q}(\eta)$ is a projection, i.e., a surjective (onto HM) idempotent in Hom (TM, HM). Consider

$$\begin{aligned}
\mathcal{L}(\xi, p^{\nabla,Q}(p^{\nabla,Q}(\eta))) &= P(\xi, p^{\nabla,Q}(\eta)) \\
&= q([\xi, p^{\nabla,Q}(\eta)]) - \nabla^Q_\xi q(p^{\nabla,Q}(\eta)) \\
&= \mathcal{L}(\xi, p^{\nabla,Q}(\eta)) - \nabla^Q_\xi 0 \\
&= \mathcal{L}(\xi, p^{\nabla,Q}(\eta))
\end{aligned}$$

which implies that $(p^{\nabla,Q})^2(\eta) = p^{\nabla,Q}(\eta)$ (again due to the nondegeneracy property of Levi form). We prove that $p^{\nabla,Q}$ is surjective. Let $\eta \in \Gamma(M, HM)$, $P(\xi, \eta) = q([\xi, \eta]) - \nabla^Q_\xi q(\eta) = \mathcal{L}(\xi, \eta) - \nabla^Q_\xi q(\eta) = \mathcal{L}(\xi, \eta) + 0 = \mathcal{L}(\xi, p^{\nabla,Q}(\eta))$ which implies $p^{\nabla,Q}(\eta) = \eta$ which means that p is a surjection.

Conversely, consider the map $p \mapsto \nabla^{p,Q}_\xi q(\eta) := q([\xi, \eta]) - \mathcal{L}(\xi, p(\eta))$, where $\xi, \eta \in \Gamma(M, HM)$. This map defines a partial linear connection on QM. To prove the correctness, take another η' such that $q(\eta) = q(\eta')$, it means that there exists a $\chi \in \Gamma(M, HM)$ such that $\eta' = \eta + \chi$. $\nabla^{p,Q}_\xi q(\eta') = q([\xi, \eta']) - \mathcal{L}(\xi, p(\eta')) = q([\xi, \eta]) - \mathcal{L}(\xi, p(\eta)) + q([\xi, \chi]) - \mathcal{L}(\xi, p(\chi)) = \nabla^{p,Q}_\xi q(\eta) + q([\xi, \chi]) - \mathcal{L}(\xi, \chi) = \nabla^{p,Q}_\xi q(\eta)$. Thus the definition is correct. Now, we shall prove that this map defines a linear connection. First, we check that $\nabla^{p,Q}_{f\xi} q(\eta) = q([f\xi, \eta]) - \mathcal{L}(f\xi, \eta) = fq([\xi, \eta]) - f\mathcal{L}(\xi, \eta) = f\nabla^{p,Q}_\xi q(\eta)$. Second, we check the Leibniz rule.

$$\begin{aligned}
\nabla^{p,Q}_\xi fq(\eta) &= q([\xi, f\eta]) - \mathcal{L}(\xi, fp(\eta)) \\
&= q([\xi, f\eta]) - f\mathcal{L}(\xi, p(\eta)) \\
&= q((\xi.f)\eta + f\xi\eta - f\eta\xi) - f\mathcal{L}(\xi, p(\eta)) \\
&= (\xi.f)q(\eta) + fq([\xi, \eta]) - f\mathcal{L}(\xi, p(\eta)) \\
&= (\xi.f)q(\eta) + f\nabla^{\pi,Q}_\xi q(\eta).
\end{aligned}$$

It is obvious that the map $p \mapsto \nabla^{p,Q}$ is inverse to the map which associate a bundle projection $p^{\nabla,Q}$ to each quotient connection ∇^Q considered above and vice versa. Thus we have proved the

Theorem 4.2.1. *There is a* 1-1 *correspondence*

$$\mathcal{A} : Proj(TM, HM) \to LinConn(QM)$$

between the set $Proj(TM, HM)$ of all bundle projections p of the tangent bundle TM to the contact bundle HM and the set $LinConn(QM)$ of all partial linear quotient connections ∇^Q.

The kernel of the projection $p^{\nabla,Q}$ is of rank one and will be denoted by Q. Thus the projection gives us a splitting $TM = Q \oplus HM$.

4.2.2 Projective structure on the set of contact connections and Transformation formulas

In this subsection, we shall define the so called projective equivalence on the set of contact connections. Given a one form $\Upsilon \in \Gamma(M, HM^*)$, the nondegeneracy of $\mathcal{L}$ implies that the definition of a bundle homomorphism $\Upsilon^\natural : QM \to HM$ given by

$$\mathcal{L}(\Upsilon^\natural(\psi), \zeta) := \Upsilon(\zeta)\psi$$

for all $\psi \in \Gamma(M, QM)$ and $\zeta \in \Gamma(M, HM)$ is correct. Due to the nondegeneracy of the Levi form $\mathcal{L}$, it is possible to define the projective structure on the set of all partial linear connections. We call two partial linear connections $\widehat{\nabla}$, ∇ *projective equivalent contact connections* if there exists a smooth one form $\Upsilon \in \Gamma(M, HM^*)$ such that

$$\widehat{\nabla}_\xi \eta - \nabla_\xi \eta = \Upsilon(\xi)\eta + \Upsilon(\eta)\xi + \Upsilon^\natural(\mathcal{L}(\xi, \eta)), \xi, \eta \in \Gamma(M, HM).$$

It is an easy computation to check that this gives a relation of equivalence on the set of all contact connections. We shall denote the class of equivalent connections by $[\nabla]$ if the contact connection ∇ belongs to this class. Now, we are ready to give the definition of projective contact structure.

Definition 4.2.1. We call the quadruple $(M, HM, \mathcal{L}, [\nabla])$ *projective contact structure* if $(M, HM, \mathcal{L})$ is a contact structure and $[\nabla]$ is the class of projective connections to which a compatible contact connection ∇ belongs.

Remark 4.2.1. It can be checked that the compatibility property is conserved within a projective contact class, i.e., if $\widehat{\nabla} \sim \nabla$ and ∇ is compatible, then $\widehat{\nabla}$ is compatible.

Transformation formulas. Let us compute how the induced quotient connection on QM changes by changing a representant of given projective class $[\nabla]$.
$\widehat{\nabla}^Q_\xi(\mathcal{L}(\eta,\zeta)) =$

$$\begin{aligned}
&= \mathcal{L}(\widehat{\nabla}_\xi\eta,\zeta) + \mathcal{L}(\eta,\widehat{\nabla}_\xi\zeta) \\
&= \mathcal{L}(\nabla_\xi\eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi + \Upsilon^\natural(\mathcal{L}(\xi,\eta)),\zeta) \\
&\quad +\mathcal{L}(\eta,\nabla_\xi\zeta + \Upsilon(\xi)\zeta + \Upsilon(\zeta)\xi + \Upsilon^\natural(\mathcal{L}(\xi,\zeta))) \\
&= \nabla^Q_\xi(\mathcal{L}(\eta,\zeta)) + \Upsilon(\xi)\mathcal{L}(\eta,\zeta) + \Upsilon(\eta)\mathcal{L}(\xi,\zeta) + \mathcal{L}(\Upsilon^\natural(\mathcal{L}(\xi,\eta)),\zeta) \\
&\quad +\Upsilon(\xi)\mathcal{L}(\eta,\zeta) + \Upsilon(\zeta)\mathcal{L}(\eta,\xi) + \mathcal{L}(\eta,\Upsilon^\natural(\mathcal{L}(\xi,\zeta))) \\
&= \nabla^Q_\xi(\mathcal{L}(\eta,\zeta)) + 2\Upsilon(\xi)\mathcal{L}(\eta,\zeta) + \Upsilon(\eta)\mathcal{L}(\xi,\zeta) + \Upsilon(\zeta)\mathcal{L}(\eta,\xi) + \Upsilon(\zeta)\mathcal{L}(\xi,\eta) \\
&\quad -\Upsilon(\eta)\mathcal{L}(\xi,\zeta) \\
&= \nabla^Q_\xi\mathcal{L}(\eta,\zeta) + 2\Upsilon(\xi)\mathcal{L}(\eta,\zeta).
\end{aligned}$$

(We used the definition of the bundle homomorphism $\Upsilon^\natural$, the quotient connection ∇^Q and the antisymmetry and bilinearity of $\mathcal{L}$.) We can write this result (*transformation law for quotient connection*) in a more convenient way

$$\widehat{\nabla}^Q_\xi\psi - \nabla^Q_\xi\psi = 2\Upsilon(\xi)\psi$$

for all $\psi \in \Gamma(M,QM)$ and $\xi \in \Gamma(M,HM)$.

The next simple quantity besides the induced quotient connection ∇^Q which transforms changing a representant of a given projective class of contact connections is the bundle projection p. Let us write the definitions of $p = p^{\nabla,Q}$ and $\widehat{p} = p^{\widehat{\nabla},Q}$

$$\mathcal{L}(\xi,\pi(\eta)) = q([\xi,\eta]) - \nabla^Q_\xi q(\eta),$$
$$\mathcal{L}(\xi,\widehat{p}(\eta)) = q([\xi,\eta]) - \widehat{\nabla}^Q_\xi q(\eta)$$

for each $\xi \in \Gamma(M,HM)$ and $\eta \in \Gamma(M,TM)$. These two equations imply that $\mathcal{L}(\xi,(\widehat{p}-p)\eta) = \nabla^Q_\xi q(\eta) - \widehat{\nabla}^Q_\xi q(\eta)$

$$= -2\Upsilon(\xi)q(\eta)$$

$$
\begin{aligned}
&= -2\mathcal{L}(\Upsilon^\natural(q(\eta)),\xi) \\
&= \mathcal{L}(\xi, 2\Upsilon^\natural q(\eta)).
\end{aligned}
$$

As a result, we obtain the following *transformation law for the contact projection*

$$
\widehat{p}(\eta) - p(\eta) = 2\Upsilon^\natural q(\eta)
$$

for all $\eta \in \Gamma(M, TM)$.

4.2.3 Torsion of projective contact structure

Let us consider a mapping $T^\nabla : \Gamma(M, HM) \times \Gamma(M, HM) \to \Gamma(M, HM)$ associated to each contact structure $(M, HM, \mathcal{L})$ and contact connection ∇ given by the formula

$$
T^\nabla(\xi,\eta) = \nabla_\xi \eta - \nabla_\eta \xi - p([\xi,\eta])
$$

where $\xi, \eta \in \Gamma(M, HM)$ and $p : TM \to HM$ is a projection associated to the induced quotient connection ∇^Q. In what follows, we shall prove the

Theorem 4.2.2. *The torsion T^∇ of a contact structure $(M, HM, \mathcal{L})$ and a contact connection ∇ remains unchanged under a change of a representant ∇ of a projective class of contact connections $[\nabla]$, i.e., it is an invariant of a projective contact structure $(M, HM, \mathcal{L}, [\nabla])$.*

Proof. Let us take two projective equivalent contact connections $\widehat{\nabla}, \nabla$ and consider appropriate torsions $\widehat{T} = T^{\widehat{\nabla}}$, $T = T^\nabla$ associated to these connections. For $\xi, \eta \in \Gamma(M, HM)$ we can compute

$$
\begin{aligned}
\widehat{T}(\xi,\eta) &= \widehat{\nabla}_\xi \eta - \widehat{\nabla}_\eta \xi - \widehat{p}([\xi,\eta]) \\
&= \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi + \Upsilon^\natural(\mathcal{L}(\xi,\eta)) \\
&\quad -\nabla_\eta \xi - \Upsilon(\eta)\xi - \Upsilon(\xi)\eta - \Upsilon^\natural(\mathcal{L}(\eta,\xi)) - \widehat{p}([\xi,\eta]) \\
&= T(\xi,\eta) + 2\Upsilon^\natural(\mathcal{L}(\xi,\eta)) + p([\xi,\eta]) - \widehat{p}([\xi,\eta]) \\
&= T(\xi,\eta) + 2\Upsilon^\natural(\mathcal{L}(\xi,\eta)) - 2\Upsilon^\natural q([\xi,\eta]) \\
&= T(\xi,\eta),
\end{aligned}
$$

where we have used the definition of Levi form, torsion and the transformation law for the contact projection $p : TM \to HM$. □

This theorem also proves the correctness of the following definition. We shall call a projective contact geometry $(M, HM, \mathcal{L}, [\nabla])$ *torsion free*, if the torsion T of $(M, HM, \mathcal{L}, [\nabla])$ vanishes.

