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Symplectic spinors and Hodge theory

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Contents

1 Introduction

I have said the 21st century might be the era of quantum mathematics or, if you like, of infinite-dimensional mathematics. What could this mean? Quantum mathematics could mean, if we get that far, 'understanding properly the analysis, geometry, topology, algebra of various non-linear function spaces', and by 'understanding properly' I mean understanding it in such a way as to get quite rigorous proofs of all the beautiful things the physicists have been speculating about.

Sir Michael Atiyah

In the literature, topics contained in this thesis are treated rather separately. From a philosophical point of view, a common thread of themes that we consider is represented by the above quotation of M. Atiyah. We are inspired by mathematical and theoretical physics.

The content of the thesis concerns analysis, differential geometry and representation theory on infinite dimensional objects. A specific infinite dimensional object which we consider is the Segal–Shale–Weil representation of the metaplectic group. This representation originated in number theory and theoretical physics. We analyze its tensor products with finite dimensional representations, induce it to metaplectic structures defined over symplectic and contact projective manifolds obtaining differential operators whose properties we study. From other point of view, differential geometry uses the infinite dimensional algebraic objects to obtain vector bundles and differential operators, that we investigate by generalizing analytic methods known currently – namely by a Hodge theory for complexes in categories of specific modules over C^*-algebras.

Results described in the thesis arose from the year 2003 to the year 2016. At the beginning, we aimed to classify all **first order invariant differential operators** acting between bundles over contact projective manifolds that are induced by those irreducible representations of the metaplectic group which have bounded multiplicities. See Krýsl [41] for a result. Similar results were achieved by Fegan [14] in the case of irreducible finite rank bundles over Riemannian manifolds equipped with a conformal structure. In both cases, for any such two bundles, there is at most one first order invariant differential operator up to a scalar multiple.[1] The condition for the existence in the case of contact projective manifolds is expressed by the highest weight of the induced representation considered as a module over a suitable simple group, by a conformal weight, and by the action of $-1 \in \mathbb{R}^{\times}$. The result is based on a decomposition of the tensor product of an irreducible highest weight $\mathfrak{sp}(2n, \mathbb{C})$-module that has bounded multiplicities with the defining representation of $\mathfrak{sp}(2n, \mathbb{C})$ into irreducible submodules. See Krýsl [40]. For similar classification results in the case of more general parabolic geometries and bundles induced by finite dimensional modules, see Slovák, Souček [71].

Our next research interest, described in the thesis, was the **de Rham sequence tensored by the Segal–Shale–Weil representation**. From the algebraic point of view, the Segal–Shale–Weil representation (SSW representation) is an L^2-globalization of the direct sum of two specific infinite dimensional Harish-Chandra $(\mathfrak{g}, K)$-modules with bounded multiplicities over the metaplectic Lie algebra, which are called completely pointed. We decompose the de Rham sequence with values in the mentioned direct sum into sections of irreducible bundles, i.e., bundles induced by irreducible representations. See Krýsl [38]. For a $2n$ dimensional symplectic manifold

[1] and up to operators of the zeroth order. See section 4.3.

with a metaplectic structure, the bundle of exterior forms of degree i, $i \leq n$, with values in the Segal–Shale–Weil representation decomposes into $2(i+1)$ irreducible bundles. For $i \geq n$, the number of such bundles is $2(2n-i+1)$. It is known that the decomposition structure of completely reducible representations is connected to the so-called Schur–Weyl–Howe dualities. See Howe [29] and Goodman, Wallach [20]. We use the decomposition of the twisted de Rham sequence to obtain a duality for the metaplectic group which acts in this case, on the exterior forms with values in the SSW representation. See Krýsl [46]. The dual partner to the metaplectic group appears to be the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$.

Any Fedosov connection (i.e., a symplectic and torsion-free connection) on a symplectic manifold with a metaplectic structure defines a covariant derivative on the symplectic spinor bundle which is the bundle induced by the Segal–Shale–Weil representation. We prove that twisted de Rham derivatives map sections of an irreducible subbundle into sections of at most three irreducible subbundles. Next, we are interested in the quite fundamental question on the structure of the curvature tensor of the symplectic spinor covariant derivative similarly as one does in the study of the curvature of a Levi-Civita or a Riemannian connection. Inspired by results of Vaisman in [75] on the curvature tensors of Fedosov connections, we derive a decomposition of the curvature tensor on symplectic spinors. See Krýsl [42]. Generalizing this decomposition, we are able to find certain subcomplexes of the twisted de Rham sequence, that are called symplectic twistor complexes in a parallel to the spin geometry. These complexes exist under specific restrictions on the curvature of the Fedosov connection. Namely, the connection is demanded to be of Ricci-type. See Krýsl [43]. Further results based on the decomposition of the curvature concern a relation of the spectrum of the symplectic spinor Dirac operator to the spectrum of the symplectic spinor Rarita–Schwinger operator. See Krýsl [39]. The symplectic Dirac operator was introduced by K. Habermann. See [23]. The next result is on symplectic Killing spinors. We prove that if there exists a non-trivial (i.e., not covariantly constant) symplectic Killing spinor, the connection is not Ricci-flat. See [45].

Since the classical theories on analysis of elliptic operators on compact manifolds are not applicable in the case of the de Rham complex twisted by the Segal–Shale–Weil representation, we tried to develop a **Hodge theory for infinite rank bundles**. We use and elaborate ideas of Mishchenko and Fomenko ([58] and [59]) on generalizations of the Atiyah–Singer index theorem to investigate cohomology groups of infinite rank elliptic complexes concerning their topological and algebraic properties. We work in the categories PH_A^* and H_A^* whose objects are pre-Hilbert C^*-modules and Hilbert C^*-modules, respectively, and whose morphisms are adjointable maps between the objects. These notions go back to the works of Kaplansky [31], Paschke [62] and Rieffel [63]. Analyzing proofs of the classical Hodge theory, we are lead to the notion of a Hodge-type complex in an additive and dagger category. We introduce a class of self-adjoint parametrix possessing complexes, and prove that any self-adjoint parametrix possessing complex in PH_A^* is of Hodge-type. Further, we prove that in H_A^* the category of self-adjoint parametrix possessing complexes coincides with the category of the Hodge-type ones. Using these results, we show that an elliptic complex on sections of finitely generated projective Hilbert C^*-bundles over compact manifolds are of Hodge-type if the images of the Laplace operators of the complex are closed. The cohomology groups of such complexes are isomorphic to the kernels of the Laplacians and they are Banach spaces with respect to the quotient topology. Further, we prove that the cohomology groups are finitely generated projective Hilbert C^*-modules. See Krýsl [47]. Using the result of Bakić and Guljaš [2] for modules over a C^*-algebra of compact operators K, we are able to remove the condition on the closed image. We prove in [51] that elliptic complexes of differential operators on finitely generated projective K-Hilbert bundles are of Hodge-type and that their cohomology groups are finitely generated projective Hilbert K-modules. See Krýsl [48], [49] and [50] for a possible application connected to the quotation of Atiyah.

We find it more appropriate to mention author's results and their context in Introduction, and treat motivations, development of important notions, and most of the references in Chapters 2 and 3. In the 2nd Chapter, we recall a definition, realization and characterization of the Segal–Shale–Weil representation. In Chapter 3, symplectic manifolds, Fedosov connections, metaplectic structures, symplectic Dirac and certain related operators are introduced. Results of K. and L. Habermann on global analysis related to these operators are mentioned in this part as well. Chapter 4 of the thesis contains own results of the applicant. We start with the appropriate representation theory and Howe-type duality. Then we treat results on the twisted de Rham derivatives, curvature of the symplectic spinor derivative and twistor complexes. Symplectic Killing spinors are defined in this part as well. We give a classification of the invariant operators for contact projective geometries together with results on the decomposition of the appropriate tensor products in the third subsection. The fourth subsection is devoted to the Hodge theory. The last part of the thesis consists of selected articles published in the period 2003–2016.

2 Symplectic spinors

The discovery of symplectic spinors as a rigorous mathematical object is attributed to I. E. Segal, D. Shale and A. Weil. See Shale [66] and Weil [80]. Segal and Shale considered the real symplectic group as a group of canonical transformations and were interested in a certain quantization of Klein–Gordon fields. Weil was interested in number theory connected to theta functions, so that he considered more general symplectic groups than the ones over the real numbers. Let us notice, that this representation appeared in works of van Hove (see Folland [15], p. 170) at the Lie algebra level and can be found in certain formulas of Fresnel in wave optics already (see Guillemin, Sternberg [22], p. 71).

When doing quantization, one has to assign to "any" function defined on the phase space of a considered classical system an operator acting on a certain function space – a Hilbert space by a rule. Usually, smooth functions are considered to represent the right class for the set of quantized functions. The prescription shall assign to the Poisson bracket of two smooth functions a multiple of the commutator of the operators assigned to the individual functions. The multiple is determined by "laws of nature". It equals to $(\imath\hbar)^{-1}$, where $\imath$ is a fixed root of -1 and $\hbar$ is the Planck constant over 2π. Thus, in the first steps, the quantization map is demanded to be a Lie algebra homomorphism up to a multiple. In further considerations, there is a freedom allowed in the sense that the image of a Poisson bracket need not be the $(\imath\hbar)^{-1}$ multiple of the commutator precisely, but the so-called deformations are allowed. (See Waldmann [77] and also Markl, Stasheff [54] for a framework of quite general deformations.) This tolerance is mainly because of the Groenewold–van Hove no go theorem (see Waldmann [77]). Analytically, deformations are convergent series in the small variable $\hbar$. Their connection to the formal deformations is given by the so called Borel lemma [77].

The state space of a classical system with finite degrees of freedom is modeled by a symplectic manifold (M,ω). The state space of a point particle moving in an n-dimensional vector space or n point particles on a line is the space $\mathbb{R}^{2n}$ or the intersection of open half-spaces in it, respectively. Considering the coordinates $q^1,\dots,q^n$, and $p_1,\dots,p_n$ on $\mathbb{R}^{2n}$ where q^i projects onto the i-th coordinate and p_i onto the $(n+i)$-th one, ω equals to $\sum_{i=1}^n dq^i \wedge dp_i$, or to its restriction to the intersection, respectively.

The group of all linear maps Φ on $\mathbb{R}^{2n}$ which preserve the symplectic form, i.e., $\Phi^*\omega=\omega$, is called the **symplectic group**. Elements of this group do not change the form of dynamic equations governing non-quantum systems – the Hamilton's equations. In this way, they coincide with linear canonical transformations used in Physics.[2] See, e.g., Arnold [1] or Marsden, Ratiu [55].

The symplectic group $G = Sp(2n,\mathbb{R})$ is an $n(2n+1)$ dimensional Lie group. Its maximal compact subgroup is isomorphic to the unitary group $U(n)$ determined by choosing a compatible positive linear complex structure, i.e., an $\mathbb{R}$-linear map $J:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ such that 1) $J^2=-1_{\mathbb{R}^{2n}}$ and 2) the bilinear map $g:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}$ given by $g(v,w)=\omega(v,Jw)$ is symmetric and positive definite, i.e., a scalar product. The unitary group can be proved diffeotopic to the circle S^1, and consequently, its first fundamental group is isomorphic to $\mathbb{Z}$. Thus, for each $m\in\mathbb{N}$, $Sp(2n,\mathbb{R})$ has a unique non-branched m-folded covering $\lambda(m): {}^m\widetilde{Sp(2n,\mathbb{R})}\to Sp(2n,\mathbb{R})$ up to a covering isomorphism. We fix an element e in the preimage of the neutral element in $Sp(2n,\mathbb{R})$ on the two fold covering. The unique Lie group such that its neutral element is e and such that the

[2]The system is supposed to be non-dissipative, i.e., its Hamiltonian function does not depend on time.

covering map is a Lie group homomorphism is called the **metaplectic group**, or if we wish, the real metaplectic group. We set $\lambda = \lambda(2)$ and $\widetilde{G} = Mp(2n, \mathbb{R}) = {}^2\,\widetilde{Sp(2n, \mathbb{R})}$. We denote the λ-preimage of $U(n)$ by $\widetilde{K}$.

So far, the construction of the metaplectic group was rather abstract. One of the basic results of the structure theory of Lie groups is that this is unavoidable indeed. By this we mean that there is no faithful representation of $Mp(2n, \mathbb{R})$ by matrices on a **finite dimensional** vector space. Otherwise said, the metaplectic group cannot be realized as a Lie subgroup of any finite dimensional general linear group. Following Knapp [32], we prove this statement.