4.3 Homogeneous model of projective contact geometry

In this section we introduce the homogeneous model of projective contact geometry, the so called *projective contact sphere.*

Let us assume the canonical basis $\{\epsilon^i\}_{i=1}^{2k}$ of the vector space $(\mathbb{R}^{2k})^*$. The linear symplectic structure $(\mathbb{R}^{2k}, \omega)$, where the symplectic form $\omega := \sum_{i=1}^{k} \epsilon^i \wedge \epsilon^{i+k}$, is called *canonical linear symplectic structure* . Consider the tautological action of $Sp(2k, \mathbb{R})$ on the canonical linear symplectic structure $(\mathbb{R}^{2k}, \omega)$. This action restricts to $\mathcal{C} := \mathbb{R}^{2k} - \{0\}$ and factors to a transitive action on $\mathcal{C}/\mathbb{R}_+$ which can be identified with S^{2k-1}.

There is a map $\pi : \mathcal{C} \to S^{2k-1}$ associating to each vector in $\mathcal{C}$ a point of sphere the S^{2k-1} which is an intersection of this sphere S^{2k-1} and a half line $\{tx, t \in \mathbb{R}_+\}$ passing through the vector $x \in \mathcal{C}$, i.e.,

$$\pi(x) := \frac{x}{|x|}, x \in \mathcal{C}$$

where $| \quad |$ is the Euclidean norm on $\mathbb{R}^{2k}$. This map defines a principle $\mathbb{R}_+$-bundle over the base S^{2k-1}. We can form a family of associated bundles $\mathcal{E}[\alpha] = \mathcal{C} \times_{\rho_\alpha} \mathbb{R}$ to this principle bundle via a representation $\rho_\alpha : \mathbb{R}_+ \to Aut(\mathbb{R}) = \mathbb{R} - \{0\}$, $\rho_\alpha(t)x = t^{-\alpha}x$ for each $t \in \mathbb{R}_+$, $x, \alpha \in \mathbb{R}$. We shall consider the mapping

$$\pi_* : \mathfrak{X}(\mathcal{C}) \to \mathfrak{X}(S^{2k-1})$$

and restrict it to invariant vector fields to obtain a mapping $\pi_* : \mathfrak{X}(\mathcal{C})^{\mathbb{R}_+} \to \mathfrak{X}(S^{2k-1})$. We shall prove in the next lemma, that the kernel of this map consists of all vector fields of the form fE where f is an invariant function, i.e.,

$$f(tx) = f(x), t \in \mathbb{R}_+, x \in \mathcal{C}$$

and E is the *Euler vector field* defined by

$$E_x := \sum_{i=1}^{n} x^i (\frac{\partial}{\partial x^i})_x, x = (x^1, \dots, x^n) \in \mathcal{C},$$

where $n := 2k$.

Lemma 4.3.1 (Kernel of π_*). *The kernel*

$$Ker(\pi_*) = \{fE; f : \mathcal{C} \to \mathbb{R}, f(tx) = f(x), t \in \mathbb{R}_+, x \in \mathcal{C}\}.$$

Proof. At first, consider $X \in \mathfrak{X}(\mathcal{C})$ such that $\pi_*(X) = 0$, i.e., $\pi_*(X)g = X(g\pi)$ for each $g \in \mathcal{C}^\infty(S^{2k-1}, \mathbb{R})$. Consider the canonical coordinates x^i, $i = 1, \ldots, n$ on $\mathcal{C}$. We shall suppose, that the vector field X is in the following form $X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$. Writing the expression $0 = X(g\pi)$ in the coordinate way, we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}(g\pi) \\ &= \sum_{i=1}^n \sum_{j=1}^n f^i \frac{\partial g}{\partial x^j} \frac{\partial \pi^j}{\partial x^i}. \end{aligned}$$

Considering the last equation holds for each $g = x^l$, $l = 1, \ldots, n$, we obtain

$$\sum_{i=1}^n f^i \frac{\partial \pi^l}{\partial x^i} = 0.$$

We use the definition of π to obtain

$$\frac{\partial \pi^l}{\partial x^i} = \frac{\delta_i^l |x| - x^l x^i / |x|}{|x|^2}.$$

Substituting this to the previous expression and multiplying the result by $|x|^2$, we obtain a set of equations

$$f^l |x|^2 - (f, x) x^l = 0, l = 1, \ldots, n.$$

From these equations it follows that

$$f^l(x) = \frac{(f(x), x)}{|x|^2} x^l, x \in \mathcal{C}.$$

Thus

$$X = \frac{(f, x)}{|x|^2} \sum_{l=1}^n x^l \frac{\partial}{\partial x^l} = \psi E,$$

for a function $\psi : \mathcal{C} \to \mathbb{R}$.

If we want to compute the kernel of the restricted mapping $\pi_* : \mathfrak{X}(\mathcal{C})^{\mathbb{R}_+} \to \mathfrak{X}(S^{2k-1})$, it is sufficient to use the previous computation with the restriction on the function $\psi(tx) = \psi(x)$. $\square$

For later use, we can write the function ψ in a form $\psi = \Upsilon(\xi)\pi$, where $\Upsilon \in \Omega^1(S^{2k-1})$ is a differential 1-form on the sphere and $\xi \in \mathfrak{X}(S^{2k-1})$.

4.3.1 Contact structure on the sphere S^{2k-1}

In this subsection we shall define a contact structure on the sphere S^{2k-1}, the so called *contact sphere.* We can define a mapping $q^\sigma : \mathfrak{X}(S^{2k-1}) \to \mathcal{C}^\infty(\mathcal{C}, \mathbb{R})$ by the formula

$$q^\sigma(\xi) := \omega(\sigma(\xi), E)$$

where $\xi \in \mathfrak{X}(S^{2k-1})$ and σ is a section of the map $\pi_* : \mathfrak{X}(C)^{\mathbb{R}_+} \to \mathfrak{X}(S^{2k-1})$, where $\mathfrak{X}(\mathcal{C})^{\mathbb{R}_+}$ denotes the vector space of $\mathbb{R}_+$-invariant vector fields on $\mathcal{C}$.

Theorem 4.3.1. *The triple $(S^{2k-1}, H, \mathcal{L})$ where $H := Ker\ (q^\sigma)$ and $\mathcal{L}$ is the Levi form is a contact structure independent of the choice of section σ.*

Proof. At first, we check that the definition is independent of the choice of the section $\sigma : \mathfrak{X}(S^{2k-1}) \to \mathfrak{X}(\mathcal{C})$ and that q is a bundle morphism. Then we check that the subbundle H is a contact subbundle. Having two sections σ, σ' of π_*, we know that $\sigma'(\xi) - \sigma(\xi)$ is in kernel of π_* thus $\sigma'(\xi) - \sigma(\xi) = fE$ for some invariant function f, see the lemma 4.3.1. As a result, we get $q^{\sigma'}(\xi) = \omega(\sigma'(\xi), E) = \omega(\sigma(\xi) + fE, E) = \omega(\sigma(\xi), E) = q^\sigma(\xi)$, using the antisymmetry of ω.

It is easy to verify that $\xi \mapsto q(\xi)$ is linear over $\mathcal{C}^\infty(S^{2k-1}, \mathbb{R})$ for $\xi \in \mathfrak{X}(S^{2k-1})$. Indeed, $f\xi \mapsto \omega(\sigma(f\xi), E) = \omega(f\sigma(\xi), E) = f\omega(\sigma(\xi), E)$ for each $f \in \mathcal{C}^\infty(S^{2k-1}, \mathbb{R})$. We know that $q(\xi)$ is a homogeneous function of degree two on $\mathcal{C}$. These functions correspond to sections of $\mathcal{E}[2]$ over S^{2k-1}. As a result we obtain that $q : TS^{2k-1} \to \mathcal{E}[2]$ is a bundle morphism covering the identity on S^{2k-1} and thus its kernel H is a smooth subbundle.

To show that H is a contact subbundle, we need to show that the Levi form $\mathcal{L} : H \times H \to \mathcal{E}[2]$ defined by $\mathcal{L}(\xi, \eta) = q([\xi, \eta])$ is nondegenerate.[2] This map equals to a mapping $\omega([\sigma(\xi), \sigma(\eta)], E)$. We know that the symplectic form is closed, thus we can write

$$\begin{aligned} 0 &= d\omega(\sigma(\xi), \sigma(\eta), E) = \\ &= \sigma(\xi)\omega(\sigma(\eta), E) - \sigma(\eta)\omega(\sigma(\xi), E) \end{aligned}$$

[2]Actually, the Levi form is defined by $\mathcal{L}(\xi, \eta) = q'(q([\xi, \eta]))$ where q' is the inverse of the bundle isomorphism $TM/H \to \mathcal{E}[2]$ as one easily checks. But for our purposes it is sufficient to consider $\mathcal{L}$ defined above.

$$
\begin{aligned}
&+ \quad E\omega(\sigma(\xi),\sigma(\eta)) - \omega([\sigma(\xi),\sigma(\eta)],E)\\
&+ \quad \omega([\sigma(\xi),E],\sigma(\eta)) - \omega([\sigma(\eta),E],\sigma(\xi)).
\end{aligned}
$$

It is easy to verify that the first two and the last two terms are zero. The first two are zero because $\xi,\eta \in H$ and thus $\omega(\sigma(\xi),E) = \omega(\sigma(\eta),E) = 0$. For the last two terms, consider the following computation $[E,\sigma(\xi)] = \mathcal{L}_E(\sigma(\xi)) = \frac{d}{dt}_{t=0} Fl_E^{t*}\sigma(\xi) = \frac{d}{dt}_{t=0} r^{e^t*}\sigma(\xi) = \frac{d}{dt}_{t=0}\sigma(\xi) = 0$. The last but one equation is a consequence of the invariance of $\sigma(\xi)$ which can be seen as follows. The invariance of $\sigma(\xi)$ is expressed by

$$
\sigma(\xi)_{r^t\sigma(u)} = (Tr^t)_{\sigma(u)} T\sigma_u \xi_u.
$$

This equation implies that

$$
r^{t*}\sigma(\xi)_{\sigma(u)} = (Tr^t)^{-1}_{\sigma(u)}\sigma(\xi)_{r^t\sigma(u)} = T\sigma_u\xi_u = \sigma(\xi)_{\sigma(u)},
$$

for each $t \in \mathbb{R}_+, u \in S^{2k-1}$.

Thus

$$
\omega([\sigma(\xi),\sigma(\eta)],E) \quad = \quad E\omega(\sigma(\xi),\sigma(\eta)). \tag{4.1}
$$

Because of the degree two homogeneity of $\omega(\sigma(\xi),\sigma(\eta))$, we obtain that the action of Euler vector field on it is equal $2\omega(\sigma(\xi),\sigma(\eta))$. But the symplectic form is nondegenerate and therefore $\mathcal{L}$ is nondegenerate too and thus H is a contact subbundle. □

The triple $(S^{2k-1}, H, \mathcal{L})$ is called *contact sphere*.

4.3.2 Projective contact structure on the sphere S^{2k-1}

In the last subsection we have defined the contact structure $(S^{2k-1}, H, \mathcal{L})$ on the sphere S^{2k-1}. In this subsection we define a projective contact structure on the contact sphere. Choose a section σ of the mapping π_*. Take $\xi,\eta \in H$. We define a partial connection

$$
\nabla^\sigma_\xi \eta := p_\sigma(\pi_*(\nabla_{\sigma(\xi)}\sigma(\eta)))
$$

where ∇ is the flat connection on $\mathbb{R}^{2k}$ and p_σ is a projection $p_\sigma : TS^{2k-1} \to H$ defined in the following way. Choose a $\theta_\sigma \in \mathfrak{X}(\mathcal{C})$ defined by the equations

$\omega(\theta_\sigma, E) = 1$ and $\omega(\theta_\sigma, \sigma(\xi)) = 0$ for any $\xi \in \mathfrak{X}(S^{2k-1})$. It can be checked that $\sigma(\xi) - \omega(\sigma(\xi), E)\theta_\sigma$ is an invariant vector field on $\mathcal{C}$, thus it descends to a vector field on S^{2k-1}. This vector field will be denoted by $p_\sigma(\xi)$.

To show that p_σ is a projection onto the contact subbundle H, we make following computations. First, take a $\xi \in \mathfrak{X}(S^{2k-1})$ and assume

$$\begin{aligned} q^\sigma p_\sigma(\xi) &= \omega(\sigma(p_\sigma(\xi)), E) \\ &= \omega(\sigma(\xi) - \omega(\sigma(\xi), E)\theta_\sigma, E) \\ &= \omega(\sigma(\xi), E) - \omega(\sigma(\xi), E)\omega(\theta_\sigma, E) = 0, \end{aligned}$$

i.e., $p_\sigma(\xi) \in H$.

Second, let us compute

$$\begin{aligned} p_\sigma(p_\sigma(\xi)) &= p_\sigma(\sigma(\xi) - \omega(\sigma(\xi), E)\theta_\sigma) \\ &= p_\sigma(\sigma(\xi)) - \omega(\sigma(\xi), E)p_\sigma(\theta_\sigma) \\ &= p_\sigma(\xi) - \omega(\sigma(\xi), E)(\theta_\sigma - \omega(\theta_\sigma, E)\theta_\sigma) \\ &= p_\sigma(\xi), \end{aligned}$$

i.e., p_σ is an idempotent.

Third, take a $\xi \in H$ and consider the vector field $\sigma(\xi) - \omega(\sigma(\xi), E)\theta_\sigma = \sigma(\xi) - q(\xi)\theta_\sigma = \sigma(\xi)$, because $H = \mathrm{Ker}\ (q)$. Thus we have proved that p_σ is a surjective (onto H) idempotent of Hom (TS^{2k-1}, H), i.e., a projection $p_\sigma : TS^{2k-1} \to H$

It is an easy exercise to compute that

$$p_{\sigma'} - p_\sigma = 2\Upsilon^\natural q,$$

where $\sigma' - \sigma = \Upsilon \pi E$. Indeed, take a $\xi \in \mathfrak{X}(S^{2k-1})$ and compute

$$\begin{aligned} p_{\sigma'}(\xi) - p_\sigma(\xi) &= \omega(\sigma'(\xi), E)\theta_{\sigma'} - \omega(\sigma(\xi), E)\theta_\sigma \\ &= \omega(\sigma(\xi) + \Upsilon(\xi)\pi E, E)\theta_{\sigma'} - \omega(\sigma(\xi), E)\theta_\sigma \\ &= \omega(\sigma(\xi), E)(\theta_{\sigma'} - \theta_\sigma) \\ &= q^\sigma(\xi)(\theta_{\sigma'} - \theta_\sigma). \end{aligned}$$

We need to prove that $q(\xi)(\theta_{\sigma'} - \theta_\sigma) = 2\Upsilon^\natural q(\xi)$. To proceed take a $\zeta \in \mathfrak{X}(S^{2k-1})$ and consider

$$\begin{aligned}
&\mathcal{L}(q(\xi)(\theta_{\sigma'} - \theta_\sigma), \sigma(\zeta)) = \\
&q(\xi)[\mathcal{L}(\theta_{\sigma'}, \sigma'(\zeta) - \Upsilon(\zeta)\pi E) - \mathcal{L}(\theta_\sigma, \sigma(\zeta))] = \\
&q(\xi)\omega([\theta_{\sigma'}, \sigma'(\zeta) - \Upsilon(\zeta)\pi E], E) - \omega([\theta_\sigma, \sigma(\zeta)], E).
\end{aligned}$$

Using the equation 4.1, we obtain

$$\begin{aligned}
&q(\xi)[E.\omega(\theta_{\sigma'}, \sigma'(\zeta)) - E.\omega(\theta_\sigma, \sigma(\zeta)) - \Upsilon(\zeta)\pi E.\omega(\theta_{\sigma'}, E)] = \\
&q(\xi)[2\omega(\theta_{\sigma'}, \sigma'(\zeta)) - 2\omega(\theta_\sigma, \sigma(\zeta)) - \Upsilon(\zeta)\pi 2\omega(\theta_{\sigma'}, E)] = \\
&-2\Upsilon(\zeta)\pi q(\xi) = \\
&\mathcal{L}(2\Upsilon^\natural q(\xi), \sigma(\zeta)).
\end{aligned}$$

The desired equation follows by the nondegenracy of the Levi form.

Let us take a section σ' and compute $\nabla^{\sigma'}$. As we know for such an element $\sigma'(\xi) = \sigma(\xi) + \Upsilon(\xi)\pi E$ holds, thus we can write

$$\begin{aligned}
\nabla^{\sigma'}_\xi \eta &= p_{\sigma'}(\pi_* \nabla_{\sigma'(\xi)} \sigma'(\eta)) \\
&= p_\sigma(\pi_* \nabla_{\sigma'(\xi)} \sigma'(\eta)) + 2\Upsilon^\natural q(\pi_* \nabla_{\sigma'(\xi)} \sigma'(\eta)).
\end{aligned}$$

At first we compute the first summand.

$$\begin{aligned}
p_\sigma \pi_*(\nabla_{\sigma'(\xi)} \sigma'(\eta)) &= p_\sigma\{\pi_*[\nabla_{\sigma(\xi)}(\sigma(\eta) + \Upsilon(\eta)\pi E) \\
&\quad + \nabla_{\Upsilon(\xi)\pi E}(\sigma(\eta) + \Upsilon(\eta)\pi E)]\} \\
&= p_\sigma\{\pi_*[\nabla_{\sigma(\xi)}\sigma(\eta) + \sigma(\xi).(\Upsilon(\eta)\pi)E + \Upsilon(\eta)\pi\nabla_{\sigma(\xi)}E \\
&\quad + \Upsilon(\xi)\pi\nabla_E(\sigma(\eta) + \Upsilon(\eta)\pi E)]\} \\
&= p_\sigma[\pi_*(\nabla_{\sigma(\xi)}\sigma(\eta))] + p_\sigma\{\pi_*[\sigma(\xi).(\Upsilon(\eta)\pi)E \\
&\quad + \Upsilon(\eta)\pi\nabla_{\sigma(\xi)}E + \Upsilon(\xi)\pi\nabla_E\sigma(\eta) \\
&\quad + \Upsilon(\xi)\pi E.(\Upsilon(\eta)\pi)E + (\Upsilon(\xi)\pi)(\Upsilon(\eta)\pi)\nabla_E E]\}.
\end{aligned}$$

Because of the flatness of the connection ∇, one can easily compute, that

$$\begin{aligned}
\nabla_{\sigma(\xi)}E &= \sigma(\xi) \\
\nabla_E\sigma(\eta) &= \nabla_{\sigma(\eta)}E + [E, \sigma(\eta)] = \sigma(\eta) \\
\nabla_E E &= E.
\end{aligned}$$

Substituting these three equations in the previous one and applying the mapping π_* (which kills the Euler vector field E), one obtains

$$p_\sigma \pi_*(\nabla_{\sigma'(\xi)}\sigma'(\eta)) = \nabla^\sigma_\xi \eta + \Upsilon(\eta)\xi + \Upsilon(\xi)\eta.$$

At second we compute the second summand.

$$\begin{aligned} 2\Upsilon^\natural q(\pi_* \nabla_{\sigma'(\xi)}\sigma'(\eta)) &= 2\Upsilon^\natural q(\nabla_{\sigma(\xi)}\sigma(\eta) + \Upsilon(\eta)\xi + \Upsilon(\xi)\eta) \\ &= 2\Upsilon^\natural q \nabla_{\sigma(\xi)}\sigma(\eta) = 2\Upsilon^\natural \omega(\nabla_{\sigma(\xi)}\sigma(\eta), E). \end{aligned}$$

Now, we use the fact that ω is covariantly constant, thus we can rewrite the last term as

$$\begin{aligned} -2\Upsilon^\natural \omega(\sigma(\eta), \nabla_{\sigma(\xi)}E) &= -2\Upsilon^\natural \omega(\sigma(\eta), \sigma(\xi)) \\ &= 2\frac{1}{2}\Upsilon^\natural \omega([\sigma(\xi), \sigma(\eta)], E) \\ &= \Upsilon^\natural \mathcal{L}(\xi, \eta) \end{aligned}$$

The second last equation is a consequence of that fact $2\omega(\sigma(\xi), \sigma(\eta)) = \omega([\sigma(\xi), \sigma(\eta)], E)$ which was derived when we have computed the nondegeneracy of the Levi form above, see the equation (4.1).

As a result of these two computations, we realized that we have obtained the relation of projective contact equivalence.

$$\nabla^{\sigma'}_\xi \eta = \nabla^\sigma_\xi \eta + \Upsilon(\eta)\xi + \Upsilon(\xi)\eta + \Upsilon^\natural \mathcal{L}(\xi, \eta)$$

Thus the choice of a section σ corresponds to a choice of projective contact representant within a projective class of connections.

Summing-up this section, we have proved that S^{2k-1} posseses a projective contact structure. At the beginning of this section, we have introduced the sphere S^{2k-1} as a homogeneous space G/P, where $G = Sp(2k, \mathbb{R})$ and P is the subgroup of G fixing an open ray in $\mathcal{C}$. It can be checked that the subgroup P is a parabolic subgroup of G and therefore $(G, S^{2k-1}, G, P, \omega)$, where ω is the Maurer-Cartan form of G, is a parabolic geometry. This leads to the definition

Definition 4.3.1. Let $G = Sp(2k, \mathbb{R})$ and P be the parabolic subgroup of G fixing an open ray in $\mathcal{C}$. Then the parabolic geometry of type (G, P) is called *projective contact geometry*.

4.4 Bernstein-Gelfand-Gelfand resolution for projective contact geometry

Contact graded symplectic algebra

Recall that a contact graded Lie algebra is a special case of a $|2|$-graded Lie algebra, see the section 2.1.

In our case of the contact grading of the algebra $\mathfrak{g} = C_k = \mathfrak{sp}(2k, \mathbb{C})$, the corresponding set $\Sigma = \{\alpha_1\}$, see Yamaguchi [31] or the section 2.1). Thus the grading is given by the pair $(C_k, \{\alpha_1\})$. In the cited work of Yamagouchi [31], the elements of the contact grading of $\mathfrak{sp}(2k, \mathbb{C})$ are written. They have the following form.

(1) $\mathfrak{g}_0 = \mathfrak{sp}(2k-2, \mathbb{C}) \oplus \mathbb{C}E$ where E is the grading element uniquely associated to the grading,

(2) $\mathfrak{g}_{-1} = \mathbb{C}^{2k-2}$,

(3) $\mathfrak{g}_{-2} = \mathbb{C}$.

Their placement in a matrix A representing an element of $\mathfrak{g}$ is displayed bellow.

$$A = \left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right)$$

To fix a notation, let us recall some basic facts on the symplectic algebra $\mathfrak{g} = \mathfrak{sp}(2k, \mathbb{C})$. Let $\{\epsilon_i\}_{i=1}^k$ be the standard basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is considered to consist of all diagonal matrices in $\mathfrak{sp}(2k, \mathbb{C})$. The set of all positive roots is $\Phi^+ = \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, 2\epsilon_i; 1 \le i \le j \le k\}$. The set of all positive roots which root spaces are in $\mathfrak{p}_+$ is $\Phi_{\mathfrak{p}_+} = \{\epsilon_1 - \epsilon_i, \epsilon_1 + \epsilon_i, 2\epsilon_1; 2 \le i \le k\}$. The corresponding root spaces of the roots in $\Phi_{\mathfrak{p}_+}$ are placed in the first row of a matrix $A \in \mathfrak{sp}(2k, \mathbb{C})$ as it is shown in the next picture (the star denote an empty space).

$\star$	$\epsilon_1 - \epsilon_2$	$\epsilon_1 - \epsilon_3$	$\ldots$	$\epsilon_1 - \epsilon_k$	$\epsilon_1 + \epsilon_k$	$\epsilon_1 + \epsilon_{k-1}$	$\ldots$	$\epsilon_1 + \epsilon_2$	$2\epsilon_1$

The fundamental weights for the symplectic algebra C_k are given by $\varpi_i = \sum_{j=1}^{i} \epsilon_j$, $i = 1, \ldots, k$.

The structure of saturated sets for $(\mathfrak{g}, \mathfrak{p})$ is in the case of contact graded symplectic algebras simple (compared with the structure of saturated sets for contact graded orthogonal algebras, for example).

Saturated sets and Hasse diagram

Theorem 4.4.1 (Saturated sets for $(C_k, \{\alpha_1\})$). *$A \subseteq \Phi_{\mathfrak{p}_+}$ is a saturated set for pair $(C_k, \{\alpha_1\})$ if and only if it is one of the following types*

(1) $A = \emptyset$;

(2) $A_i = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_1 - \epsilon_i\}$, $i = 2, \dots, k$;

(3) $A_{k+1} = A_k \cup \{2\epsilon_1\}$;

(4) $A_i = A_{k+1} \cup \{\epsilon_1 + \epsilon_k, \dots, \epsilon_1 + \epsilon_{2k-i+2}\}$, $i = k+2, \dots, 2k$.