Theorem 1: The connected double cover $Mp(2n, \mathbb{R})$ does not have a realization as a Lie subgroup of $GL(W)$ for a finite dimensional vector space W.

Proof. Let us suppose that there exists a faithful representation $\eta' : \widetilde{G} \to \mathrm{Aut}(W)$ of the metaplectic group on a finite dimensional space W. This representation gives rise to a faithful representation $\eta : \widetilde{G} \to \mathrm{Aut}(W^{\mathbb{C}})$ on the complexified vector space $W^{\mathbb{C}}$. This map is injective by its construction. The corresponding Lie algebra representation, i.e., the map $\eta_* : \widetilde{\mathfrak{g}} \to \mathrm{End}(W^{\mathbb{C}})$ is well defined because of the finite dimension of $W^{\mathbb{C}}$. Consequently, we have the commutative diagram

$$\begin{tikzcd} \widetilde{\mathfrak{g}} \arrow[r, "\eta_*"] \arrow[d, "\lambda_*"] & \mathrm{End}(W^{\mathbb{C}}) \\ \mathfrak{g} \arrow[r, "j'"] & \mathfrak{sp}(2n, \mathbb{C}) \arrow[u, "\phi'"'] \end{tikzcd}$$

where $\mathfrak{g}$ is the Lie algebra of the appropriate symplectic group, j' is the natural inclusion and ϕ' is defined by $\phi'(A + \imath B) = \eta_* \lambda_*^{-1}(A) + \imath \eta_* \lambda_*^{-1}(B)$, $A, B \in \mathfrak{g}$. Taking the exponential of the Lie algebra $\mathfrak{sp}(2n, \mathbb{C}) \subseteq \mathrm{End}(\mathbb{C}^{2n})$, we get the group $Sp(2n, \mathbb{C})$. Because this group is simply connected, we get a representation $\phi : Sp(2n, \mathbb{C}) \to \mathrm{Aut}(W^{\mathbb{C}})$ which integrates ϕ' in the sense that $\phi_* = \phi'$. Because λ_*, η_* and also ϕ_* are derivatives at 1 of the corresponding Lie groups representations, and j' is the derivative at $1 \in G$ of the canonical inclusion $j : G \to Sp(2n, \mathbb{C})$, we obtain a corresponding diagram at Lie groups level which is commutative as well. Especially, we have $\eta = \phi \circ j \circ \lambda$. However, the right hand side of this expression is not injective whereas the left hand side is. This is a contradiction. $\square$

Remark: The complex orthogonal group $SO(n, \mathbb{C})$ is not simply connected, so that the above proof does not apply for $G = SO(n, \mathbb{R})$ and its connected double cover $\widetilde{G} = Spin(n, \mathbb{R})$. If it applied, $Spin(n, \mathbb{R})$ would not have any faithful finite dimensional representation.

2.1 The Segal–Shale–Weil representation

For the canonical symplectic vector space $(\mathbb{R}^{2n}, \omega)$, a group $H(n) = \mathbb{R}^{2n} \times \mathbb{R}$ is assigned in which the group law is given by

$$(v, t) \cdot (w, s) = (v + w, s + t + \frac{1}{2}\omega(v, w))$$

where $(v, t), (w, s) \in H(n)$. This is the so called **Heisenberg group** $H(n)$ of dimension $2n + 1$. Let us set $L = \mathbb{R}^n \times \{0\} \times \{0\} \subseteq H(n)$ and $L' = \{0\} \times \mathbb{R}^n \times \{0\}$ for the vector space of initial space and for the vector space of initial impulse conditions, respectively. In particular, the symplectic vector space $\mathbb{R}^{2n}$ is isomorphic to the direct sum $L \oplus L'$.

For any $u' \in H(n)$, we have a unique $t \in \mathbb{R}$ and $q \in L$, $p \in L'$ such that $u' = (q, p, t)$. The **Schrödinger representation** Sch of the Heisenberg group $Sch : H(n) \to U(L^2(L))$ is given by

$$(Sch((q, p, t))f)(x) = e^{2\pi\imath t + \pi\imath\omega(q,p) + 2\pi\imath\omega(x,p)} f(x + q)$$

where $q, x \in L$, $p \in L'$, $t \in \mathbb{R}$, and $f \in L^2(L)$. It is an irreducible representation. See Folland [15]. (By $\omega(x, p)$, we mean $\omega((x, 0), (0, p))$ and similarly for $\omega(q, p)$.)

We may "twist" this representation in the following way. For any $g \in G$, we set $l_g : H(n) \to H(n)$, $l_g(u, t) = (gu, t)$, where $u \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. Consequently, we obtain a family of representations $Sch \circ l_g$ of the Heisenberg group $H(n)$ parametrized by elements g of the symplectic group G. The action of the center of $H(n)$ is the same for each member of the family $(Sch \circ l_g)_{g \in G}$. Namely, $(Sch \circ l_g)(0, 0, t) = e^{2\pi\imath t}$, $t \in \mathbb{R}$. Let us recall the Stone–von Neumann theorem. For a proof, we refer to Folland [15] or Wallach [78].

Theorem 2 (Stone–von Neumann): Let T be an irreducible unitary representation of the Heisenberg group on a separable infinite dimensional Hilbert space H. Then T is unitarily equivalent to the Schrödinger representation.

By Stone–von Neumann theorem, we find a unitary operator $C_g : L^2(L) \to L^2(L)$ that intertwines $Sch \circ l_g$ and Sch for each $g \in G$.[3] By Schur lemma for irreducible unitary representations (see Knapp [33]), we see that there is a function $m : G \times G \to U(1)$ such that $C_g C_{g'} = m(g, g')C_{gg'}$, $g, g' \in G$. In particular, $g \mapsto C_g$ is a projective representation of $Sp(2n, \mathbb{R})$ on the Hilbert space $L^2(L)$. It was proved by Shale in [66] and Weil in [80] that it is possible to lift the cocycle m and the projective representation $g \mapsto C_g$ of G to the metaplectic group to obtain a true representation of the 2-fold cover. We denote this representation by σ and call it the **Segal–Shale–Weil representation**. Note that some authors call it the symplectic spinor, metaplectic or oscillator representation. The representation is unitary and faithful. See, e.g., Weil [80], Borel, Wallach [5], Folland [15], Moeglin et al. [60], Habermann, Habermann [26] or Howe [30].

The "essential" uniqueness of the Segal–Shale–Weil representation with respect to the choice of a representation of the Heisenberg group is expressed in the next theorem.

Theorem 3: Let $T : H(n) \to U(W)$ be an irreducible unitary representation of the Heisenberg group on a Hilbert space W and $\sigma' : \widetilde{G} \to U(W)$ be a non-trivial unitary representation of the metaplectic group such that for all $(v, t) \in H(n)$ and $g \in \widetilde{G}$

$$\sigma'(g)T(v, t)\sigma'(g)^{-1} = T(\lambda(g)v, t).$$

Then there exists a deck transformation γ of λ, such that σ' is equivalent either to $\sigma \circ \gamma$ or to $\sigma^* \circ \gamma$, where $\sigma^*(g) = \tau\sigma(g)\tau$ and $(\tau(f))(x) = \overline{f(x)}$, $x \in \mathbb{R}^n$, $g \in \widetilde{G}$ and $f \in L^2(\mathbb{R}^n)$.
Proof. See Wallach [78], p. 224. □

Remark: Let us recall that a deck transformation γ is any continuous map which satisfies $\lambda \circ \gamma = \lambda$. Note that in the case of the symplectic group covered by the metaplectic group, a deck transformation is either the identity map or the map "interchanging" the leaves of the metaplectic group.

[3] We say that $C : W \to W$ intertwines a representation $T : H \to \mathrm{Aut}(W)$ of the group H if $C \circ T(h) = T(h) \circ C$ for each $h \in H$.

2.2 Realization of symplectic spinors

There are several different objects that one could call a symplectic basis. We choose the one which is convenient for considerations in projective contact geometry. (See Yamaguchi [82] for a similar choice.) If (V, ω) is a symplectic vector space of dimension $2n$ over a field $\mathbb{k}$ of characteristic zero, we call a basis $(e_i)_{i=1}^{2n}$ of V a **symplectic basis** if $\omega(e_i, e_j) = \delta_{i,2n+1-j}$ for $1 \leq i \leq n$ and $1 \leq j \leq 2n$, and $\omega(e_i, e_j) = -\delta_{i,2n+1-j}$ for $n+1 \leq i \leq 2n$ and $1 \leq j \leq 2n$. Thus, with respect to a symplectic basis, the matrix of the symplectic form is

$$(\omega_{ij}) = \left(\begin{array}{c|c} 0 & K \\ \hline -K & 0 \end{array}\right)$$

where K is the following $n \times n$ matrix

$$K = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}.$$

Further, we denote by ω^{ij}, $i, j = 1, \dots, 2n$, the coordinates which satisfy $\sum_{k=1}^{2n} \omega_{ik}\omega^{jk} = \delta_i^j$ for each $i, j = 1, \dots, 2n$. They define a bilinear form $\omega^* : V^* \times V^* \to \mathbb{k}$, e.g., by setting $\omega^* = \sum_{i,j=1}^{2n} \omega^{ij} e_i \wedge e_j$. We use ω_{ij} and ω^{ij} to **rise** and **lower** indices of tensors over V. For coordinates $K_{ab\dots c\dots d}{}^{rs\dots t\dots u}$ of a tensor K on V, we denote the expression $\sum_{c=1}^{2n} \omega^{ic} K_{ab\dots c\dots d}{}^{rs\dots t}$ by $K_{ab\dots}{}^{i}{}_{\dots d}{}^{rs\dots t}$ and $\sum_{t=1}^{2n} K_{ab\dots c}{}^{rs\dots t\dots u}\omega_{ti}$ by $K_{ab\dots c}{}^{rs\dots}{}_{i}{}^{\dots u}$ and similarly for other types of tensors and in the geometric setting when we consider tensor fields on symplectic manifolds.

Remark: Let $(\mathbb{R}^{2n}, \omega)$ be the canonical symplectic vector space introduced at the beginning of this Chapter. Then the canonical arithmetic basis of $\mathbb{R}^{2n}$ is not a symplectic basis according to our definition unless $n = 1$.

Let us denote the λ-preimage of $g \in Sp(2n, \mathbb{R})$ by $\widetilde{g}$. Suppose $A, B \in M_n(\mathbb{R})$, A is invertible and $B^t = B$. We define the following representation of $\widetilde{G}$ on $L^2(\mathbb{R}^n)$

$$\begin{aligned}
(\sigma(h_1)f)(x) &= \pm e^{-\pi\imath g_0(Bx,x)/2} f(x) \text{ for any } h_1 \in \widetilde{g_1},\ g_1 = \left(\begin{array}{c|c} 1 & 0 \\ \hline B & 1 \end{array}\right) \\
(\sigma(h_2)f)(x) &= \sqrt{\det A^{-1}} f(A^{-1}x) \text{ for any } h_2 \in \widetilde{g_2},\ g_2 = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A^{-1t} \end{array}\right) \\
(\sigma(h_3)f)(x) &= \pm \imath^n e^{\pi\imath n/4} (\mathcal{F}f)(x) \text{ for any } h_3 \in \widetilde{g_3},\ g_3 = J_0 = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right)
\end{aligned}$$

where $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The $\pm$ signs and the square roots in the definition of $\sigma(h_i)$ depend on the specific element in the preimage of g_i. The coordinates of g_i, $i = 1, 2, 3$, are considered with respect to the canonical basis of $\mathbb{R}^{2n}$. See Folland [15]. Notice that we use the Fourier transform defined by $(\mathcal{F}f)(y) = \int_{x \in \mathbb{R}^n} e^{-2\pi\imath g_0(x,y)} f(x) \mathrm{d}x$, $y \in \mathbb{R}^n$, with respect to the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^n$ induced by the scalar product $g_0(x, y) = \omega(x, J_0 y)$, $(x, 0), (y, 0) \in \mathbb{R}^n \times \{0\} \simeq L$. Elements of type g_1, g_2 and g_3 generate $Sp(V, \omega)$. See Folland [15]. Note that in Habermann, Habermann [26], a different convention for the Fourier transform is used. Note that $L^2(\mathbb{R}^n)$ decomposes into the direct sum $L^2(\mathbb{R}^n)_+ \oplus L^2(\mathbb{R}^n)_-$ of irreducible $\widetilde{G}$-modules of the even and of the odd functions in $L^2(\mathbb{R}^n)$. For a proof that σ is a representation, see Folland [15] or Wallach [78] for instance. For a proof that σ intertwines the Schrödinger representation of the Heisenberg group, see Wallach [78], Habermann, Habermann [26] or Folland [15]. A proof that $L^2(L)_\pm$ are irreducible is contained in Folland [15].

Taking the derivative σ_* at the unit element of $\widetilde{G}$ of the representation σ restricted to smooth vectors in $L^2(L)$, we get the representation $\sigma_* : \widetilde{\mathfrak{g}} \to \mathrm{End}\,(S)$ of the Lie algebra of the metaplectic group on the vector space $S = S(L)$ of Schwartz functions on L. See Borel, Wallach [5] and Folland [15] where the smooth vectors are determined. Note that, we have $S \simeq S_+ \oplus S_-$ similarly as in the previous decomposition. For $n \times n$ real matrices $B = B^t$, $C = C^t$ and A, we have (see Folland [15])

$$\begin{aligned}
\sigma_*(X) &= \frac{1}{4\pi\imath}\sum_{i,j=1}^{n} B_{ij}\frac{\partial^2}{\partial x^i \partial x^j} \text{ for } X = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \\
\sigma_*(Y) &= -\pi\imath \sum_{i,j=1}^{n} C_{ij}x^i x^j \text{ for } Y = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \\
\sigma_*(Z) &= -\sum_{i,j=1}^{n} A_{ij}x^j \frac{\partial}{\partial x^i} - \frac{1}{2}\sum_{i=1}^{n} A_{ii} \text{ for } Z = \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}.
\end{aligned}$$

It follows that

$$\sigma_*(J_0) = \imath \sum_{i=1}^{n}\left(\frac{1}{4\pi}\frac{\partial^2}{\partial (x^i)^2} - \pi (x^i)^2\right).$$

Definition 1: For any $m \in \mathbb{N}_0$, we set $h_m(x) = \frac{2^{1/4}}{\sqrt{m!}}(\frac{-1}{2\sqrt{\pi}})^m e^{\pi x^2}\frac{d^m}{dx^m}(e^{-2\pi x^2})$. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, we define the Hermite function h_α with index $\alpha = (\alpha_1, \ldots, \alpha_n)$ by $h_\alpha(x^1, \ldots, x^n) = h_{\alpha_1}(x^1)\ldots h_{\alpha_n}(x^n)$, $(x^1, \ldots, x^n) \in \mathbb{R}^n$.

Remark: For Hermite functions, see Whittaker, Watson [81] and Folland [15]. We use the convention of Folland [15]. Especially, $h_0(x) = 2^{1/4}e^{-\pi x^2}$.

Well known properties of Hermite functions make us able to derive that for any $\alpha = (\alpha_1, \ldots$ $\ldots, \alpha_n) \in \mathbb{N}_0^n$

$$\sigma_*(J_0)h_\alpha = -\imath(|\alpha| + \frac{n}{2})h_\alpha$$

where $|\alpha| = \sum_{i=1}^n \alpha_i$. Thus, the Hermite functions are eigenfunctions of $\sigma_*(J_0)$.

2.3 Weyl algebra and Symplectic spinor multiplication

Let $\mathbb{k}$ be a field of characteristic zero. For any $n \in \mathbb{N}$, the **Weyl algebra** W_n over $\mathbb{k}$ is the associative algebra generated by elements $1 \in \mathbb{k}$, $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfying to the relations $1a_i = a_i 1$, $1b_i = b_i 1$, $a_i b_j - b_j a_i = -\delta_{ij}1$, $a_i a_j = a_j a_i$, $b_i b_j = b_j b_i$, $1 \leq i, j \leq n$. It is known that W_n has a faithful representation on the space of polynomials $\mathbb{k}[q^1, \ldots, q^n]$ given by $1 \mapsto 1$ (multiplication by 1), $a_i \mapsto q^i$ and $b_i \mapsto \frac{\partial}{\partial q^i}$, where q^i denotes the multiplication of a polynomial by q^i and $\frac{\partial}{\partial q^i}$ is the partial derivative in the i-th variable. See, e.g., Björk [3].

Any associative algebra A over field $\mathbb{k}$ can be equipped with the commutator

$$[\,,\,] : A \times A \to A$$

defined by $[x, y] = xy - yx,\ x, y \in A$, making it a Lie algebra. The **Heisenberg Lie algebra** H_n is the real vector space $\mathbb{R}^{2n+1}[q^1, \ldots, q^n, p_1, \ldots, p_n, t]$ with the Lie bracket

$$[\,,\,] : H_n \times H_n \to \{0\} \times \{0\} \times \mathbb{R} \subseteq H_n$$

prescribed on basis by $[\partial_t, \partial_{q^i}] = [\partial_t, \partial_{p_i}] = [\partial_{q^i}, \partial_{q^j}] = [\partial_{p_i}, \partial_{p_j}] = 0$ and $[\partial_{q^i}, \partial_{p_j}] = -\delta_{ij}\partial_t$, $1 \leq i, j \leq n$. Note that $[\,,\,]$ is not the Lie bracket of vector fields in this case. It is the Lie algebra of the Heisenberg group $H(n)$ and it is isomorphic (as a Lie algebra) to

$$W_n(1) = \{\tau 1 + \sum_{i=1}^{n}(\alpha_i a_i + \beta_i b_i) |\, \tau, \alpha_i, \beta_i \in \mathbb{R},\, i = 1, \ldots, n\} \subseteq W_n$$

equipped with the commutator as the Lie algebra bracket. An isomorphism can be given on a basis by $\partial_t \mapsto 1$, $\partial_{q^i} \mapsto a_i$, $\partial_{p_i} \mapsto b_i$, $i = 1, \ldots, n$.

For a symplectic vector space (V, ω) of dimension $2n$ over $\mathbb{R}$, let us choose a symplectic basis $(e_i)_{i=1}^{2n}$ and consider the tensor algebra

$$A = T(V^{\mathbb{C}}) = \mathbb{C} \oplus V^{\mathbb{C}} \oplus (V^{\mathbb{C}} \otimes V^{\mathbb{C}}) \oplus \cdots .$$

Let us set $sCliff(V, \omega) = A/I$, where I is the two sided ideal generated over A by elements $v \otimes w - w \otimes v + \imath\omega(v, w)$, $v, w \in V^{\mathbb{C}}$. The complex associative algebra $sCliff(V, \omega)$ is called the **symplectic Clifford algebra**. Let us consider the map $1 \mapsto 1$, $e_{n+i} \mapsto -a_i$ and $e_{n+1-i} \mapsto \imath b_i$, $i = 1, \ldots, n$, which extends to a homomorphism of associative algebras $sCliff(V, \omega)$ and W_n for $\mathbb{k} = \mathbb{C}$. It is not difficult to see that this map is an isomorphism onto W_n. Summing up, W_n and $sCliff(V, \omega)$ are isomorphic as associative algebras. The Heisenberg Lie algebra H_n embeds homomorphically into $sCliff(V, \omega)$ (considered as a Lie algebra with respect to the commutator) via $\partial_t \mapsto 1$, $\partial_{q^i} \mapsto -e_{n+i}$ and $\partial_{p_i} \mapsto \imath e_{n+1-i}$, $i = 1, \ldots, n$.

Remark: Note that there is an isomorphism of the Heisenberg Lie algebra H_n with $\mathbb{k}_1[q^1, \ldots,$ $\ldots q^n, p_1, \ldots, p_n]$, the space of degree one polynomials in q^i, p_i $(i = 1, \ldots, n)$, equipped with the Poisson bracket

$$\{f, g\}_P = \sum_{i,j=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)$$

where $f, g \in \mathbb{k}_1[q^1, \ldots, q^n, p_1, \ldots, p_n]$.

We come to the following important definition.

Definition 2: Let $(e_i)_{i=1}^{2n}$ be a symplectic basis of (V, ω). For $i = 1, \ldots, n$ and $f \in S$, we set

$$e_i \cdot f = \imath x^i f \quad \text{and} \quad e_{i+n} \cdot f = \frac{\partial f}{\partial x^{n-i+1}}$$

and extend it linearly to V. The map $\cdot : V \times S \to S$ is called the **symplectic spinor multiplication**.

Remark: In the preceding definition, $f \in S(\mathbb{R}^n)$ and x^i denotes the projection onto the i-th coordinate in $\mathbb{R}^n$. Note that the symplectic spinor multiplication depends on the choice of a symplectic basis. Because of its equivariant properties (see Habermann [26], p. 13), one can use the multiplication on the level of bundles. In this case, we denote it by the dot as well. Note that the equivariance of the symplectic Clifford multiplication with respect to the Segal–Shale–Weil representation makes the definitions of the symplectic spinor Dirac, the second symplectic spinor Dirac and the associated operator correct.

3 Symplectic spinors in differential geometry

Let us recall that a **symplectic manifold** is a manifold equipped with a closed non-degenerate exterior differential 2-form ω.

One of the big achievements of Bernhard Riemann in geometry is a definition of the curvature (Krümmungsmaß) in an arbitrary dimension. After publishing of his Habilitationsschrift, Levi-Civita and Riemannian connections became fundamental objects for metric geometries. Intrinsic notions and properties (such as straight lines, angle deficits, parallelism etc.) of many geometries known in that time can be defined and investigated by means of them. Using these connections, one can find out quite easily, whether the given manifold is locally isometric to the Euclidean space.

Definition 3: Let (M, ω) be a symplectic manifold. An affine connection ∇ on M is called symplectic if $\nabla\omega = 0$. Such a connection is called a **Fedosov connection** if it is torsion-free.

For symplectic connections, see, e.g., Libermann [52], Tondeur [74], Vaisman [75] and Gelfand, Retakh, Shubin [19]. In contrast to Riemannian geometry, we have the following theorem which goes back to Tondeur [74]. See Vaisman [75] for a proof.

Theorem 4: The space of Fedosov connections on a symplectic manifold (M, ω) is isomorphic to an affine space modeled on the infinite dimensional vector space $\Gamma(S^3TM)$, where S^3TM denotes the third symmetric product of the tangent bundle of M.

Remark: Note that due to a theorem of Darboux (see McDuff, Salamon [56]), all symplectic manifolds of equal dimension are locally equivalent. In particular, symplectic connections cannot serve for distinguishing of symplectic manifolds in the local sense. From the eighties of the last century, symplectic connections gained an important role in mathematical physics. They became crucial for quantization procedures. See Fedosov [13] and Waldmann [77].

Let (M^{2n}, ω) be a symplectic manifold and ∇ be a Fedosov connection. The **curvature tensor field** of ∇ is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

where $X, Y, Z \in \mathfrak{X}(M)$. A **local symplectic frame** $(U, (e_i)_{i=1}^{2n})$ of (M, ω) is an open subset U in M and a sequence of vector fields e_i on U such that $((e_i)_m)_{i=1}^{2n}$ is a symplectic basis of (T_mM, ω_m) for each $m \in U$.

Let $(U, (e_i)_{i=1}^{2n})$ be a local symplectic frame. For $X = \sum_{i=1}^{2n} X^i e_i, Y = \sum_{i=1}^{2n} Y^i e_i, Z =$

$\sum_{i=1}^{2n} Z^i e_i, V = \sum_{i=1}^{2n} V^i e_i \in \mathfrak{X}(M)$, $X^i, Y^i, Z^i, V^i \in \mathcal{C}^\infty(U)$, and $i, j, k, l = 1, \ldots, 2n$, we set

$$\begin{aligned}
R_{ijkl} &= \omega(R(e_k, e_l)e_j, e_i) \\
\sigma(X, Y) &= \mathrm{Tr}(V \mapsto R(V, X)Y),\ V \in \mathfrak{X}(M) \\
\sigma_{ij} &= \sigma(e_i, e_j) \\
\sigma_{ijkl} &= \frac{1}{2(n+1)}(\omega_{il}\sigma_{jk} - \omega_{ij}\sigma_{lk} + \omega_{jl}\sigma_{ik} - \omega_{jl}\sigma_{ik} + 2\sigma_{ij}\omega_{kl}) \\
\widetilde{\sigma}(X, Y, Z, V) &= \sum_{i,j,k,l=1}^{2n} \sigma_{ijkl} X^i Y^j Z^k V^l \\
W &= R - \widetilde{\sigma}
\end{aligned}$$

where at the last row, R represents the $(4, 0)$-type tensor field $\sum_{i,j,k,l=1}^{2n} R_{ijkl}\epsilon^i \otimes \epsilon^j \otimes \epsilon^k \otimes \epsilon^l$ and $(\epsilon^i)_{i=1}^{2n}$ is the frame dual to $(e_i)_{i=1}^{2n}$.