Proof. First, we shall show that each of the sets written above is a saturated set, i.e., that conditions (R1) and (R2) in theorem 2.3.1 are satisfied. To check (R1), consider $\epsilon_1 - \epsilon_j, \epsilon_1 - \epsilon_l \in A_i$, $i = 2, \dots, k$ then $(\epsilon_1 - \epsilon_j) + (\epsilon_1 - \epsilon_l) \in \Phi$ is an empty condition, thus $A_i, i = 2, \dots, k$ is a saturated set. The same is true for A_{k+1}. To check (R1) for A_i, $i = k+2, \dots, 2k$, consider $(\epsilon_1 - \epsilon_j) + (\epsilon_1 + \epsilon_l)$. This element is in Φ if and only if $j = l$. In this case this element equals $2\epsilon_1$, which is an element of A_i, $i = k+2, \dots, 2k$, so again the condition (R1) is satisfied. To check (R2) it necessary to decompose each element into a sum of positive roots. $\epsilon_1 - \epsilon_l = (\epsilon_1 - \epsilon_j) + (\epsilon_j - \epsilon_l)$ where $2 \leq j \leq l \leq k$. Due to the construction of A_i we know that if $\epsilon_1 - \epsilon_l \in A_i$ then $\epsilon_1 - \epsilon_j \in A_i$ for all $2 \leq j \leq l$ thus the condition (R2) is satisfied. The next decomposition is $\epsilon_1 + \epsilon_l = (\epsilon_1 - \epsilon_j) + (\epsilon_j + \epsilon_l)$ where $2 \leq j \leq l \leq k$. If $\epsilon_1 + \epsilon_l \in A_i$, $i = k+2, \dots, 2k$ then $\epsilon_1 - \epsilon_j \in A_i$, thus (R2) is satisfied. The last decomposition to be considered is $2\epsilon_1 = (\epsilon_1 - \epsilon_j) + (\epsilon_1 + \epsilon_j)$. Again from the construction of A_i we know that if $2\epsilon_1 \in A_i$ then $\epsilon_1 - \epsilon_j \in A_i$, $i = k+1, \dots, 2k$.

Second, we shall prove that the sets written in the statement of this theorem build all saturated sets for $(C_k, \{\alpha_1\})$. The empty set is a saturated set. If $A \neq \emptyset$ then there is a root $\alpha \in \Phi_{\mathfrak{p}_+}$ which is an element of A.

(1) If $\alpha = \epsilon_1 + \epsilon_j \in A$ then because of the decomposition $\epsilon_1 + \epsilon_j = (\epsilon_1 + \epsilon_l) + (\epsilon_j - \epsilon_l)$, $2 \leq j \leq l \leq k$ we know that $\epsilon_1 + \epsilon_l \in A$, $2 \leq j \leq l \leq k$. The second decomposition $\epsilon_1 + \epsilon_j = (\epsilon_1 - \epsilon_j) + 2\epsilon_j$ for $2 \leq j \leq k$ shows that $\epsilon_1 - \epsilon_j \in A$, $2 \leq j \leq k$ due to the condition (R2). Due to the (R1), we obtain $2\epsilon_1 = \epsilon_1 - \epsilon_l + \epsilon_1 + \epsilon_l$, $2 \leq l \leq k$, i.e., that $2\epsilon_1 \in A$. Thus A has the shape A_i, $i = k+2, \ldots, 2k$.

(2) If $\alpha = 2\epsilon_1 \in A$, then the decomposition $2\epsilon_1 = (\epsilon_1 + \epsilon_j) + (\epsilon_1 - \epsilon_j)$ shows that $\epsilon_1 + \epsilon_j \in A$ or $\epsilon_1 - \epsilon_j \in A$ for $2 \leq j \leq k$.

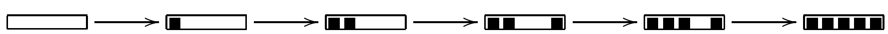
   (2.1) If $\epsilon_1 + \epsilon_j \in A$ we achieve the previous case.

   (2.2) If $\epsilon_1 - \epsilon_j$, and $\epsilon_1 + \epsilon_j \notin A$, $2 \leq j \leq k$ we get that $A$is of the shape A_{k+1}.

(3) If $\alpha = \epsilon_1 - \epsilon_j \in A$ then $\epsilon_1 - \epsilon_j = (\epsilon_1 - \epsilon_l) + (\epsilon_l - \epsilon_j)$, $2 \leq l \leq j \leq k$ shows that $\epsilon_1 - \epsilon_l \in A$ for $2 \leq l \leq j \leq k$, thus A is of the shape A_i for $2 \leq i \leq k$.

□

Example: The Hasse diagram for $(\mathfrak{sp}(6, \mathbb{C}), \{\alpha_1\})$ is of the following shape:



Each vertex of this diagram represents the first row of a matrix in $\mathfrak{sp}(6, \mathbb{C})$. There is a black box in the previous Hasse diagram if and only if the root corresponding to a (in such way marked) root space belongs to the saturated set (see the previous picture of displacement of root spaces in the matrix).

In the next theorem, we prove that the previous example is typical, i.e., that the shape of the Hasse diagram is similar to the shape of the Hasse diagram written in the previous example.

Theorem 4.4.2 (Hasse diagram for $(C_k, \{\alpha_1\})$). *There are exactly following labelled arrows in the Hasse diagram for the pair $(C_k, \{\alpha_1\})$:*

$$A_i \overset{\epsilon_1 - \epsilon_{i+1}}{\longrightarrow} A_{i+1}, i = 2, \ldots, k;$$

$$A_k \overset{2\epsilon_1}{\longrightarrow} A_{k+1};$$

$$A_i \overset{\epsilon_1 + \epsilon_{2k-i+1}}{\longrightarrow} A_{i+1}, i = k+1, \ldots, 2k-1.$$

Proof. We know from the lemma 2.3.1 on arrows in Hasse diagram, that we need to compute $|A_i| - |A_j|$ for all $2 \leq i \leq j \leq 2k$.

First, take $2 \leq i < j \leq k$. In this case, $|A_j| - |A_i| = |\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_1 - \epsilon_j\}| - |\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_1 - \epsilon_i\}| = |\{\epsilon_1 - \epsilon_{i+1}, \ldots, \epsilon_1 - \epsilon_j\}| = (j-i)\epsilon_1 - (\epsilon_{i+1} + \ldots + \epsilon_j)$. This element is a multiple of a positive root the corresponding root space of which is in the parabolic part, i.e., it belongs to $\Phi_{\mathfrak{p}_+}$ if and only if $j = i+1$.

Second, consider $2 \leq i \leq k$ and the difference $|A_{k+1}| - |A_i| = |\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_1 - \epsilon_k, 2\epsilon_1\}| - |\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_1 - \epsilon_i\}| = |\{\epsilon_1 - \epsilon_{i+1}, \ldots, \epsilon_1 - \epsilon_k, 2\epsilon_1\}| = (k-i+2)\epsilon_1 - (\epsilon_{i+1} + \ldots + \epsilon_k)$. The only possible relation between i and k to make this difference a multiple of a root of $\Phi_{\mathfrak{p}_+}$ is $i = k$. In this case the difference is $1.(2\epsilon_1)$.

Third, consider $2 \leq i \leq k < k+1 < j \leq 2k$. In this case, it is easy to obtain that the difference cannot be a multiple of a root in $\Phi_{\mathfrak{p}_+}$ for any choice of i, j satisfying the previous relation.

Fourth, $k+2 \leq j \leq 2k$. The difference equals $|\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_1 - \epsilon_k, 2\epsilon_1, \epsilon_1 + \epsilon_k, \ldots, \epsilon_1 + \epsilon_{2k-i+2}\}| - |\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_1 - \epsilon_k, 2\epsilon_1\}| = |\{\epsilon_1 + \epsilon_k, \ldots, \epsilon_1 - \epsilon_{2k-i+2}\}| = (k-i+3)\epsilon_1 + (\epsilon_k + \epsilon_{2k-i+2})$. The last written expression is a multiple of an element of $\Phi_{\mathfrak{p}_+}$ if and only if $i = k+2$. In this case, the difference equals $\epsilon_1 + \epsilon_k$.

Fifth, consider $k+2 \leq i < j \leq 2k$ in this last case, we obtain a similar result as in the first case, i.e., the difference is acceptable if and only if $j = i+1$. In this case the difference equals $\epsilon_1 - \epsilon_{2k-i+1}$. $\square$

Bernstein-Gelfand-Gelfand diagram for contact graded symplectic algebra

In this paragraph we compute the vertices of the BGG diagram for a given $\mathfrak{g}$-dominant weight. See the remark 2.3.1 for comments on the computation of a BGG diagram.

Example: Let us compute the BGG diagram for $(\mathfrak{sp}(8, \mathbb{C}), \{\alpha_1\})$ and a dominant integral weight $\lambda = \sum_{i=1}^{4} \lambda_i \varpi_i$. An explicit computation gives the following BGG diagram written in the basis of dominant weights.

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \xrightarrow{\epsilon_1 - \epsilon_2} (-\lambda_1, \lambda_1 + \lambda_2, \lambda_3, \lambda_4) \xrightarrow{\epsilon_1 - \epsilon_3} (-\lambda_1 - \lambda_2, \lambda_1, \lambda_2 + \lambda_3, \lambda_4) \xrightarrow{\epsilon_1 - \epsilon_4}$$

$$(-\lambda_1 - \lambda_2 - \lambda_3, \lambda_1, \lambda_2, \lambda_3 + \lambda_4) \xrightarrow{2\epsilon_1} (-\lambda_1 - \lambda_2 - \lambda_3 - 2\lambda_4, \lambda_1, \lambda_2, \lambda_3 + \lambda_4) \xrightarrow{\epsilon_1 + \epsilon_4}$$

$$(-\lambda_1-\lambda_2-2\lambda_3-2\lambda_4,\lambda_1,\lambda_2+\lambda_3,\lambda_4) \stackrel{\epsilon_1+\epsilon_3}{\longrightarrow} (-\lambda_1-2\lambda_2-2\lambda_3-2\lambda_4,\lambda_1+\lambda_2,\lambda_3,\lambda_4)$$
$$\stackrel{\epsilon_1+\epsilon_2}{\longrightarrow} (-\lambda_1-2\lambda_2-2\lambda_3-2\lambda_4,\lambda_2,\lambda_3,\lambda_4).$$

Let us denote the vertex in BGG diagram for $(C_k, \{\alpha_1\})$ and a weight λ corresponding to the set A_i, $i = 2, \ldots, 2k$ in the (isomorphic) Hasse diagram by $A_i(\lambda)$. Let us define following vectors written in the coordinates of the basis of the fundamental weights.

$$\gamma_j := (-\sum_{i=1}^{j-1}\lambda_i, \lambda_1, \ldots, \lambda_{j-2}, \lambda_{j-1}+\lambda_j, \lambda_{j+1}, \ldots, \lambda_k), \quad j = 1, \ldots, k;$$

$$\gamma_j := (-\sum_{i=1}^{k}\lambda_i - \sum_{i=2k-j+1}^{k}\lambda_i, \lambda_1, \ldots, \lambda_{2k-j-1}, \lambda_{2k-j}+\lambda_{2k-j+1}, \lambda_{2k-j+2}, \ldots, \lambda_k),$$
$j = k+1, \ldots, 2k$.

The last written vector for $j = 2k$ is to be understood as equal to $(-\lambda_1 - 2\sum_{i=2}^{k}\lambda_i, \lambda_1, \ldots, \lambda_k)$.

Theorem 4.4.3 (BGG diagram for $(C_k, \{\alpha_1\}, \lambda)$). *The BGG diagram for $(C_k, \{\alpha_1\})$ and an integral dominant weight λ for C_k has the following vertices written in the basis of fundamental weights:*

$$A_i(\lambda) = \gamma_i, \quad i = 1, \ldots, 2k.$$

There is an oriented labelled arrow between two vertices $A_i(\lambda)$ and $A_j(\lambda)$ if and only if $j = i+1$, $i = 1, \ldots, 2k-1$ and $j = 2, \ldots, 2k$. The labels of oriented arrows are the same as those in the Hasse diagram.