Definition 4: We call W the **symplectic Weyl curvature**. The $(2, 0)$-type tensor field σ is called the **symplectic Ricci curvature**. A symplectic manifold with a Fedosov connection is called of Ricci-type if $W = 0$ and it is called Ricci-flat if $\sigma = 0$.

Let (M, ω) be a symplectic manifold. We set

$$Q = \{f \text{ is a symplectic basis of } (T_m M, \omega_m) |\, m \in M\}$$

and call it the **symplectic repère bundle**. For any $f = (e_1, \ldots, e_{2n}) \in Q$, we denote by $\pi_Q(f)$ the unique point $m \in M$ such that each vector in f belongs to $T_m M$. The topology on Q is the coarsest one for which π_Q is continuous. It can be seen that $\pi_Q : Q \to M$ is a principal $Sp(2n, \mathbb{R})$-bundle.

Definition 5: A pair (P, Λ) is called a **metaplectic structure** if $\pi_P : P \to M$ is a principal $Mp(2n, \mathbb{R})$-bundle over M and $\Lambda : P \to Q$ is a principal bundle homomorphism such that the following diagram commutes. The horizontal arrows denote the actions of $\widetilde{G}$ and G, respectively.

$$\begin{tikzcd}
P \times \widetilde{G} \arrow[r] \arrow[dd, "\Lambda\times\lambda"] & P \arrow[dd, "\Lambda"] \arrow[dr, "\pi_P"] & \\
 & & M \\
Q \times G \arrow[r] & Q \arrow[ur, "\pi_Q"'] &
\end{tikzcd}$$

A **compatible positive almost complex structure** J on a symplectic manifold (M, ω) is any endomorphism $J : TM \to TM$ such that $J^2 = -1_{TM}$ and such that $g(X, Y) = \omega(X, JY)$, $X, Y \in \mathfrak{X}(M)$, is a Riemannian metric. In particular, g is a symmetric tensor field. Note that J is an isometry and a symplectomorphism as well. A compatible positive almost complex structure always exists on a symplectic manifold (M, ω). See, e.g., McDuff, Salamon [56], pp. 63 and 70, for a proof.

Remark: Note that a Kähler manifold can be defined as a symplectic manifold equipped with a Fedosov connection ∇ and a compatible positive almost complex structure J such that $\nabla J = 0$, i.e., J is ∇-flat. Especially, any Kähler manifold is symplectic. The first example of a compact symplectic manifold which does not admit any Kähler structure was given by Thurston [72]. He was inspired by a review note of Libermann [53] who comments a mistake in an article of Guggenheimer [21]. See also the review [28] of the Guggenheimer's article by Hodge.

In the following theorem, a condition for the existence of a metaplectic structure is given.

Theorem 5: Let (M, ω) be a symplectic manifold and J be a compatible positive almost complex structure. Then (M, ω) possesses a metaplectic structure if and only if the second Stiefel-Whitney class $w_2(TM)$ of TM vanishes if and only if the first Chern class $c_1(TM) \in H^2(M, \mathbb{Z})$ of (TM, J) is even.
Proof. See Kostant [36] and Forger, Hess [16], p. 270. □

Remark: An element $a \in H^2(M, \mathbb{Z})$ is called even if there is an element $b \in H^2(M, \mathbb{Z})$ such that $a = 2b$. By a Chern class of (TM, J), we mean the Chern class of the complexification $TM^{\mathbb{C}}$ defined with the help of the compatible positive almost complex structure J. See Milnor, Stasheff [57].

3.1 Habermann's symplectic Dirac and associated second order operator

We introduce the symplectic Dirac operators and the associated second order operator of K. Habermann. Note that there exists a complex version of the metaplectic structure (so-called Mp^c-structure), and also of the mentioned operators. See Robinson, Rawnsley [64] and Cahen, Gutt, La Fuente Gravy and Rawnsley [10]. Let us notice that Mp^c structures exist globally on any symplectic manifold (see [64]). Generalizations of many results of Habermann, Habermann in [26] to the Mp^c-case are straightforward (see [10]).

Definition 6: Let (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure (P, Λ). The associated bundle $\mathcal{S} = P \times_\sigma S$ is called the **symplectic spinor** or the **Kostant's bundle**. Its smooth sections are called **symplectic spinor fields**.

After introducing the Kostant's bundle, we can set up definitions of the differential operators.

Definition 7: Let ∇ be a symplectic connection on a symplectic manifold (M, ω) admitting a metaplectic structure (P, Λ). Consider the principal connection $TQ \to \mathfrak{sp}(2n, \mathbb{R})$ induced by ∇ and its lift $Z : TP \to \widetilde{\mathfrak{g}}$ to the metaplectic structure. The associated covariant derivative $\nabla^S : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S} \otimes T^*M)$ on symplectic spinor fields is called the **symplectic spinor covariant derivative**. Let $(U, (e_i)_{i=1}^{2n})$ be a local symplectic frame. The operator $D : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ defined for any $\phi \in \Gamma(\mathcal{S})$ by

$$D\phi = \sum_{i,j=1}^{2n} \omega^{ij} e_i \cdot \nabla^S_{e_j} \phi$$

is called the **(Habermann's) symplectic spinor Dirac operator**.

Let J be a compatible positive almost complex structure on a symplectic manifold (M, ω). A **local unitary frame** is a local symplectic frame which is orthogonal with respect to the associated Riemann tensor $g(X, Y) = \omega(X, JY)$, $X, Y \in \mathfrak{X}(M)$.

Definition 8: Let J be a compatible positive almost complex structure on a symplectic manifold which admits a metaplectic structure and $(U, (e_i)_{i=1}^{2n})$ be a local unitary frame. The operator $\widetilde{D} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ defined for any $\phi \in \Gamma(\mathcal{S})$ by

$$\widetilde{D}\phi = \sum_{i=1}^{2n} (Je_i) \cdot \nabla^S_{e_i} \phi$$

is called the **second symplectic spinor Dirac operator**. The operator $\mathfrak{P} = \imath[\widetilde{D}, D]$ is called the **associated second order operator**.

Remark: The associated second order operator $\mathfrak{P}$ is elliptic in the sense that its principal symbol $\sigma(\mathfrak{P}, \xi) : \mathcal{S} \to \mathcal{S}$ is a bundle isomorphism for any non-zero cotangent vector $\xi \in T^*M$. See Habermann, Habermann [26], p. 68.

For symplectic spinor covariant derivative ∇^S and a chosen compatible positive almost complex structure, one defines the formal adjoint $(\nabla^S)^* : \Gamma(\mathcal{S} \otimes T^*M) \to \Gamma(\mathcal{S})$ of ∇^S. See Habermann, Habermann [26].

Definition 9: The **Bochner-Laplace operator** on symplectic spinors $\Delta^S : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ is the composition $\Delta^S = (\nabla^S)^* \circ \nabla^S$.

Definition 10: The curvature tensor field R^S on symplectic spinors induced by a Fedosov connection ∇ is defined by

$$R^S(X,Y)\phi = \nabla^S_X \nabla^S_Y \phi - \nabla^S_Y \nabla^S_X \phi - \nabla^S_{[X,Y]}\phi$$

where $X, Y \in \mathfrak{X}(M)$, $\phi \in \Gamma(\mathcal{S})$ and ∇^S is the symplectic spinor derivative.

In the next theorem, a relation of the associated second order operator $\mathfrak{P}$ to the Bochner-Laplace operator Δ^S on symplectic spinors is described. It is derived by K. Habermann, and it is a parallel to the well known Weitzenböck's and Lichnerowicz's formulas for the Laplace operator of the de Rham differentials (Hodge-Laplace) and the Laplace operator of a Levi-Civita connection (Bochner-Laplace); and for the square of the Dirac operator and the Laplace operator of a Lichnerowicz connection on spinors (Lichnerowicz-Laplace), respectively. See, e.g., Friedrich [18] for the latter formula. We present a version of the Habermann's theorem for Kähler manifold. See Habermann, Habermann [26] for more general versions.

Theorem 6: Let (M, ω, J) be a Kähler manifold and $(U, (e_i)_{i=1}^{2n})$ be a local unitary frame. Then for any $\phi \in \Gamma(\mathcal{S})$

$$\mathfrak{P}\phi = \Delta^S \phi + \imath \sum_{i,j=1}^{2n} (Je_i) \cdot e_j \cdot R^S(e_i, e_j)\phi.$$

Proof. See Habermann, Habermann [26]. $\square$

For complex manifolds of complex dimension one[4], Habermann obtains the following consequence of the formula in Theorem 6.

[4]i.e., Riemann surfaces

Theorem 7: If M is a Riemann surface of genus $g \geq 2$, ω is a volume form on M, and (P, Λ) is a metaplectic structure, then the kernel of the associated second order operator is trivial.
Proof. See Habermann [25]. $\square$

Remark: In [24] and [25], Habermann proves that for T^2 $(g = 1)$ and the trivial metaplectic structure, the null space for $\mathfrak{P}$ is isomorphic to the Schwartz space $S = S(\mathbb{R})$. In the case of the (trivial) metaplectic structure on the sphere, the kernel of the associated second order operator is rather complicated. See Habermann [25] or Habermann, Habermann [26]. In the case of genus $g = 1$ and non-trivial metaplectic structures, the kernel of $\mathfrak{P}$ is trivial as well. For it, see Habermann [25].

For further results on spectra and null-spaces of the introduced operators, see Brasch, Habermann, Habermann [6], Cahen, La Fuente Gravy, Gutt, Rawnsley [10] and Korman [35]. The key features used are the Weitzenböck-type formula (Theorem 6) and an orthogonal decomposition of the Kostant's bundle. To our knowledge, this decomposition was used firstly by Habermann in this context. It is derived from a $\widetilde{K}$-isomorphism between $L^2(\mathbb{R}^n)$ and the Hilbert orthogonal sum $\bigoplus_{m=0}^{\infty} \mathfrak{H}_m$ of the spaces

$$\mathfrak{H}_m = \bigoplus_{\alpha, |\alpha| \leq m} \mathbb{C} h_\alpha, \ m \in \mathbb{N}_0.$$

Recall that $\widetilde{K}$ denotes the preimage in the metaplectic group of the unitary group $U(n)$ by the covering λ. (See Habermann, Habermann [26], p. 18 for a description of the isomorphism.)

3.2 Quantization by symplectic spinors

For a symplectic manifold (M, ω) and a smooth function f on M, we denote by X_f the vector field ω-dual to df, i.e., such a vector field for which

$$\omega(X_f, Y) = (df)Y$$

for any $Y \in \mathfrak{X}(M)$. It is called the Hamiltonian vector field of f. A vector field is called symplectic if its flow preserves the symplectic form. Any Hamiltonian vector field is symplectic but not vice versa. For a study of these notions, we refer to the monograph McDuff, Salamon [56]. Note that in this formalism, a Poisson bracket of two smooth functions f, g on M is defined by

$$\{f, g\}_P = \omega(X_f, X_g).$$

Let (M, ω) be a symplectic manifold admitting a metaplectic structure. For a symplectic vector field Y, let L_Y denote the Lie derivative on the sections of the Kostant's bundle in direction Y. See Habermann, Klein [27] and Kolář, Michor, Slovák [34].

Definition 11: Let (M, ω) be a symplectic manifold admitting a metaplectic structure. For a smooth function f on M, we define a map $\mathfrak{q}(f) : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ by

$$\mathfrak{q}(f)\phi = -\imath\hbar L_{X_f}\phi$$

for any $\phi \in \Gamma(\mathcal{S})$. We call $\mathfrak{q} : f \mapsto \mathfrak{q}(f)$ the **Habermann's map**.