Proof. We prove the statement only for $i = 1, \ldots, k$. A similar proof can be done for $i = k+1, \ldots, 2k$. We prove the statement by induction on i. I. is immediately. II. Suppose we have proved the statement for i. Consider

$$\begin{aligned} A_{i+1}(\lambda) &= A_i(\lambda) - 2\frac{(A_i(\lambda), \epsilon_1-\epsilon_{i+1})}{(\epsilon_1-\epsilon_{i+1}, \epsilon_1-\epsilon_{i+1})}(\epsilon_1-\epsilon_{i+1}) \\ &= \gamma_i - 2\frac{(\gamma_i, \epsilon_1-\epsilon_{i+1})}{2}(\epsilon_1-\epsilon_{i+1}) \\ &= \gamma_i - (\gamma_i, \epsilon_1-\epsilon_{i+1})(\epsilon_1+\epsilon_{i+1}) \end{aligned}$$

After a short computation (using only the prescriptions for γ_i), one obtains that $(\gamma_i, \epsilon_1-\epsilon_{i+1}) = \lambda_i$. Substituting $\epsilon_1-\epsilon_{i+1} = \varpi_1-\varpi_{i+1}+\varpi_i$ one immediately obtains $A_{i+1}(\lambda) = \gamma_{i+1}$ what we have had to compute. $\square$

4.4.1 BGG diagram for the contact graded real form CI_k

The only real form of C_k admitting the contact grading is the split real form CI_k denoted by the Satake diagram in which all of the nodes are white, see Yamaguchi [31] and 2.1.

Let $\mathbb{V}$ be a real representation of CI_k, we call it *real representation with the highest weight* λ if its complexification contains a complex irreducible representation with the highest weight λ. This definition is correct, see Zhida, Dagan [32] and subsection 2.2.2 for comments. The Maltsev height $m(\lambda)$ of a complex representation with the highest weight $\lambda = (\lambda_1, \ldots, \lambda_l)$ is equal to

$$m(\lambda) = \sum i(2l - i)\lambda_i,$$

where the sum goes over all black nodes in the Satake diagram of a real form of C_k, see Goodman, Wallach [14] (5.1.8 Exercises, where the Maltsev height is computed for the compact form and is denoted by h^0 there). For the split real case, where all the nodes are white, we conclude that the Maltsev height $m(\lambda) = 0$. Next, we know that the weighted Satake diagram of CI_k is symmetric because there are no (nontrivial) arrows; thus the symmetry $s\nu = id$. Hence we know that all such representation are of real type, due to the theorem 2.2.3. This means in particular that their complexification is an irreducible $CI_k \otimes \mathbb{C}$-module. Using that we obtain a

Corollary(BGG diagram for $(CI_k, \{\alpha_1\}, \lambda)$**).:** The BGG diagram for $(CI_k, \{\alpha_1\}, \lambda)$ looks the same as the BGG diagram for $(C_k, \{\alpha_1\}, \lambda)$.

Proof. It is only necessary to remark that the complexification of each node of the diagram gives an irreducible representation, i.e., we get the same diagram as in the complex case.□

Chapter 5

Contact odd orthogonal geometry

The geometrical structure of contact orthogonal geometries of odd rank is well known, the homogeneous model of these geometries are isotropic Grassmannians, see Yamaguchi [31] for example. Therefore we omit the treatment of geometrical structures and begin with the algebraic and combinatorial aspects related to these geometries like BGG and Hasse diagrams.

5.1 Bernstein-Gelfand-Gelfand resolution for contact odd orthogonal geometry

5.1.1 Contact graded orthogonal algebra of odd rank

To fix a notation, let us recall some basic facts on the structure theory of odd dimensional orthogonal algebras $B_l = \mathfrak{so}(2l+1, \mathbb{C})$. Let us choose an arbitrary Cartan subalgebra $\mathfrak{h}$ of the algebra B_l. Let the set $\{\varepsilon_1, ..., \varepsilon_l\}$ be the standard basis of the dual of $\mathfrak{h}$. We assume that this is an orthogonal basis with respect to the Killing-Cartan form $(,) : \mathfrak{h} \to \mathbb{C}$. Then it is possible to choose the system of simple roots in the following way:

(1) $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, l-1,$

(2) $\alpha_l = \varepsilon_l.$

Now, we give a description of the root spaces of the parabolic subalgebra associated to the contact graded algebra $(B_l, \{\alpha_2\})$ in terms of matrices.

We shall use following symbols: let $i-j$, k, $i+j$ denote the roots $\varepsilon_i - \varepsilon_j$, ε_k, $\varepsilon_i + \varepsilon_j$ respectively for suitable i, j, k.

It is possible to choose the defining bilinear form of the algebra B_l in such a way that the root spaces are placed in the matrix as it is shown at the following picture:

$1-3$	$1-4$	$\ldots$	$1-l$	1	$1+l$	$\ldots$	$1+3$	$1+2$
$2-3$	$2-4$	$\ldots$	$2-l$	2	$2+l$	$\ldots$	$2+3$	$\star$

As a shorthand, we shall use the corresponding notation:

(1) $a_i := \varepsilon_1 - \varepsilon_{i+1}, 1 \leq i \leq l-1$; $a_l := \varepsilon_1$; $a_{l+i} := \varepsilon_1 + \varepsilon_{l-i+1}, 1 \leq i \leq l-1$;

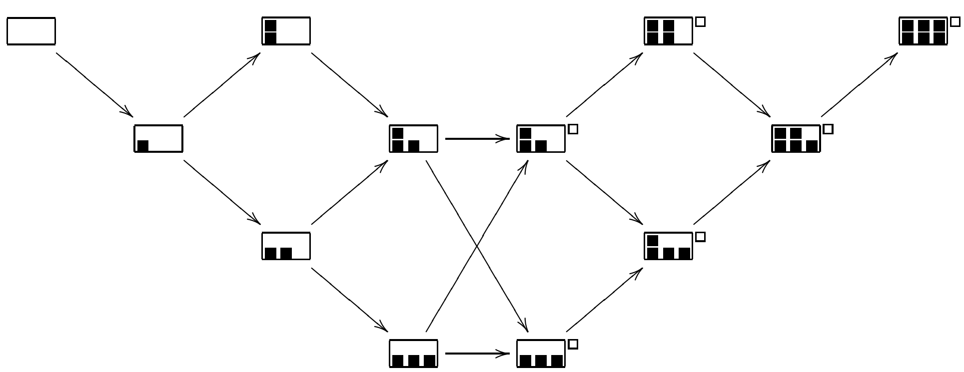
(2) $b_i := \varepsilon_2 - \varepsilon_{i+1}, 1 \leq i \leq l-1$; $b_l := \varepsilon_2$; $b_{l+i} := \varepsilon_2 + \varepsilon_{l-i+1}, 1 \leq i \leq l-2$.

It is easy to compute the sets Φ_{a_i} and Φ_{b_i}, where $\Phi_\alpha = Q(\sigma_\alpha)$.

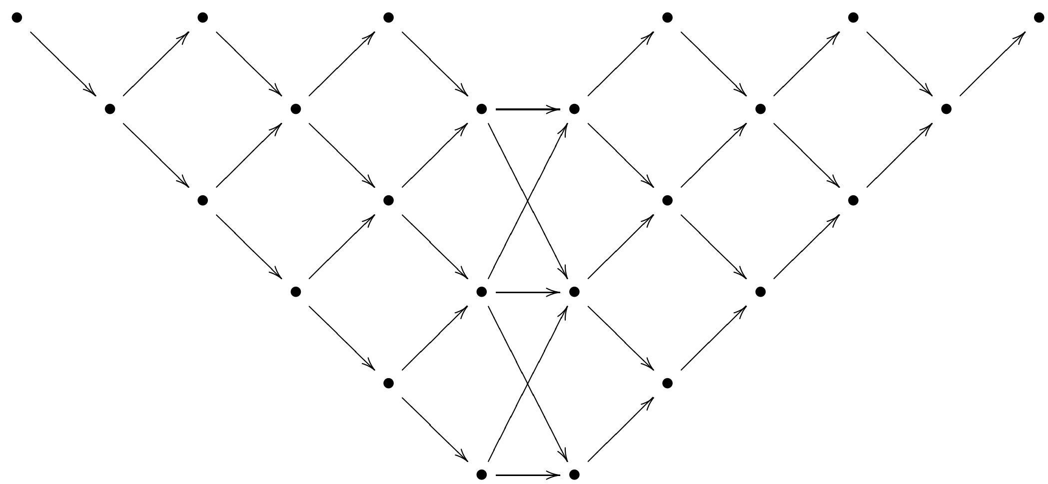
5.1.2 Saturated sets and the Hasse diagram for $(B_l, \{\alpha_2\})$

We shall start drawing some Hasse diagrams for odd dimensional orthogonal algebras with the contact grading. One can derive these examples by the definition of the Hasse diagram. Then we shall prove a theorem on the Hasse diagram for general odd dimensional contact graded orthogonal algebra.

Example: Let us write the Hasse diagram for the pair $(B_3, \{\alpha_2\})$.



There is the Hasse diagram for $(B_4, \{\alpha_2\})$ at the following picture.



Remark 5.1.1. The examples of the Hasse diagrams for B_3 and B_4 make it easy to see how the shape of the Hasse diagram for B_l is changing with increasing l.

The following lemma describes the form of the saturated sets in a general situation (see the special cases for illustration above).

Lemma 5.1.1 (Saturated sets for $(B_l, \{\alpha_2\})$). *A set A is a saturated set for the contact graded algebra $(B_l, \{\alpha_2\})$ if and only if A is one of the two following types:*

(1) $A^l_{1,ij} := (\{a_k; k = 1, \ldots, i\} \cup \{b_k; k = 1, \ldots, j\}) \cap \Phi_{\mathfrak{p}_+}$, *for* $1 \leq i \leq j \leq 2l-2$ *and* $i + j \leq 2l - 1$,

(2) $A^l_{2,ij} := (\{a_k; k = 1, \ldots, i\} \cup \{b_k; k = 1, \ldots, j\} \cup \{\varepsilon_1 + \varepsilon_2\}) \cap \Phi_{\mathfrak{p}_+}$, *for* $1 \leq i \leq j \leq 2l - 2$ *and* $i + j \geq 2l - 1$.

Proof. Firstly, we shall prove that each set of type $A^l_{k,ij}$ for i, j, k satisfying the conditions in the formulation of the lemma, is a saturated set. To check (R1), take $\alpha, \beta \in A^l_{1,ij}$ and $\alpha+\beta \in \Phi$. We should make a conclusion that $\alpha+\beta \in A^l_{1,ij}$. Because of $\alpha + \beta$ never belongs to Φ, the implication is trivially satisfied. Now, we do the same for the sets of type $A^l_{2,ij}$. The only pairs $\alpha, \beta \in A^l_{r,ij}$ for which $\alpha+\beta \in \Phi$, are those for which $\alpha+\beta = \varepsilon_1+\varepsilon_2$. But the sum $\varepsilon_1+\varepsilon_2$ is contained in $A^l_{2,ij}$ for each i, j. We shall show that the condition (R2) holds. We shall go through the list of all decompositions of an element of $A^l_{1,ij}$ or $A^l_{2,ij}$ into two positive roots.

(1) $\varepsilon_1 - \varepsilon_k = (\varepsilon_1 - \varepsilon_m) + (\varepsilon_m - \varepsilon_k)$; the element $\varepsilon_m - \varepsilon_k$ is a positive root, if and only if the relation $m < k$ is satisfied. From the definition of $A^l_{r,ij}$ it follows that if $\varepsilon_1 - \varepsilon_k \in A^l_{r,ij}$ and $m < k$ then $\varepsilon_1 - \varepsilon_m \in A^l_{r,ij}$.

(2) $\varepsilon_2 - \varepsilon_k = (\varepsilon_2 - \varepsilon_m) + (\varepsilon_m - \varepsilon_k)$; the element $\varepsilon_m - \varepsilon_k$ is a positive root, if and only if the relation $m < k$ is satisfied. From the definition of $A^l_{r,ij}$ it follows that if $\varepsilon_2 - \varepsilon_k \in A^l_{r,ij}$ and $m < k$, then $\varepsilon_2 - \varepsilon_m \in A^l_{r,ij}$.

(3) $\varepsilon_1 = (\varepsilon_1 - \varepsilon_m) + \varepsilon_m$, it is obvious that $\varepsilon_1 - \varepsilon_m \in A^l_{r,ij}$, if $\varepsilon_1 \in A^l_{r,ij}$.