Due to the properties of L_X, it is clear that $\mathfrak{q}$ maps into the vector space endomorphisms of $\Gamma(\mathcal{S})$

$$\mathfrak{q} : \mathcal{C}^\infty(M) \to \mathrm{End}(\Gamma(\mathcal{S})).$$

Moreover, Habermann derives the following theorem.

Theorem 8: Let (M,ω) be a symplectic manifold admitting a metaplectic struture. Then for any $f,g \in \mathcal{C}^\infty(M)$, the Habermann's map satisfies

$$[\mathfrak{q}(f),\mathfrak{q}(g)] = \imath\hbar\,\mathfrak{q}(\{f,g\}_P).$$

Proof. See Habermann, Habermann [26]. □

Remark: The Habermann's map $\mathfrak{q}$ satisfies the quantization condition (see Introduction) and thus, it gives an example of a non-deformed quantization. By this we mean that $\mathfrak{q}$ is a morphism of Poisson algebras $(\mathcal{C}^\infty(M), \{,\})$ and $(\mathrm{End}\,(\Gamma(\mathcal{S})), [,])$ up to a multiple. However notice that usually, a quantization is demanded to be a map on smooth functions $\mathcal{C}^\infty(M)$ defined on the phase space M into the space of operators on the vector space $L^2(N)$ of L^2-functions or L^2-sections of a line bundle over N where N denotes the Riemannian manifold of the configuration space. See Souriau [70] and Blau [4] for conditions on quantization maps, their constructions and examples.

4 Author's results in Symplectic spinor geometry

We present results achieved by the author in differential geometry concerning symplectic spinors that we consider important and relevant. We start with a chapter on representational theoretical, or if we wish equivariant, properties of exterior differential forms with values in symplectic spinors.

4.1 Decomposition of tensor products and a Howe-type duality

Let $\mathfrak{g}$ be the Lie algebra of symplectic group $Sp(2n,\mathbb{R})$, $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, Δ^+ a choice of positive roots, and $\{\varpi_i\}_{i=1}^n$ the set of fundamental weights with respect to these choices. Let us denote the irreducible complex highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ by $L(\lambda)$. For any $\lambda = \sum_{i=1}^n \lambda_i \varpi_i$, we set $L(\lambda_1,\ldots,\lambda_n) = L(\lambda)$. For $i = 0,\ldots,2n$, we denote by σ^i the tensor product representation of the complexified symplectic Lie algebra $\mathfrak{g}^{\mathbb{C}}$ on $E^i = \bigwedge^i V^* \otimes S$, i.e., $\sigma^i : \mathfrak{g}^{\mathbb{C}} \to \mathrm{End}\,(E^i)$ and $\sigma^i(X)(\alpha \otimes s) = \lambda_*^{\wedge i}(X)\alpha \otimes s + \alpha \otimes \sigma_*(X)s$ for any $X \in \mathfrak{g}^{\mathbb{C}}, \alpha \in \bigwedge^i V^*$ and $s \in S$, where $\lambda_*^{\wedge i}$ denotes the action of $\mathfrak{g}^{\mathbb{C}}$ on $\bigwedge^i V^*$. We consider $E = \bigoplus_{i=0}^{2n} E^i$ equipped with the direct sum representation $\sigma^\bullet(X) = (\sigma^0(X),\ldots,\sigma^{2n}(X))$, $X \in \mathfrak{g}^{\mathbb{C}}$. Let us notice that here, σ_* denotes the complex linear extension of the representation $\sigma_* : \mathfrak{g} \to \mathrm{End}(S)$ considered above.

Remark: Note that there is a misprint in Krýsl [46]. Namely, the "action" of $\mathfrak{g}$ on E (denoted by $\mathbb{W}$ there) is prescribed by $X(\alpha \otimes s) = \lambda_*^{\wedge i}(X)\alpha \otimes \sigma_*(X)s$ for $X \in \mathfrak{g}$, $\alpha \in \bigwedge^i V$, $s \in S$, and $i = 0,\ldots,2n$. Actually, we meant the standard tensor product representation as given above, i.e., $X(\alpha \otimes s) = \lambda_*^{\wedge i}(X)\alpha \otimes s + \alpha \otimes \sigma_*(X)s$. However, the results in [46] are derived for the correct action $\sigma^\bullet$ defined above.

Definition 12: Let us set $\Xi = \{(i,j_i)|\, i = 0,\ldots,n, j_i = 0,\ldots,i\} \cup \{(i,j_i)|\, i = n+1,\ldots,2n, j_i = 0,\ldots,2n-i\}$, $\mathrm{sgn}(+) = 0$, $\mathrm{sgn}(-) = 1$, and

$$E_\pm^{ij} = L(\underbrace{\frac{1}{2},\cdots,\frac{1}{2}}_{j},\underbrace{-\frac{1}{2},\cdots,-\frac{1}{2}}_{n-j-1},-1+\frac{1}{2}(-1)^{i+j+\mathrm{sgn}(\pm)})$$

for $i = 0, \ldots, n-1$, $j = 0, \ldots, i$ and $i = n$, $j = 0, \ldots, n-1$. For $i = j = n$, we set $E_+^{nn} = L(\frac{1}{2}, \cdots, \frac{1}{2})$ and $E_-^{nn} = L(\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{5}{2})$. For $i = n+1, \ldots, 2n$ and $j = 0, \ldots, 2n-i$, we set $E_\pm^{ij} = E_\pm^{(2n-i)j}$. For any $(i,j) \in \mathbb{Z} \times \mathbb{Z} \setminus \Xi$, we define $E_\pm^{ij} = 0$. Finally for any $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, we set $E^{ij} = E_+^{ij} \oplus E_-^{ij}$. For $(i,j) \in \Xi$, the $\mathfrak{g}^\mathbb{C}$-modules $E_\pm^{ij}$ are called **higher symplectic spinor modules** and their elements **higher symplectic spinors**.

Theorem 9: The following decomposition into irreducible $\mathfrak{g}^\mathbb{C}$-modules holds

$$\bigwedge^\bullet V^* \otimes S_\pm = \bigoplus_{(i,j) \in \Xi} E_\pm^{ij}.$$

Proof. Krýsl [46]. □

Remark: The decomposition holds also on the level of minimal and hyperfunction globalizations since the corresponding globalization functors are adjoint functors to the Harish-Chandra forgetful functor. See Vogan [76] and Casselmann [12]. It holds also for smooth Fréchet globalization $\widetilde{G} \to \mathrm{Aut}(S)$. By abuse of notation, we shall denote the tensor product representation of $\widetilde{G}$ on E by $\sigma^\bullet$ as well. The above decomposition holds also when V^* is replaced by V since the symplectic form gives an isomorphism of the appropriate representations of $\mathfrak{g}^\mathbb{C}$.

Definition 13: For $i = 0, \ldots, 2n$, we denote the uniquely determined equivariant projections of $\bigwedge^i V \otimes S^\pm \to E_\pm^{ij} \subseteq \bigwedge^i V \otimes S_\pm$ by $p_\pm^{ij}$ and the projections $p_+^{ij} + p_-^{ij}$ onto E^{ij} by p^{ij}, $(i,j) \in \mathbb{Z} \times \mathbb{Z}$.

Let us recall a definition of the simple Lie superalgebra $\mathfrak{osp}(1|2)$. It is generated by elements e^+, e^-, h, f^+, f^- satisfying the following relations

$$\begin{aligned} [h, e^\pm] &= \pm e^\pm & [e^+, e^-] &= 2h \\ [h, f^\pm] &= \pm\frac{1}{2} f^\pm & \{f^+, f^-\} &= \frac{1}{2} h \\ [e^\pm, f^\mp] &= -f^\pm & \{f^\pm, f^\pm\} &= \pm\frac{1}{2} e^\pm \end{aligned}$$

where $\{,\}$ denotes the anticommutator, i.e., $\{a,b\} = ab + ba$, $a, b \in \mathfrak{osp}(1|2)$.

We give a $\mathbb{Z}_2$-grading to the vector space $E = \bigwedge^\bullet V \otimes S$ by setting $E_0 = \bigoplus_{i=0}^n \bigwedge^{2i} V \otimes S$, $E_1 = \bigoplus_{i=1}^n \bigwedge^{2i-1} V \otimes S$ and $E = E_0 \oplus E_1$. Further, we choose a symplectic basis $(e_i)_{i=1}^{2n}$ of (V, ω) and denote its dual basis by $(\epsilon^i)_{i=1}^{2n} \subseteq V^*$. The Lie superalgebra $\mathfrak{osp}(1|2)$ has a representation $\rho : \mathfrak{osp}(1|2) \to \mathrm{End}(E)$ on the superspace E given by

$$\rho(f^+)(\alpha \otimes s) = \frac{\imath}{2} \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes e_i \cdot s \quad \text{and} \quad \rho(f^-)(\alpha \otimes s) = \frac{1}{2} \sum_{i=1}^{2n} \omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot s$$

where $\alpha \in \bigwedge^\bullet V^*$, $s \in S$, and ι_v denotes the contraction by the vector v. Consequently, elements e^+, e^- and h act by

$$\rho(e^\pm) = \pm 2\{\rho(f^\pm), \rho(f^\pm)\} \quad \text{and} \quad \rho(h) = \frac{1}{2}[\rho(e^+), \rho(e^-)]$$

where $\{\,,\,\}$ and $[\,,\,]$ denote the anticommutator and the commutator on the associative algebra $\mathrm{End}(E)$, respectively.

The following theorem is parallel to the Schur and Weyl dualities for tensor representations of $GL(n,\mathbb{C})$ and $SO(n,\mathbb{C})$, respectively. See Howe [29] where they are treated.

Theorem 10: The following $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{osp}(1|2)$-module isomorphism holds

$$\bigwedge^{\bullet} V^* \otimes S \simeq \bigoplus_{i=0}^{n} (E_+^{ii} \otimes F_i) \oplus \bigoplus_{i=0}^{n} (E_-^{ii} \otimes F_i)$$

where $F_i = \mathbb{C}^{2n-2i+1}$ and $\rho_i : \mathfrak{osp}(1|2) \to \mathrm{End}(F_i)$ is given on a basis $(b_j)_{j=i}^{2n-i}$ of F_i by prescriptions

$$\begin{aligned} &\rho_i(f^+)(b_j) = A(n,i+1,j)b_{j+1} & &\rho_i(f^-)(b_j) = b_{j-1} \\ &\rho_i(h) = 2\{\rho_i(f^+),\rho_i(f^-)\} \quad \text{and} & &\rho_i(e^\pm) = \pm 2\{\rho_i(f^\pm),\rho_i(f^\pm)\} \end{aligned}$$

where $i = 0,\dots,n$ and $A(n,i,j) = \frac{(-1)^{i-j}+1}{16}(j-i) + \frac{(-1)^{i-j+1}+1}{16}(i+j-2n-1)$.
Proof. See Krýsl [46]. $\square$

Remark: In the preceding definition, if an index exceeds its allowed range, the object is considered to be zero. Thus, e.g., b_{2n-i+1} or b_{i-2} are zero vectors.

Theorem 11: For $i = 0,\dots,n$, representations F_i are irreducible.
Proof. See Krýsl [46]. $\square$

Remark: Representations ρ_i in Theorem 10 depend on the choice of a basis, but not their equivalence class. As follows from Theorem 11, the multiplicity of $E_\pm^{ii}$ in the $\mathfrak{g}^{\mathbb{C}}$-module E is $2n-2i+1$ for $i = 0,\dots,n$.

4.2 Differential geometry of higher symplectic spinors

For any symplectic manifold (M,ω) admitting a metaplectic structure (P,Λ), the decomposition from Theorem 9 can be lifted to the associated bundle $\mathcal{E} = P \times_{\sigma^\bullet} E$.

Remark: Since S is a smooth globalization, we may consider E as a representation of the metaplectic group as well.

Definition 14: Let (M,ω) be a symplectic manifold admitting a metaplectic structure (P,Λ). For any $(i,j) \in \mathbb{Z}\times\mathbb{Z}$, we set $\mathcal{E}^{ij} = P \times_{\widetilde{G}} E^{ij}$ and call it the **higher symplectic spinor bundle** and elements of its section spaces the **higher symplectic spinor fields** if $(i,j) \in \Xi$.

We keep denoting the lifts of the projections $\bigwedge^i V^* \otimes S \to E^{ij}$ to $\Gamma(\mathcal{E}^i) \to \Gamma(\mathcal{E}^{ij})$ by p^{ij}, where $\mathcal{E}^i = P \times_{\sigma^i} E^i$.