(4) $\varepsilon_2 = (\varepsilon_2 - \varepsilon_m) + \varepsilon_m$, it is obvious that $\varepsilon_2 - \varepsilon_m \in A^l_{r,ij}$, if $\varepsilon_2 \in A^l_{r4,ij}$

(5) $\varepsilon_1 + \varepsilon_k = (\varepsilon_1 + \varepsilon_m) + (\varepsilon_k - \varepsilon_m), m > k$; if $\varepsilon_1 + \varepsilon_k \in A^l_{r,ij}$ then $\varepsilon_1 + \varepsilon_m \in A^l_{r,ij}$.

(6) $\varepsilon_2 + \varepsilon_k = (\varepsilon_2 + \varepsilon_m) + (\varepsilon_k - \varepsilon_m), m > k$; if $\varepsilon_2 + \varepsilon_k \in A^l_{r,ij}$ then $\varepsilon_2 + \varepsilon_m \in A^l_{r,ij}$.

(7) The only remaining root is $\varepsilon_1 + \varepsilon_2 \in A^l_{2,ij}$. We can decompose it into two sums: $\varepsilon_1 + \varepsilon_2$ or $(\varepsilon_1 - \varepsilon_k) + (\varepsilon_2 + \varepsilon_k)$. In each decomposition, there is a summand which belongs to $A^l_{2,ij}$.

Secondly, we are going to check, that there are no other saturated sets then those that are written in the formulation of this lemma. Let A be a saturated set for $(B_l, \{\alpha_2\})$.

(1) First, let us suppose that $\varepsilon_1 + \varepsilon_2 \notin A$.

(1.1) If $\varepsilon_1 - \varepsilon_k \in A$ then the decomposition $\varepsilon_1 - \varepsilon_k = (\varepsilon_1 - \varepsilon_j) + (\varepsilon_j - \varepsilon_k)$ for $2 < j < k$ implies $\varepsilon_1 - \varepsilon_j \in A$.

(1.2) If $\varepsilon_1 \in A$ then the decomposition $\varepsilon_1 = (\varepsilon_1 - \varepsilon_j) + \varepsilon_j$ implies $\varepsilon_1 - \varepsilon_j \in A$ for $j = 3, \ldots, l$.

(1.3) If $\varepsilon_1 + \varepsilon_k \in A$ for $k \geq 3$ then $\varepsilon_1 \in A$. The decomposition $\varepsilon_1 + \varepsilon_k = (\varepsilon_1 - \varepsilon_j) + (\varepsilon_j + \varepsilon_k)$ implies $\varepsilon_1 - \varepsilon_j \in A$ for $j = 3, \ldots, k-1$.

(1.4) A similar inspection could be done for "b-roots."

(1.5) The following decomposition $\varepsilon_1 + \varepsilon_k = (\varepsilon_1 - \varepsilon_2) + (\varepsilon_2 + \varepsilon_k)$ implies that if $\varepsilon_1 + \varepsilon_k \in A$ then $\varepsilon_2 + \varepsilon_k \in A$. This implies the inequality $i \leq j$ in the formulation of this lemma.

(1.6) We shall show that the inequality $i + j \leq 2l - 1$ holds. Suppose $i+j \geq 2l$ to get a contradiction. Thus if $\varepsilon_2+\varepsilon_k \in A$ and $\varepsilon_2-\varepsilon_{k+1} \notin A$, then $\varepsilon_1 - \varepsilon_k \in A$ and consequently $\varepsilon_1 + \varepsilon_2 \in A$, a contradiction. If $\varepsilon_2 \in A$ and $\varepsilon_2 + \varepsilon_j \notin A$, $j = 3, \ldots, l$, then $\varepsilon_1 \in A$ and consequently $\varepsilon_1 + \varepsilon_2 \in A$, a contradiction again. Because there is no other possibility different from those we have discussed, the claim of this item is proved.

(2) The case $\varepsilon_1+\varepsilon_2 \in A$ is quite similar. We shall comment only the inequality $i + j \geq 2l - 1$. If $\varepsilon_1 + \varepsilon_2 \in A$ then $\varepsilon_2 \in A$ or $\varepsilon_1 \in A$. Because $\varepsilon_1 \in A$ implies $\varepsilon_2 \in A$ (due to the inequality $i \leq j$), the root ε_2 is in A in the case of that $\varepsilon_1 + \varepsilon_2 \in A$. Therefore $j \geq l$ and we shall discuss two cases. First, suppose that $\varepsilon_2 + \varepsilon_k \in A$ and $\varepsilon_2 + \varepsilon_{k+1} \notin A$. From the decomposition $\varepsilon_1 + \varepsilon_2 = \varepsilon_2 + \varepsilon_{k+1} + \varepsilon_1 - \varepsilon_{k+1}$, it follows that $\varepsilon_1 - \varepsilon_{k+1} \in A$. Thus $i+j \geq 2l-1$. Second, suppose that $\varepsilon_2 \in A$ and $\varepsilon_2+\varepsilon_k \notin A$ for $k = 3, \ldots l$. From the decomposition $\varepsilon_1 + \varepsilon_2 = (\varepsilon_2 + \varepsilon_l) + (\varepsilon_1 - \varepsilon_l)$, it follows that $\varepsilon_1 - \varepsilon_l \in A$ and therefore $i + j \geq 2l - 1$.$\Box$

To describe the structure of arrows in the Hasse diagram, we shall need the lemma 2.3.1.

Theorem 5.1.1 (Hasse diagram for $(B_l, \{\alpha_2\})$). *The Hasse diagram for the parabolic algebra $(B_l, \{\alpha_2\})$ has the following structure: The set of all vertices V^l may be identified with the set $\{A^l_{1,ij}\} \cup \{A^l_{2,ij}\}$, for i, j written in the previous lemma 5.1.1. There are following arrows in the Hasse diagram*

(1) $A^l_{1,ij} \xrightarrow{b_{j+1}} A^l_{1,i(j+1)}$, *where* $i + j < 2l - 1$, $1 \leq i \leq j \leq 2l - 2$,

(2) $A^l_{1,ij} \xrightarrow{a_{i+1}} A^l_{1,(i+1)j}$, *where* $i + j < 2l - 1$, $1 \leq i \leq j \leq 2l - 2$,

(3) $A^l_{1,ij} \xrightarrow{a_{i+1}} A^l_{2,(i+1)(j-1)}$, *where* $i + j = 2l - 1, i = 1, \ldots, 2l - 2$,

(4) $A^l_{1,ij} \xrightarrow{a_{2l-1}} A^l_{2,ij}$, *where* $i + j = 2l - 1, i = 1, \ldots, 2l - 2$,

(5) $A^l_{1,ij} \xrightarrow{b_{j+1}} A^l_{2,(i-1)(j+1)}$, *where* $i + j = 2l - 1, i = 1, \ldots, 2l - 2$,

(6) $A^l_{2,ij} \xrightarrow{b_{j+1}} A^l_{2,i(j+1)}$, *where* $i + j > 2l - 1, 1 \leq i \leq j \leq 2l - 2$,

(7) $A^l_{2,ij} \xrightarrow{a_{i+1}} A^l_{2,(i+1)j}$, *where* $i+j > 2l-1, 1 \leq i \leq j \leq 2l-2$.

Proof. We use the lemma 2.3.1. It is easy to compute the first two coordinates of $|A^l_{k,ij}|$ in the $\{\varepsilon_r\}_{r=1}^l$ basis. Let us denote the projection onto the s^{th} coordinate of an element of the subalgebra $\mathfrak{h}$ by $pr_s : \mathfrak{h} \to \mathbb{C}$, $1 \leq s \leq l$. Regarding the structure of saturated sets, we obtain

(1) $pr_1(|A^l_{1,ij}|) = i-1$; $pr_2|A^l_{1,ij}| = j-1$

(2) $pr_1(|A^l_{2,ij}|) = i$; $pr_2|A^l_{2,ij}| = j$, for all admissible i, j.

Clearly $pr_s(A^l_{r,ij}) \in \{0, \pm 1, \pm 2\}$, for $s > 2$ and all admissible r, i, j. We already know that the only possible labels of arrows in the Hasse diagram, are $\varepsilon_1 \pm \varepsilon_k$, $\varepsilon_2 \pm \varepsilon_k$, ε_1, ε_2. Second, we know that the number of elements of sets which are joined by an arrow differs exactly by one. If we add a $a-$root or a $b-$root, respectively the first or the second coordinate, respectively of $|A^l_{k,ij}|$ increases of one. Other coordinates of $|A^l_{k,ij}|$ either decrease (if the root a_j, $j < l$ was added into the set) or increase (if the root a_j, $j > l$ was added into the set) or do not change. From that, it is clear that all arrows occurring in the Hasse diagram must be exactly those that described in the formulation of this theorem. The inspection of this fact is very simple and it suffers only a carefully handling with indices and using the facts on projections written above.

The only problem could be caused by handling with the root $\varepsilon_1 + \varepsilon_2$. An arrow labelled by this root could by placed only between those saturated sets A, B for which the coordinates of $|A|$ and $|B|$ differ by 1 at the two first positions and are the same on the other one. Thus these sets differs only by the element $\varepsilon_1 + \varepsilon_2$. From this it is evident that these arrows are exactly of the fourth case mentioned in the formulation of this theorem. □

5.1.3 BGG diagrams for $(B_l, \{\alpha_2\})$ and its real forms

We shall compute the BGG diagrams for contact graded orthogonal Lie algebra of odd rank and its real form. At the first part, we shall deal with the complex case, in the second part of this subsection with the real one. In that part we shall use some basic facts on representation theory of real forms of simple Lie algebras, see the section 2.2.2.

BGG diagrams - the complex case

See the remark 2.3.1 for comments on the computation of a BGG diagram. Let us fix a notation for writing of weights which occur in the BGG diagram for $(\mathfrak{p}, \mathfrak{g})$. For a vector (in fact, we shall use this notation only for $\mathfrak{g}-$dominant weights) $\beta = (c_1, \ldots, c_l)$ written in the basis of fundamental weights, let us define the following vectors ("$\gamma-$vectors"):

For $1 \leq i \leq j \leq l$,

$$\gamma_{1,ij} := (\sum_{k=i}^{j} c_k, -\sum_{k=1}^{j} c_k, c_1, \ldots, c_l).$$

For $1 \leq i \leq j \leq 2l-2$ and $j > l$,

$$\gamma_{1,ij} := (\sum_{k=i}^{l} c_k + \sum_{k=1}^{j-l} c_{l-k}, -(\sum_{k=1}^{l} c_k + \sum_{k=1}^{j-l} c_{l-k}), c_1, \ldots, c_l).$$

Now, we define the following vectors ("$\beta-$vectors"):

(1) $\beta^l_{1,ij} :=$
$(\gamma^1_{ij}, \ldots, \gamma^i_{1,ij}, \gamma^{i+1}_{1,ij} + \gamma^{i+2}_{1,ij}, \gamma^{i+3}_{1,ij}, \ldots, \gamma^{j+1}_{1,ij}, \gamma^{j+2}_{1,ij} + \gamma^{j+3}_{1,ij}, \gamma^{j+4}_{1,ij}, \ldots, \gamma^{l+2}_{1,ij})$,
$1 \leq i \leq j \leq 2l-2$ and $j < l-1$.

(2) $\beta^l_{1,ij} :=$
$(\gamma^1_{ij}, \ldots, \gamma^i_{1,ij}, \gamma^{i+1}_{1,ij} + \gamma^{i+2}_{1,ij}, \gamma^{i+3}_{1,ij}, \ldots, \gamma^{2l-j}_{1,ij}, \gamma^{2l-j+1}_{1,ij} + \gamma^{2l-j+2}_{1,ij}, \gamma^{2l-j+3}_{1,ij}, \ldots, \gamma^{l+2}_{1,ij})$,
$1 \leq i \leq j \leq 2l-2$ and $j > l$.

(3) $\beta^l_{1,ij} := (\gamma^1_{ij}, \ldots, \gamma^i_{1,ij}, \gamma^{i+1}_{1,ij} + \gamma^{i+2}_{1,ij}, \gamma^{i+3}_{1,ij}, \ldots, \gamma^l_{1,ij}, 2\gamma^{l+1}_{1,ij} + \gamma^{l+2}_{1,ij})$, $i = 1, \ldots, l-2$, $j = l-1, l$.