4.2.1 Curvature, higher curvature and symplectic twistor complexes

For a Fedosov connection ∇ on a symplectic manifold (M,ω) admitting a metaplectic structure, we consider the exterior covariant derivative d^{∇^S} for the induced symplectic spinor derivative ∇^S. See, e.g., Kolář, Michor, Slovák [34] for a general construction of such derivatives.

Theorem 12: Let (M,ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection. Then for any $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, the restriction of the exterior symplectic spinor derivative satisfies

$$d^{\nabla^S} : \Gamma(\mathcal{E}^{ij}) \to \Gamma(\mathcal{E}^{i+1,j-1}) \oplus \Gamma(\mathcal{E}^{i+1,j}) \oplus \Gamma(\mathcal{E}^{i+1,j+1}).$$

Proof. See Krýsl [38]. □

Remark: In particular, sections of each higher symplectic spinor bundle are mapped into sections of at most three higher symplectic spinor bundles. Note that in the case of orthogonal spinors in pseudo-Riemannian geometry, the target space structure of the exterior covariant derivative is similar. See Slupinski [68].

Let $(e_i)_{i=1}^{2n}$ be a local symplectic frame on (M,ω) and $(\epsilon^i)_{i=1}^{2n}$ be its dual symplectic coframe. Recall that above, we defined the symplectic Ricci and symplectic Weyl curvature tensor fields. Let us denote by σ^S the endomorphism of the symplectic spinor bundle defined for any $\phi \in \mathcal{S}$ by

$$\sigma^S\phi = \frac{\imath}{2}\sum_{i,j,k,l=1}^{2n} \sigma^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_i \cdot e_j \cdot \phi.$$

Similarly we set

$$W^S\phi = \frac{\imath}{2}\sum_{i,j,k,l=1}^{2n} W^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_i \cdot e_j \cdot \phi.$$

Recall that

$$\bigwedge^2 T^*M \otimes \mathcal{S} = \mathcal{E}^{20} \oplus \mathcal{E}^{21} \oplus \mathcal{E}^{22}$$

according to Theorem 9.

In the next theorem, components of R^S in $\mathcal{E}^{20}$, $\mathcal{E}^{21}$ and $\mathcal{E}^{22}$ are found. We notice that

1) we use the summation convention, i.e., if two indices occur which are labeled by the same letter, we sum over it without denoting the sum explicitly and
2) instead of $e_i \cdot e_j \cdot$, we write $e_{ij} \cdot$ and similarly for a higher number of indices.

Theorem 13: Let $n > 1$, (M^{2n},ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection. Then for any $\phi \in \Gamma(\mathcal{S})$, $\sigma^S\phi \in \Gamma(\mathcal{E}^{20} \oplus \mathcal{E}^{21})$ and $W^S\phi \in \Gamma(\mathcal{E}^{21} \oplus \mathcal{E}^{22})$. Moreover, we have the following projection formulas

$$\begin{aligned}
p^{20}R^S\phi &= \frac{\imath}{2n}\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi \\
p^{21}R^S\phi &= \frac{\imath}{n+1}\sigma^{ij}\epsilon^k \wedge \epsilon^l \otimes (\omega_{il}e_{kj} \cdot - \frac{1}{2n}\omega_{kl}e_{ij} \cdot)\phi - \frac{\imath}{1-n}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij} \cdot \phi \\
p^{22}R^S\phi &= \frac{\imath}{2}W^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi + \frac{\imath}{1-n}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij} \cdot \phi.
\end{aligned}$$

Proof. See Krýsl [42]. $\square$

Remark: Note that for $n = 1$, $E^{21} = E^{22} = 0$, so that there is no Weyl component of the curvature tensor of a Fedosov connection in this dimension. The formula for p^{20} holds also for $n = 1$.

Definition 15: For $(i, j), (i + 1, k) \in \Xi$, $a = 0, \ldots, n - 1$ and $b = n, \ldots, 2n - 1$, let us set

$$D^{ij}_{i+1,k} = p^{i+1,k} d^{\nabla^S}_{|\Gamma(\mathcal{E}^{ij})} : \Gamma(\mathcal{E}^{ij}) \to \Gamma(\mathcal{E}^{i+1,k}), \quad T_a = D^{aa}_{a+1,a+1} \quad \text{and} \quad T_b = D^{b,2n-b}_{b+1,2n-b-1}.$$

The operators T_i, $i = 0, \ldots, 2n - 1$, are called the **symplectic twistor operators**.

Let (V, ω) be a symplectic vector space, $(e_i)^{2n}_{i=1}$ be a symplectic basis, $(\epsilon^i)^{2n}_{i=1}$ be a basis of V^* dual to $(e_i)^{2n}_{i=1}$, and $\sigma \in S^2V^*$ be a bilinear form. For $\alpha \in \bigwedge^\bullet V^*$ and $s \in S$, we set

$$\Sigma^\sigma(\alpha \otimes s) = \sum_{i,j=1}^{2n} \sigma^i{}_j \epsilon^j \wedge \alpha \otimes e_i \cdot s$$

and

$$\Theta^\sigma(\alpha \otimes s) = \sum_{i,j=1}^{2n} \alpha \otimes \sigma^{ij} e_i \cdot e_j \cdot s.$$

We keep denoting the corresponding tensors on symplectic spinor bundles by the same symbols. In this case, the the symplectic Ricci curvature tensor field plays the role of the tensor σ.

We use abbreviations

$$E^\pm = \rho(e^\pm) : E \to E \text{ and } F^\pm = \rho(f^\pm) : E \to E.$$

Let (M, ω) be a symplectic manifold which admits a metaplectic structure and ∇ be a Fedosov connection of Ricci-type. For a higher symplectic spinor field $\phi \in \Gamma(\mathcal{E})$, we have (see Krýsl [43]) the following formula

$$R^E\phi = \frac{1}{n+1}(E^+\Theta^\sigma + 2F^+\Sigma^\sigma)\phi.$$

Remark: By the higher curvature, we understand the curvature of ∇^S on higher symplectic spinors, i.e., $R^E = d^{\nabla^S} \circ d^{\nabla^S}$.

The above formula is used for proving the next theorem.

Theorem 14: Let $n > 1$, (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection of Ricci-type. Then

$$0 \longrightarrow \Gamma(\mathcal{E}^{00}) \xrightarrow{T_0} \Gamma(\mathcal{E}^{11}) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-1}} \Gamma(\mathcal{E}^{nn}) \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow \Gamma(\mathcal{E}^{nn}) \xrightarrow{T_n} \Gamma(\mathcal{E}^{n+1,n+1}) \xrightarrow{T_{n+1}} \cdots \xrightarrow{T_{2n-1}} \Gamma(\mathcal{E}^{2n,2n}) \longrightarrow 0$$

are complexes.
Proof. See Krýsl [43]. $\square$

We call the complexes from Theorem 14 the **symplectic twistor complexes**.

Theorem 15: Let $n > 1$, (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection of Ricci-type. Then

$$0 \longrightarrow \Gamma(\mathcal{E}^{00}) \xrightarrow{T_0} \cdots \xrightarrow{T_{n-2}} \Gamma(\mathcal{E}^{n-1,n-1}) \xrightarrow{T_n T_{n-1}} \Gamma(\mathcal{E}^{n+1,n+1}) \xrightarrow{T_{n+1}} \cdots \xrightarrow{T_{2n-1}} \Gamma(\mathcal{E}^{2n,2n}) \longrightarrow 0$$

is a complex.
Proof. See Krýsl [43]. □

Definition 16: Let $(\mathcal{F}^i \to M)_{i \in \mathbb{Z}}$ be a sequence of vector bundles over a smooth manifold M, $D^\bullet = (\Gamma(\mathcal{F}^i), D_i : \Gamma(\mathcal{F}^i) \to \Gamma(\mathcal{F}^{i+1}))_{i \in \mathbb{Z}}$ be a complex of pseudodifferential operators and for each $\xi \in T^*M$, let $\sigma(D)(\xi)^\bullet = (\mathcal{F}_i, \sigma(D_i, \xi) : \mathcal{F}^i \to \mathcal{F}^{i+1})_{i \in \mathbb{Z}}$ be the complex of symbols evaluated in ξ which is associated to the complex $D^\bullet$. We call $D^\bullet$ **elliptic** if $\sigma(D)(\xi)^\bullet$ is an exact sequence in the category of vector bundles for any $\xi \in T^*M \setminus \{0\}$.

Remark: Note that in homological algebra, the above complexes are usually called cochain complexes.

Theorem 16: Let $n > 1$, (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection of Ricci-type. Then the complexes

$$0 \longrightarrow \Gamma(\mathcal{E}^0) \xrightarrow{T_0} \Gamma(\mathcal{E}^1) \xrightarrow{T_1} \cdots \xrightarrow{T_{n-2}} \Gamma(\mathcal{E}^{n-1}) \text{ and}$$

$$\Gamma(\mathcal{E}^n) \xrightarrow{T_n} \Gamma(\mathcal{E}^{n+1}) \xrightarrow{T_{n+1}} \cdots \xrightarrow{T_{2n-1}} \Gamma(\mathcal{E}^{2n}) \longrightarrow 0$$

are elliptic.
Proof. See Krýsl [44]. □

4.2.2 Symplectic spinor Dirac, twistor and Rarita–Schwinger operators

Definition 17: Let (M, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection. The operators

$$\mathfrak{D} = F^- \circ D^{00}_{10} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}) \qquad \text{and } \mathfrak{R} = F^- \circ D^{11}_{21} : \Gamma(\mathcal{E}^{11}) \to \Gamma(\mathcal{E}^{11})$$

are called the **symplectic spinor Dirac** and the **symplectic spinor Rarita–Schwinger operator**, respectively.

Remark: $\mathfrak{D}$ is the 1/2 multiple of the Habermann's symplectic spinor Dirac operator.

Let us denote the set of eigenvectors of a vector space endomorphism $G : W \to W$ by eigen(G) and the set of its eigenvalues by *spec*(G). Recall that by an eigenvalue, we mean simply a complex number μ, for which there is a nonzero $w \in W$, such that $Gw = \mu w$. (We do not investigate spectra from the functional analysis point of view.)

Definition 18: A **symplectic Killing spinor field** is any not everywhere zero section $\phi \in \Gamma(\mathcal{S})$ for which there exists $\mu \in \mathbb{C}$ such that

$$\nabla^S_X \phi = \mu X \cdot \phi$$

for each $X \in \mathfrak{X}(M)$. (The dot denotes the symplectic Clifford multiplication.) The set of symplectic Killing spinor fields is denoted by kill. Number μ from the above equation is called the **symplectic Killing spinor number** and its set is denoted by *kill*.

Remark: The equation for a symplectic Killing spinor field can be written also as

$$\nabla^S \phi = -2\mu\imath F^+ \phi.$$

Remark: Note that there is a misprint in the abstract in Krýsl [39]. Namely, we write there that $-\imath l\lambda$ is not a symplectic Killing number instead of $\frac{\imath\lambda}{l}$ is not a symplectic Killing spinor number. In that paper, l denotes the half of the dimension of the corresponding symplectic manifold.

Theorem 17: If (M, ω) is a symplectic manifold admitting a metaplectic structure and ∇ is a Fedosov connection, then

$$\mathrm{kill} = \mathrm{Ker}\, T_0 \cap \mathrm{Ker}\, \mathfrak{D}.$$

Proof. See Krýsl [45]. □

Theorem 18: Let (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection with Ricci tensor σ. Let ϕ be a symplectic Killing spinor field to the symplectic Killing spinor number μ. Then in a local symplectic frame $(U, (e_i)_{i=1}^{2n})$, we have

$$\Theta^\sigma \phi = 2\mu^2 n\phi.$$

Proof. See Krýsl [45]. □

As a consequence of this theorem, we have

Theorem 19: Let (M, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Ricci-flat Fedosov connection. Then $kill = \{0\}$ and any symplectic Killing spinor field on M is locally covariantly constant.
Proof. See Krýsl [45]. □

Remark: By a locally covariantly constant field ϕ, we mean $\nabla^S \phi = 0$ which implies that ϕ is locally constant if the Kostant's bundle is trivial.

Theorem 20: Let $n > 1$, (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a flat Fedosov connection. Then

(1) If $\mu \in spec(\mathfrak{D}) \setminus \frac{\imath n}{2} kill$, then $\frac{n-1}{n}\mu \in spec(\mathfrak{R})$.

(2) If $\phi \in \mathrm{eigen}(\mathfrak{D}) \setminus \mathrm{kill}$, then $T_0\phi \in \mathrm{eigen}(\mathfrak{R})$.

Proof. See Krýsl [39]. □

Remark: For any $\lambda \in \mathbb{C}$, $\lambda kill$ denotes the number set $\{\lambda\alpha,\, \alpha \in kill\}$.

4.3 First order invariant operators in projective contact geometry

Some of the results described above can be modified to get information for contact projective manifolds which are more complicated objects to handle than the symplectic ones. Contact manifolds are models for time-dependent Hamiltonian mechanics. The adjective 'projective' is related to the fact that we want to deal with unparametrized geodesics rather than with the ones with a fixed parametrization. Connections that we consider are partial in the sense that they act on sections of the contact bundle only.

Definition 19: A **contact manifold** is a manifold M together with a corank one subbundle HM (contact bundle) of the tangent bundle TM which is not integrable in the Frobenius sense in any point of the manifold, i.e., for each $m \in M$, there are $\eta_m, \zeta_m \in H_m M$ such that $[\eta_m, \zeta_m] \notin HM$.

Equivalently, HM is a contact bundle if and only if the Levi bracket

$$L(X,Y) = q([X,Y])$$

is non-degenerate. Here $X, Y \in \Gamma(HM)$ and $q : TM \to QM = TM/HM$ denotes the quotient projection onto QM. The Levi bracket induces a tensor field which we denote by the same letter $L : \bigwedge^2 HM \to QM$.

Definition 20: For a contact manifold (M, HM), a partial connection $\nabla : \Gamma(HM) \times \Gamma(HM) \to \Gamma(HM)$ is called a **contact connection** if the associated exterior covariant derivative d^∇ on $\Gamma(\bigwedge^2 HM)$ preserves the kernel of the Levi form, i.e., $d^\nabla_\zeta(\operatorname{Ker} L) \subseteq \operatorname{Ker} L$ for any $\zeta \in HM$. The set of contact connections is denoted by $\mathcal{C}_M$. A **contact projective manifold** is a contact manifold (M, HM) together with a set S_M of contact connections for which the following holds. If $\nabla^1, \nabla^2 \in S_M$, there exists a differential one-form $\Upsilon \in \Gamma(HM^*)$ such that for any $X, Y \in \Gamma(HM)$

$$\nabla^1_X Y - \nabla^2_X Y = \Upsilon(X)Y + \Upsilon(Y)X + \Upsilon^\sharp(L(X,Y))$$

where $\Upsilon^\sharp : QM \to HM$ is a bundle morphism defined by $L(\Upsilon^\sharp(\eta), \zeta) = \Upsilon(\zeta)\eta$, $\zeta \in QM$ and $\eta \in HM$. Morphisms between contact projective manifolds (M, HM, S_M) and (N, HN, S_N) are local diffeomorphisms $f : M \to N$ such that $f_*(HM) = HN$, and for any $\nabla \in S_N$, the pull-back connection $f^*\nabla \in S_M$.

Remark: For a contact projective manifold (M, HM, S_M), it is easy to see that the relation $R = S_M \times S_M \subseteq \mathcal{C}_M \times \mathcal{C}_M$ on the set of contact connections $\mathcal{C}_M$ is an equivalence.

Let (V, ω) be a real symplectic vector space of dimension $2n+2$ and $(e_i)_{i=1}^{2n+2}$ be a symplectic basis. The action of the symplectic group G' of (V, ω) on the projectivization of V is transitive and its stabilizer P' is a parabolic subgroup of G'. We denote the preimages of G' and P' by the covering $\lambda' : Mp(2n+2, \mathbb{R}) \to Sp(2n+2, \mathbb{R})$ by $\widetilde{G}'$ and $\widetilde{P}'$, respectively.

Definition 24: A **projective contact Cartan geometry** is a Cartan geometry $(\mathcal{G}', \vartheta)$ whose model is the Klein geometry $G' \to G'/P'$ with G' and P' as introduced above. We say that a Cartan geometry is a **metaplectic projective contact Cartan geometry** if it is modeled on the Klein geometry $\widetilde{G}'/\widetilde{P}'$.

Remark: For Cartan geometries, see Sharpe [67] and Čap, Slovák [11]. In Čap, Slovák [11], a theorem is proved on an equivalence of the category of the so-called regular normal projective contact Cartan geometries and the category of regular normal projective contact manifolds. See Čap, Slovák [11], pp. 277 and 410. See also Fox [17].

The Levi part $\widetilde{G_0}$ of $\widetilde{P}'$ is isomorphic $Mp(2n,\mathbb{R}) \times \mathbb{R}^{\times}$ with the semisimple part $\widetilde{G_0^{ss}} \simeq \widetilde{G} = Mp(2n,\mathbb{R})$ and the center isomorphic to the multiplicative group $\mathbb{R}^{\times}$. The Lie algebra $\mathfrak{p}'$ of $\widetilde{P}'$ is graded, $\mathfrak{p}' = (\mathfrak{sp}(2n,\mathbb{R}) \oplus \mathbb{R}) \oplus \mathbb{R}^{2n} \oplus \mathbb{R}$ with $\mathfrak{g}_0 \simeq \mathfrak{sp}(2n,\mathbb{R}) \oplus \mathbb{R}$, $\mathfrak{g}_1 \simeq \mathbb{R}^{2n}$ and $\mathfrak{g}_2 \simeq \mathbb{R}$. We denote the Lie algebra of $\widetilde{G}'$ by $\mathfrak{g}'$ and identify it with the Lie algebra $\mathfrak{sp}(2n+2,\mathbb{R})$. The semi-simple part $\mathfrak{g}_0^{ss}$ of $\mathfrak{g}_0$ is isomorphic $\mathfrak{sp}(2n,\mathbb{R})$. We denote it by $\mathfrak{g}$ in order to be consistent with the preceding sections. The grading of $\mathfrak{g}' = \bigoplus_{i=-2}^{2} \mathfrak{g}_i$, $\mathfrak{g}_{-2} \simeq \mathfrak{g}_2$ and $\mathfrak{g}_{-1} \simeq \mathfrak{g}_1$, can be visualized with respect to the basis $(e_i)_{i=1}^{2n+2}$ by the following block diagonal matrix of type $(1,n,1) \times (1,n,1)$

$$\mathfrak{g} = \left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right).$$

The center of the Lie algebra $\mathfrak{g}_0$ is generated by

$$Gr = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right)$$

which is usually called the **grading element** because of the property $[Gr, X] = jX$ for each $X \in \mathfrak{g}_j$ and $j = -2, \ldots, 2$.

Let $\kappa : (\mathfrak{g}^{\mathbb{C}})^* \times (\mathfrak{g}^{\mathbb{C}})^* \to \mathbb{C}$ be the dual form to the Killing form of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(2n,\mathbb{C})$. We choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ and a set of positive roots obtaining the set of fundamental weights $\{\varpi_i\}_{i=1}^{n}$ for $\mathfrak{g}^{\mathbb{C}}$. Further, we set $\langle X, Y\rangle = (4n+4)\kappa(X,Y)$, $X, Y \in (\mathfrak{g}^{\mathbb{C}})^*$, and define

$$c_{\lambda\nu}^{\mu} = \frac{1}{2}[\langle\lambda, \lambda + 2\delta\rangle + \langle\nu, \nu + 2\delta\rangle - \langle\mu, \mu + 2\delta\rangle]$$

for any $\lambda, \mu, \nu \in \mathfrak{h}^*$, where δ is the sum of fundamental weights, or equivalently, the half-sum of positive roots. For any $\mu \in \mathfrak{h}^*$, we set

$$A = \{\sum_{i=1}^{n} \lambda_i \varpi_i |\, \lambda_i \in \mathbb{N}_0, i = 1, \ldots, n-1, \lambda_n + 2\lambda_{n-1} + 3 > 0, \lambda_n \in \mathbb{Z} + \frac{1}{2}\} \subseteq \mathfrak{h}^* \text{ and}$$

$$A_\mu = A \cap \{\mu + \nu |\, \nu = \pm\epsilon_i, i = 1, \ldots, n\}$$

where $\epsilon_1 = \varpi_1$, $\epsilon_i = \varpi_i - \varpi_{i-1}, i = 2, \ldots, n$.

Considering $\mathbb{C}^{2n}$ with the defining representation of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(2n,\mathbb{C})$, i.e., $\mathbb{C}^{2n} = L(\varpi_1)$, we have the following decomposition.

Theorem 21: For any $\mu \in A$, the following decomposition into irreducible $\mathfrak{g}^{\mathbb{C}}$-modules

$$L(\mu) \otimes \mathbb{C}^{2n} = \bigoplus_{\lambda \in A_\mu} L(\lambda)$$

holds.

Proof. See Krýsl [40]. □

Remark: The above decomposition has the same form when we consider the algebra $\mathfrak{sp}(2n,\mathbb{R})$ instead of $\mathfrak{sp}(2n,\mathbb{C})$.

The set $\{L(\lambda)|\, \lambda \in A\}$ coincides with the set of all infinite dimensional irreducible $\mathfrak{g}^{\mathbb{C}}$-modules with bounded multiplicities, i.e., those irreducible $\mathfrak{sp}(2n,\mathbb{C})$-modules W for which there exists a bound $l \in \mathbb{N}$ such that for any weight ν, $\dim W_\nu \leq l$.[5] See Britten, Hooper, Lemire [8] and Britten, Lemire [9].

In the next four steps, we define $\widetilde{P}$-modules $\mathbf{L}(\lambda, c, \gamma)$ for any $\lambda \in A$, $c \in \mathbb{C}$ and $\gamma \in \mathbb{Z}_2$.

1) Let $\mathbb{S}$ and $\mathbb{S}_+$ be the $\mathfrak{g}^{\mathbb{C}}$-modules of smooth $\widetilde{K}$-finite vectors of the $Mp(2n,\mathbb{R})$-modules $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)_+$, respectively. Recall that $L^2(\mathbb{R}^n)$ denotes the Segal–Shale–Weil module and $L^2(\mathbb{R}^n)_+$ is the submodule of even functions in $L^2(\mathbb{R}^n)$. For any $\lambda \in A$, there is an irreducible finite dimensional $\mathfrak{g}^{\mathbb{C}}$-module $F(\nu)$ with highest weight $\nu \in \mathfrak{h}^*$ such that $L(\lambda)$ is an irreducible summand in $\mathbb{S}_+ \otimes F(\nu) = \bigoplus_{i=1}^k \mathbb{S}_i$. For it, see Britten, Lemire [9]. Otherwise said, there exists a $j \in \{1, \ldots, k\}$ such that $L(\lambda) \simeq \mathbb{S}_j$. The tensor product of the smooth globalization $S = S(\mathbb{R}^n)$ of $\mathbb{S}$ with $F(\nu)$ decomposes into a finite number of irreducible $\widetilde{G}$-submodules in the corresponding way
$$S_+ \otimes F(\nu) = \bigoplus_{i=1}^k S_i$$
i.e., $\mathbb{S}_i$ is the $\mathfrak{g}^{\mathbb{C}}$-module of smooth $\widetilde{K}$-finite vectors in S_i. We set $\mathbf{L}(\lambda) = S_j$, obtaining a $\widetilde{G}$-module.

2) We let the element $\exp(Gr) \in \widetilde{G_0}$ act by the scalar $\exp(c)$ (the conformal weight) on $\mathbf{L}(\lambda)$ and denote the resulting structure by $\mathbf{L}(\lambda, c)$.

3) Let us consider the element $(1, -1) \in Sp(2n,\mathbb{R}) \times \mathbb{R}^\times \subseteq \lambda'(\widetilde{G_0}) \subseteq P$ and the preimage $\Gamma = {\lambda'}^{-1}((1,-1)) \subseteq \widetilde{G_0} \simeq Mp(2n,\mathbb{R}) \times \mathbb{R}^\times$. Let us suppose that the element in Γ the first component of which is the neutral element $e \in Mp(2n,\mathbb{R})$ acts by $\gamma \in \mathbb{Z}_2$ on $\mathbf{L}(\lambda, c)$.