(4) $\beta^l_{1,(l-1)j} := (\gamma^1_{ij}, \ldots, \gamma^{l-1}_{1,ij}, 2\gamma^l_{1,ij} + 2\gamma^{l+1}_{1,ij} + \gamma^{l+2}_{1,ij})$, $j = l-1, l$.

Let us define $\beta^l_{2,ij}$. In the case of $i + j \geq 2l - 1$, the vector $\gamma_{2,ij}$ is equal to $\gamma_{1,ij}$, only with the exception of its second coordinate. The second coordinate of this "$\gamma-$vector" is equal to $-(\sum_{k=1}^{l} c_k + \sum_{k=2}^{l-1} c_k)$. The vector $\beta^l_{2,ij}$ is constructed in the same way as in the case of $\beta_{1,(2l-j+1)(2l-i+1)}$ but using the vector $\gamma_{2,(2l-j+1)(2l-i+1)}$ defined above.

Thus the betas are symmetric w.r. to the vertical symmetry axis of the Hasse diagram with the only exception, namely the coefficient above the crossed node $\{\alpha_2\}$.

Let us denote the weight in the BGG diagram for β placed at the position corresponding to the saturated set $A^l_{k,ij}$ in the Hasse diagram by $A^l_{k,ij}(\beta)$.

With help of this notation we can formulate the following theorem on the complex BGG-diagram.

Theorem 5.1.2 (BGG diagram for $(B_l, \{\alpha_2\}, \beta)$**).** *Let* $\beta = \sum_{k=1}^{l} c_k \varpi_k$ *be a* $\mathfrak{g}$*-dominant integral weight.*

(1) If $1 \leq i \leq j \leq 2l-2$, $i+j \leq 2l-1$ *then* $A^l_{1,ij}(\beta) = \beta^l_{1,ij}$,

(2) If $1 \leq i \leq j \leq 2l-2$, $i+j \geq 2l-1$ *then* $A^l_{2,ij}(\beta) = \beta^l_{2,ij}$.

Proof. We prove this theorem for the case 1 only. The second case could be proved in a similar way. We shall use an induction on $k := i+j$. I. The first step $k=2$ is clear, $A^l_{1,11} = \beta = \beta^l_{1,ij}$. II. Let us suppose that the theorem is true for all i, j, s.t. $i+j=k$. From the structure of the Hasse diagram which is isomorphic to the BGG diagram, it follows that it suffices to prove that $\sigma_{\varepsilon_1-\varepsilon_{i+2}}(\beta^l_{1,ij}) = \beta_{1,(i+1)j}$ and $\sigma_{\varepsilon_2-\varepsilon_{i+2}}(\beta^l_{1,ij}) = \beta_{1,i(j+1)}$. Let us compute

$$\sigma_{\varepsilon_1-\varepsilon_{i+2}}(\beta^l_{1,ij}) = \beta^l_{1,ij} - 2\frac{(\varepsilon_1-\varepsilon_{i+2}, \beta^l_{1,ij})}{(\varepsilon_1-\varepsilon_{i+2}, \varepsilon_1-\varepsilon_{i+2})}(\varepsilon_1-\varepsilon_{i+2}).$$

From the structure theory of the odd orthogonal algebras, it is well known that

(1) $\varpi_i = \varepsilon_1 + \ldots + \varepsilon_i, i = 1, \ldots l-1$; $\varpi_l = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_l)$,

(2) $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, \ldots l-1$; $\alpha_l = \varepsilon_l$.

By substituting that relations into the previous formula for the reflection (using the fact that $\{\varepsilon_i\}_{i=1}^l$ is an orthogonal basis), we obtain $\sigma_{\varepsilon_1-\varepsilon_{i+2}}(\beta^l_{1,ij}) = \beta_{1,(i+1)j}$ after a straightforward computation regarding the recipe of the construction of $\beta^l_{r,ij}$ described above. In a similar way, one can prove the remaining induction step, from (i,j) to $(i,j+1)$.$\square$

BGG diagrams - the real case

We shall denote the Satake diagram of type BI (see Goodman, Wallach [14], pp. 544) by (B_l, p) if there are l nodes in the diagram and the last p nodes are black. A weighted Satake diagram of type (B_l, p) with p black nodes will be denoted by $(\lambda_1, \ldots, \lambda_l|p)$ if the numbers $\lambda_1, \ldots, \lambda_l$ are written above the nodes of such a diagram.

From the structure theory of simple real Lie algebras, it follows that the only admissible real forms of odd dimensional orthogonal algebras admitting the contact grading are represented by Satake diagrams of the type BI the first black node of which (counting from the left side) may be the third one, i.e $\{\alpha_3\}$, see the section 2.1.

We shall denote the real representation by $\mathbb{V}_\beta$, and call it *real representation with the highest weight* β if its complexification contains a complex irreducible representation with the highest weight β. This definition is correct, see Zhida, Dagan [32] and the subsection 2.2.2 for comments. We shall use the same notation as for a weighted Satake diagram $\beta = (\lambda_1, \ldots, \lambda_l|p)$ as for the highest weight of the corresponding real representation.

The Maltsev height $m(\beta)$ in the case of $\beta = (\lambda_1, ..., \lambda_l|p)$ is equal to

$$m(\beta) = \sum_{k=p}^{l-1} i(2l + 1 - i)\lambda_k + \frac{l(l+1)}{2}\lambda_l,$$

see Goodman, Wallach [14] (5.1.8 Exercises, the Maltsev height is computed for the compact form and is denoted by h^0 there). The symmetry $s\nu$ of the Satake diagram of BI is trivial, i.e., we know that $s\nu(\lambda) = \lambda$.

Theorem 5.1.3. *The complexification $\mathbb{V}_\beta \otimes_{\mathbb{R}} \mathbb{C}$ of the real representation $\mathbb{V}_\beta$ of the corresponding real form (B_l, p) splits (into two self-dual representations) if and only if the complexification $\mathbb{V}_{\beta'} \otimes \mathbb{C}$ of the representation $\mathbb{V}_{\beta'}$ with the highest weight $\beta' = A^l_{k,ij}(\beta)$ does.*

Proof. From the previous theorem 5.1.2 on the complex BGG diagram, we know that the number λ'_l above the last node in the weighted Dynkin diagram of the complex representation $\mathbb{V}_{\beta'} \otimes_{\mathbb{R}} \mathbb{C}$ of B_l (which is placed at the position

$A^l_{k,ij}$) differs from the number λ_l over the last node of the Dynkin diagram at the position $A^l_{1,11}$ in this BGG diagram only by an even number, say $2k$ (see the last components in the β-vectors defined above). From the form of the Maltsev height for $\beta = (\lambda_1, \ldots, \lambda_l, p)$ given above, it follows that $m(\beta) = \frac{l(l+1)}{2}\lambda_l$ mod 2. Thus $m(A^l_{k,ij}(\beta)) = \frac{l(l+1)}{2}\lambda'_l$ mod $2 = \frac{l(l+1)}{2}(2k + \lambda_l)$ mod $2 = \frac{l(l+1)}{2}\lambda_l$ mod $2 = m(\beta)$. We have already mentioned that $s\nu(\lambda) = \lambda$ for each λ. Thus $\mathbb{V}_\beta$ is of quaternionic or real type if and only if $\mathbb{V}_{\beta'}$ is of quaternionic or real type, respectively, see the theorem 2.2.3. This together with the remark implies the statement we had to prove. $\square$

Theorem 5.1.4 (BGG diagram for $((B_l, p), \{\alpha_2\}, \beta)$**).** *The BGG diagram for the contact graded real form* $((B_l, p)\{\alpha_2\})$ *and a* $\mathfrak{g}$*-dominant integral weight has the same form as the BGG diagram in the complex case (for the same weight).*

Proof. If we complexify each representation in the real BGG diagram, we should get either a sum of two self-dual complex irreducible representations or a complex irreducible representation. According to the previous corollary, we get the mentioned sum at the first position $A_{1,11}$, if and only if we get it at any position of the BGG diagram. Analogously for the irreducible complexification. $\square$

Chapter 6

Invariant differential operators for fields with values in some standard cyclic modules

The aim of this section is to rewrite the theory of first order invariant differential operators in parabolic geometries for the case of certain infinite dimensional standard cyclic modules.

6.1 General theory of first order invariant differential operators

Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra, $\mathfrak{p}$ the associated parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{p}_+$ the positive part, $\mathfrak{g}_-$ the negative part of $\mathfrak{g}$, for details see 2.1. Let $\mathbb{E}$, $\mathbb{F}$ be $\mathfrak{p}$-modules. Denote the action of $\mathfrak{p}$ on $\mathbb{E}$ by λ, $\lambda : \mathfrak{p} \to \mathrm{End}(\mathbb{E})$. Further, let $J^1\mathbb{E}$ be the first jet prolongation of the $\mathfrak{p}$-module $\mathbb{E}$ associated to the $|k|$-graded Lie algebra $\mathfrak{g}$. Let us recall that the induced action of $\mathfrak{p}$ on $J^1\mathbb{E}$ is given by

$$Z.(v, \phi) = (\lambda(Z)v, \lambda(Z) \circ \phi - \phi \circ ad_-(Z) + \lambda(ad_{\mathfrak{p}}(Z)(-)v))$$

for $Z \in \mathfrak{p}$, $v \in \mathbb{E}$ and $\phi \in \mathfrak{g}_-^* \otimes \mathbb{E}$, see the section 2.6.

It is a well known fact that $\mathfrak{g}_{-i}^* \simeq \mathfrak{g}_i$ for $i = 1, \ldots, k$ as $\mathfrak{g}_0$-modules. Let $\mathfrak{p}_+^2$ denote the space $[\mathfrak{p}_+, \mathfrak{p}_+] = \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k$. We call the space $J_R^1\mathbb{E} = \mathbb{E} \oplus (\mathfrak{g}_-^* \otimes$

$\mathbb{E})/(\{0\} \oplus (\mathfrak{p}_+^2 \otimes \mathbb{E})) \simeq \mathbb{E} \oplus (\mathfrak{g}_{-1}^* \otimes \mathbb{E}) \simeq \mathbb{E} \oplus (\mathfrak{g}_1 \otimes \mathbb{E})$ the *space of restricted jets.* This space carries the structure of $\mathfrak{p}$-module inherited by factorization. Let us denote the bases of $\mathfrak{g}_{\pm 1}$ by $\{\eta^{\alpha'}\}, \{\xi_{\alpha'}\}$.

Theorem 6.1.1. *Let $\mathbb{E}, \mathbb{F}$ be irreducible $\mathfrak{p}$-modules with the trivial action of $\mathfrak{p}_+$. Let $\Psi : J^1\mathbb{E} \to \mathbb{F}$ be a $\mathfrak{g}_0$-module homomorphism. Then Ψ is a $\mathfrak{p}$-module homomorphism if and only if Ψ factors through the restricted jets $J^1_R\mathbb{E}$ and for all $Z \in \mathfrak{g}_1$ and $v_0 \in \mathbb{E}$*

$$\Psi(\sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}].v_0) = 0.$$

Proof. To prove that Ψ is a $\mathfrak{p}$-module homomorphism it sufficient and necessary to show that it is a $\mathfrak{p}_+$-module homomorphism, because we suppose that it is a $\mathfrak{g}_0$-module homomorphism, see the section 2.6. Suppose that Ψ is a $\mathfrak{p}$-module homomorphism. Let A be the image of the restriction of the induced action of $\mathfrak{p}$ to $\mathfrak{p}_+$ on $J^1\mathbb{E}$, i.e., for an element $w \in J^1\mathbb{E}$ the relation $w \in A$ holds if and only if there is some $Z \in \mathfrak{p}_+$ such that $w = Z.v$ for some $v \in J^1\mathbb{E}$. Now, assume the value of Ψ on an element $w \in A$. We can write $\Psi(w) = \Psi(Z.v)$ for some $Z \in \mathfrak{p}_+$ and $v \in J^1\mathbb{E}$. Because Ψ is a $\mathfrak{p}_+$-module homomorphism the last expression equals $Z.\Psi(v) = 0$ because of the triviality of the representation of $\mathfrak{p}_+$ on $\mathbb{F}$. Thus we have proved that Ψ vanishes on A (the image of the action of $\mathfrak{p}_+$ on $J^1\mathbb{E}$).

Now, we prove that Ψ factors through the restricted jets. Let us fix dual basis $\{\xi_\alpha\}$, $\{\eta^\alpha\}$ of $\mathfrak{g}_-$ and $\mathfrak{p}_+$. We know that for the induced action we can write

$$Z.(v_0, Y \otimes v_1) = (\lambda(Z)v_0, Y \otimes \lambda(Z)v_1 + [Z, Y] \otimes v_1 + \sum_\alpha \eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{p}} v_0)$$

for all $v_0, v_1 \in \mathbb{E}$, $Z \in \mathfrak{g}_i$, $i > 0$, $Y \in \mathfrak{g}_-^*$. Inserting $v_0 = 0$ we obtain

$$Z.(0, Y \otimes v_1) = [Z, Y] \otimes v_1.$$

For $Z_i \in \mathfrak{g}_i$, $v_i \in \mathbb{E}$, $i = 1, \dots, k$, we shall compute $\Psi(\sum_{i=1}^k Z_i \otimes v_i)$. Because $\mathfrak{g}_1$ generates $\mathfrak{p}_+$ there are $X_i \in \mathfrak{g}_1$, $Y_i \in \mathfrak{g}_{i-1}$ for $i = 2, \dots, k$ such that $Z_i = [X_i, Y_i]$. Thus we can write $\Psi(\sum_{i=1}^k Z_i \otimes v_i) = \Psi(Z_1 \otimes v_1 + \sum_{i=2}^k [X_i, Y_i] \otimes v_i) =$

$\Psi(Z_1 \otimes v_1) + \Psi(X_2.(0, Y_2 \otimes v_2)) + \ldots + \Psi(X_k.(0, Y_k \otimes v_k)) = \Psi(Z_1 \otimes v_1)$. The terms $\Psi(X_k.(0, Y_k \otimes v_k)) = 0$ for $k = 2, \ldots, k$ because Ψ vanish on the image of the $\mathfrak{p}_+$ action on $J^1\mathbb{E}$. Thus we have proved that Ψ factors through the restricted jets. Looking at the induced action of $\mathfrak{p}_+$ on $J^1\mathbb{E}$ we derive the condition

$$\Psi(\sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}].v_0) = 0.$$

The opposite direction of the implication in the statement of this theorem is obvious. □

To further purposes let as define an endomorphism

$$\Phi : \mathfrak{g}_1 \otimes \mathbb{E} \to \mathfrak{g}_1 \otimes \mathbb{E}$$

$$\Phi(Z \otimes s) := \sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}].s$$

for $Z \in \mathfrak{g}_1, s \in \mathbb{E}$.

Thus we can reformulate the last theorem as

Theorem 6.1.2. *Let $\mathbb{E}, \mathbb{F}$ be irreducible $\mathfrak{p}$-modules. Let $\Psi : J^1\mathbb{E} \to \mathbb{F}$ be a $\mathfrak{g}_0$-module homomorphism. Then Ψ is a $\mathfrak{p}$-module homomorphism if and only if Ψ factors through the restricted jets $J^1_R\mathbb{E}$ and $\Psi_{|Im(\Phi)} = 0$.*

Computing of the mapping Φ. In this paragraph, we would like to compute the mapping Φ with help of the universal Casimir element.

(1) First, let us make some assumptions on the Lie algebra $\mathfrak{g}$. Suppose that the subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ has an one dimensional center. This center is necessarily generated by the grading element E of the $|k|$-graded Lie algebra $\mathfrak{g}$. Thus we can decompose $\mathfrak{g} = \mathfrak{g}_0^{ss} \oplus \mathbb{C}E$ where the part $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ denotes the semisimple part of $\mathfrak{g}$, see the section 2.1. The Killing form B of the Lie algebra $\mathfrak{g}$ when restricted to $\mathfrak{g}_0$ is nondegenerate too, see the book Čap, Slovák [7]. Let us normalize the Killing form B by the condition $B(E, E) = 1$ and denote this resulting nondegenerate invariant form on $\mathfrak{g}_0$ by $(,) : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{C}$. It is easy to compute that the decomposition $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathbb{C}E$ is an orthogonal decomposition. Indeed, take an arbitrary $X \in \mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ in the form $X = [U, V]$ for a $U, V \in \mathfrak{g}_0$ and compute

$(E, X) = (E, [U, V]) = ([E, U], V) = 0$ because E is the grading element and $U \in \mathfrak{g}_0$.

Let us denote the basis of $\mathfrak{g}_0^{ss}$ by $\{Y_a\}_{a=1}^k$ and the dual basis with respect to $(,)$ by $\{Y'_a\}_{a=1}^k$. Sometimes we will denote the element E by Y_{k+1}.

(2) Now, we derive the following

Lemma 6.1.1. *Let $\mathbb{V}$ be representation of a semisimple $|k|$-graded Lie algebra $\mathfrak{g}$ then*

$$\Phi(Z \otimes s) = \sum_{a=1}^{k} Y'_a.Z \otimes Y_a.s$$

for each $Z \in \mathfrak{g}_1$ and $s \in \mathbb{V}$.

Proof. We use the invariance of the Killing form $[Z, \xi_{\alpha'}] = \sum_a (Y'_a, [Z, \xi_{\alpha'}])Y_a = \sum_a ([Y'_a, Z], \xi_{\alpha'})Y_a$ to compute the value $\Phi(Z \otimes s)$.

$$\begin{aligned}
\Phi(Z \otimes s) &= \sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}].s \\
&= \sum_{\alpha'} \eta^{\alpha'} \otimes \sum_a (Y'_a, [Z, \xi_{\alpha'}])Y_a.s \\
&= \sum_{\alpha'} \eta^{\alpha'} \otimes \sum_a ([Y'_a, Z], \xi_{\alpha'})Y_a.s \\
&= \sum_{\alpha'} \sum_a ([Y'_a, Z], \xi_{\alpha'})\eta^{\alpha'} \otimes Y_a.s \\
&= \sum_a Y'_a.Z \otimes Y_a.s.
\end{aligned}$$

□

(3) Now, we make some assumptions on the representations we shall consider. We will consider that $\mathbb{V}$ is an irreducible standard cyclic $\mathfrak{g}$-module with the highest weight λ considered as a $\mathfrak{g}_0^{ss}$-module. Further, we assume that $\mathfrak{g}_1 \otimes \mathbb{V}$ decomposes into a finite direct sum of irreducible $\mathfrak{g}_0^{ss}$-modules without multiplicities and denote by π_μ the projection $\pi_\mu : \mathfrak{g}_1 \otimes \mathbb{V} \to \mathbb{V}_\mu$ where $\mathbb{V}_\mu$ is the representation with highest weight μ which occurs in the decomposition of the completely reducible tensor product $\mathfrak{g}_1 \otimes \mathbb{V}$. Let us suppose that the representation of the center $\mathbb{C}E$ of $\mathfrak{g}_0$ is given by $E.v := wv$ for each $v \in \mathbb{V}$ and a $w \in \mathbb{C}$. So we are given a representation of the whole $\mathfrak{g}_0$ which is characterized by the tuple (λ, w). The complex

number w is often called *conformal weight*. Finally, we assume that $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0^{ss}$-module with the highest weight α.

(4) In order to compute the mapping Φ let us evaluate the following expression $\sum_{a=1}^{k+1}(Y_a'Y_a)(Z \otimes s)$ for $s \in \mathbb{V}$ and $Z \in \mathfrak{g}_1$.

$$\begin{aligned}\sum_{a=1}^{k+1}(Y_a'Y_a).(Z\otimes s) &= \sum_{a=1}^{k+1}(Y_a'Y_a).Z\otimes s + Z\otimes\sum_{a=1}^{k+1}(Y_a'Y_a).s + \\ & \sum_{a=1}^{k+1} Y_a'.Z\otimes Y_a.s + \sum_{a=1}^{k+1} Y_a.Z\otimes Y_a'.s \\ &= \sum_{a=1}^{k+1}(Y_a'Y_a).Z\otimes s + Z\otimes\sum_{a=1}^{k+1}(Y_a'Y_a).s + \\ & 2\Phi(Z\otimes s), \end{aligned} \tag{6.1}$$

where we have used the lemma 6.1.1 above. Now, we would like to compute the first two terms of the last written equation using the universal Casimir element, see theorem 2.2.2.

$$\begin{aligned}\sum_{a=1}^{k+1}(Y_a'Y_a).Z\otimes s &= \sum_{a=1}^{k}(Y_a'Y_a).Z\otimes s + (E'E).Z\otimes s = \\ &= (\alpha,\alpha+2\delta)Z\otimes s + Z\otimes s\end{aligned} \tag{6.2}$$

$$\begin{aligned}Z\otimes\sum_{a=1}^{k+1}(Y_a'Y_a).s &= Z\otimes\sum_{a=1}^{k}(Y_a'.Y_a).s + Z\otimes(E'E).s = \\ &= (\lambda,\lambda+2\delta)Z\otimes s + w^2 Z\otimes s\end{aligned} \tag{6.3}$$

Let us compute the L.H.S. of 6.1

$$\begin{aligned}\sum_{a=1}^{k+1}(Y_a'Y_a).(Z\otimes s) &= \sum_{a=1}^{k+1}\sum_{\mu}(Y_a'Y_a).\pi_\mu(Z\otimes s) \\ &= \sum_{a=1}^{k}\sum_{\mu}(Y_a'Y_a)\pi_\mu(Z\otimes s) + \sum_{\mu}(E'E)\pi_\mu(Z\otimes s) \\ &= \sum_{\mu}(\mu,\mu+2\delta)\pi_\mu(Z\otimes s) + \sum_{\mu}\pi_\mu(E'E).(Z\otimes s)\end{aligned}$$

$$
\begin{aligned}
&= \sum_\mu (\mu, \mu+2\delta)\pi_\mu(Z\otimes s) + \sum_\mu \pi_\mu[(E'E).Z\otimes s + \\
&\quad E'.Z\otimes E.s + Z\otimes (E'E).s + E.Z\otimes E'.s] \\
&= \sum_\mu (\mu, \mu+2\delta)\pi_\mu(Z\otimes s) \\
&\quad + \sum_\mu \pi_\mu[Z\otimes s + 2wZ\otimes s + w^2 Z\otimes s] \qquad (6.4)
\end{aligned}
$$

Substituting these equations 6.2, 6.3 and 6.4 into the equation 6.1 we obtain

$$
\sum_\mu (\mu, \mu+2\delta)\pi_\mu(Z\otimes s) + 2\sum_\mu w\pi_\mu(Z\otimes s) + \sum_\mu w^2\pi_\mu(Z\otimes s) =
$$
$$
= 2\Phi(Z\otimes s) + (\alpha, \alpha+2\delta)Z\otimes s + Z\otimes s + (\lambda, \lambda+2\delta)Z\otimes s + w^2 Z\otimes s.
$$

As a result we obtain

$$
\Phi(Z\otimes s) = \sum_\mu (w - c^\mu_{\lambda\alpha})\pi_\mu(Z\otimes s),
$$

where

$$
c^\mu_{\lambda\alpha} = \frac{1}{2}[(\lambda, \lambda+2\delta) + (\alpha, \alpha+2\delta) - (\mu, \mu+2\delta)].
$$

We state this result as a theorem formulating explicitly the assumptions we have made.

Theorem 6.1.3. *Let $\mathfrak{g}$ be a $|k|$-graded simple Lie algebra such that the subalgebra $\mathfrak{g}_0$ has an one dimensional center $\mathbb{C}E$. Let $\mathbb{V}$ be an irreducible $\mathfrak{g}_0$-module with the highest weight λ when considered as a $\mathfrak{g}_0^{ss}$-module. Let the grading element E acts by the complex number w (conformal weight). Further, let $\mathfrak{g}_1$ be an irreducible $\mathfrak{g}_0^{ss}$-module with the highest weight α. Assume that the tensor product $\mathfrak{g}_1\otimes\mathbb{V}$ decomposes into a finite direct sum of irreducible $\mathfrak{g}_0^{ss}$-modules and has no multiplicities then*

$$
\Phi(Z\otimes s) = \sum_\mu (w - c^\mu_{\lambda\alpha})\pi_\mu(Z\otimes s),
$$

where

$$
c^\mu_{\lambda\alpha} = \frac{1}{2}[(\lambda, \lambda+2\delta) + (\alpha, \alpha+2\delta) - (\mu, \mu+2\delta)].
$$

Proof. See the analyzes above this theorem. □

Remark 6.1.1. In the future, we would like to use the last written theorem in the case of projective contact geometries and harmonic modules. It can be done when one knows a relationship between the representations of the Lie algebra G_0 and those of $\mathfrak{g}_0$.

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