4) Finally, the preimage ${\lambda'}^{-1}(G_+) \subseteq \widetilde{P}$ of the unipotent part G_+ of P is supposed to act by the identity on $\mathbf{L}(\lambda, c)$. We denote the resulting admissible $\widetilde{P}$-module by $\mathbf{L}(\lambda, c, \gamma)$. (See Vogan [76] for the admissibility condition.)

For details on notions in the next definition, see Slovák, Souček [71].

Definition 25: Let $\mathfrak{G} = (\mathcal{G} \to M, \vartheta)$ be a Cartan geometry of type (G, H) and $\mathcal{E}, \mathcal{F} \to M$ be vector bundles associated to the principal H-bundle $\mathcal{G} \to M$. We call a vector space homomorphism $D : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ a **first order invariant differential operator** if there is a bundle homomorphism $\Phi : J^1\mathcal{E} \to \mathcal{F}$ such that $Ds = \Phi(s, \nabla^\vartheta s)$ for any section $s \in \Gamma(\mathcal{E})$, where $J^1\mathcal{E}$ denotes the first jet prolongation of $\mathcal{E} \to M$ and ∇^ϑ is the invariant derivative for $\mathfrak{G}$.

[5] By W_ν we mean the wight space $W_\nu = \{w \in W|\, H \cdot w = \nu(H)w \text{ for any } H \in \mathfrak{h}\}$.

It is convenient to divide the vector space of first order invariant differential operators by those bundle homomorphisms between $J^1\mathcal{E}$ and $\mathcal{F}$ which act trivially on the tangent space part of $J^1\mathcal{E}$. We call the resulting vector space the space of **first order invariant operators up to the zeroth order** and denote it by $\mathrm{Diff}^1_{\mathfrak{G}}(\mathcal{E},\mathcal{F})$.

Remark: Between any bundles induced by irreducible bounded multiplicities representations introduced above, there is at most one such an invariant operator up to a multiple and up to the operators of zeroth order. An equivalent condition for its existence is given in the next theorem. The author obtained it at the infinitesimal level when writing his dissertation thesis already. See [37].

Theorem 22: Let $(\mathcal{G} \to M^{2n+1}, \vartheta)$ be a metaplectic contact projective Cartan geometry, (λ, c, γ), $(\mu, d, \gamma') \in A \times \mathbb{C} \times \mathbb{Z}_2$, and $\mathcal{E} = \mathcal{G} \times_{\widetilde{P}} \mathbf{L}(\lambda, c, \gamma)$ and $\mathcal{F} = \mathcal{G} \times_{\widetilde{P}} \mathbf{L}(\mu, d, \gamma')$ be the corresponding vector bundles over M. Then the space

$$\mathrm{Diff}^1_{(\mathcal{G}\to M,\vartheta)}(\mathcal{E},\mathcal{F}) \simeq \begin{cases} \mathbb{C} & \text{if } \mu \in A_\lambda, c = d - 1 = c^{\mu}_{\lambda\varpi_1} \text{ and } \gamma = \gamma' \\ 0 & \text{in other cases.} \end{cases}$$

Proof. See Krýsl [41]. □

4.4 Hodge theory over C^*-algebras

An additive category is called **dagger** if it is equipped with a contravariant functor $*$ which is the identity on the objects, it is involutive on morphisms, $**F = F$, and it preserves the identity morphisms, i.e., $*\mathrm{Id}_C = \mathrm{Id}_C$ for any object C. No compatibility with the additive structure is demanded. See Brinkmann, Puppe [7]. For a morphism F, we denote $*F$ by F^*. For any additive category $\mathcal{C}$, we denote the category of its complexes by $\mathfrak{K}(\mathcal{C})$. If $\mathcal{C}$ is an additive and dagger category and $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}} \in \mathfrak{K}(\mathcal{C})$, we set $\Delta_i = d_i^* d_i + d_{i-1}d_{i-1}^*$, $i \in \mathbb{Z}$, and call it the i-th **Laplace operator**.

Definition 26: Let $\mathcal{C}$ be an additive and dagger category. We call a complex $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}} \in \mathfrak{K}(\mathcal{C})$ of **Hodge-type** if for each $i \in \mathbb{Z}$

$$U^i = \mathrm{Ker}\,\Delta_i \oplus \mathrm{Im}\,d_{i-1} \oplus \mathrm{Im}\,d_i^*.$$

We call $d^\bullet$ **self-adjoint parametrix possessing** if for each i, there exist morphisms $G_i : U^i \to U^i$ and $P_i : U^i \to U^i$ such that $\mathrm{Id}_{U^i} = G_i\Delta_i + P_i$, $\mathrm{Id}_{U^i} = \Delta_i G_i + P_i$, $\Delta_i P_i = 0$ and $P_i = P_i^*$.

Remark: In the preceding definition, we suppose that the images of the chain maps, the images of their adjoints, and the kernels of the Laplacians exist as objects in the additive and dagger category $\mathcal{C}$. The sign $\oplus$ denotes the biproduct in $\mathcal{C}$. See Weibel [79], p. 425.

The first two equations from the definition of a self-adjoint parametrix possessing complex are called the parametrix equations. Morphisms P_i from the above definition are idempotent as can be seen by composing the first equation with P_i from the right and using the equation $\Delta_i P_i = 0$. In particular, they are projections. The operators G_i are called the Green operators.

Definition 27: Let $(A, *_A, |\,|_A)$ be a C^*-algebra and A^+ be the positive cone of A, i.e., the set of all hermitian elements ($*_A a = a$) in A whose spectrum is contained in the non-negative real numbers. A tuple $(U, (,))$ is called a **pre-Hilbert A-module** if U is a right module over the complex associative algebra A, and $(,) : U \times U \to A$ is an A-sesquilinear map such that for all $u, v \in U$, $(u,v) = *_A(v,u)$, $(u,u) \in A^+$, and $(u,u) = 0$ implies $u = 0$. A pre-Hilbert module is called a **Hilbert A-module** if it is complete with respect to the norm $|u| = \sqrt{|(u,u)|_A}$, $u \in U$. A pre-Hilbert A-module morphism between $(U, (,)_U)$ and $(V, (,)_V)$ is any continuous A-linear map $F : U \to V$.

Remark: We consider that $(,)$ is antilinear in the left variable and linear in the right one as it is usual in physics.

An adjoint of a morphism $F : U \to V$ acting between pre-Hilbert modules $(U, (,)_U)$ and $(V, (,)_V)$ is a morphism $F^* : V \to U$ that satisfies the condition $(Fu, v)_V = (u, F^*v)_U$ for any $u \in U$ and $v \in V$. The category of pre-Hilbert and Hilbert C^*-modules and adjointable morphisms is an additive and dagger category. The dagger functor is the adjoint on morphisms. For any C^*-algebra A, we denote the categories of pre-Hilbert A-modules and Hilbert A-modules and adjointable morphisms by PH_A^* and H_A^*, respectively. In both of these cases, the dagger structure is compatible with the additive structure.

To any complex $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}} \in \mathfrak{K}(PH_A^*)$, the cohomology groups $H^i(d^\bullet) = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$ are assigned which are A-modules and which we consider to be equipped with the canonical quotient topology. They are pre-Hilbert A-modules with respect to the restriction of $(,)_{U_i}$ to $\operatorname{Ker} d_i$ if and only if $\operatorname{Im} d_{i-1}$ has an A-orthogonal complement in $\operatorname{Ker} d_i$.

We have the following

Theorem 23: Let $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}}$ be a self-adjoint parametrix possessing complex in PH_A^*. Then for any $i \in \mathbb{Z}$

1) $d^\bullet$ is of Hodge-type
2) $H^i(d^\bullet)$ is isomorphic to $\operatorname{Ker} \Delta_i$ as a pre-Hilbert A-module
3) $\operatorname{Ker} d_i = \operatorname{Ker} \Delta_i \oplus \operatorname{Im} d_{i-1}$
4) $\operatorname{Ker} d_i^* = \operatorname{Ker} \Delta_{i+1} \oplus \operatorname{Im} d_{i+1}^*$
5) $\operatorname{Im} \Delta_i = \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$.

Proof. See Krýsl [50]. □

Remark: If the image of d_{i-1} is not closed, the quotient topology on the cohomology group $H^i(d^\bullet)$ is non-Hausdorff and in particular, it is not in PH_A^*. See, e.g., von Neumann [61] on the relevance of topology for state spaces. See also Krýsl [51] for further references and for a relevance of our topological observation (Theorem 23 item 2) to the basic principles of the so-called Becchi–Rouet–Stora–Tyutin (BRST) quantization.

Theorem 24: Let $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}}$ be a complex of Hodge-type in H_A^*, then $d^\bullet$ is self-adjoint parametrix possessing.
Proof. See Krýsl [51]. □

Definition 28: Let M be a smooth manifold, A be a C^*-algebra and $\mathcal{F} \to M$ be a Banach bundle with a smooth atlas such that each of its maps targets onto a fixed Hilbert A-module (the typical fiber). If the transition functions of the atlas are Hilbert A-module automorphisms, we call $\mathcal{F} \to M$ an A-Hilbert bundle. We call an A-Hilbert bundle $\mathcal{F} \to M$ **finitely generated projective** if the typical fiber is a finitely generated projective Hilbert A-module.

For further information on analysis on C^*-Hilbert bundles, we refer to Solovyov, Troitsky [69], Troitsky [73] and Schick [65]. In the paper of Troitsky, complexes are treated with an allowance of the so-called 'compact' perturbations.

Theorem 25: Let M be a compact manifold, A be a C^*-algebra and $D^\bullet = (\Gamma(\mathcal{F}^i), D_i)_{i\in\mathbb{Z}}$ be an elliptic complex on finitely generated projective A-Hilbert bundles over M. Let for each $i \in \mathbb{Z}$, the image of Δ_i be closed in $\Gamma(\mathcal{F}^i)$. Then for any $i \in \mathbb{Z}$

1) $D^\bullet$ is of Hodge-type
2) $H^i(D^\bullet)$ is a finitely generated projective Hilbert A-module isomorphic to $\operatorname{Ker}\Delta_i$ as a Hilbert A-module
3) $\operatorname{Ker} D_i = \operatorname{Ker} \triangle_i \oplus \operatorname{Im} D_{i-1}$
4) $\operatorname{Ker} D_i^* = \operatorname{Ker} \triangle_{i+1} \oplus \operatorname{Im} D_{i+1}^*$
5) $\operatorname{Im} \Delta_i = \operatorname{Im} D_{i-1} \oplus \operatorname{Im} D_i^*$.

Proof. See Krýsl [50]. $\square$

Let H be a Hilbert space. Any C^*-subalgebra of the C^*-algebra of compact operators on H is called a **C^*-algebra of compact operators**.

For C^*-algebras of compact operators, we have the following analogue of the Hodge theory for elliptic complexes of operators on finite rank vector bundles over compact manifolds.

Theorem 26: Let M be a compact manifold, K be a C^*-algebra of compact operators and $D^\bullet = (\Gamma(\mathcal{F}^i), D_i)_{i\in\mathbb{Z}}$ be an elliptic complex on finitely generated projective K-Hilbert bundles over M. If $D^\bullet$ is elliptic, then for each $i \in \mathbb{Z}$

1) $D^\bullet$ is of Hodge-type
2) The cohomology group $H^i(D^\bullet)$ is a finitely generated projective Hilbert K-module isomorphic to the Hilbert K-module $\operatorname{Ker}\Delta_i$.
3) $\operatorname{Ker} D_i = \operatorname{Ker} \Delta_i \oplus \operatorname{Im} D_{i-1}$
4) $\operatorname{Ker} D_i^* = \operatorname{Ker} \Delta_{i+1} \oplus \operatorname{Im} D_{i+1}^*$
5) $\operatorname{Im} \Delta_i = \operatorname{Im} D_{i-1} \oplus \operatorname{Im} D_i^*$

Proof. See [51]. $\square$

Remark: In particular, we see that the cohomology groups share properties of the fibers.

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5 Selected author's articles

Decomposition of a tensor product of a higher symplectic spinor module and the defining representation of $\mathfrak{sp}(2n, \mathbb{C})$, Journal of Lie Theory, Vol. 17 (1) (2007), 63–72.

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Decomposition of a Tensor Product of a Higher Symplectic Spinor Module and the Defining Representation of $\mathfrak{sp}(2n,\mathbb{C})$

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Abstract. Let $L(\lambda)$ be the irreducible highest weight $\mathfrak{sp}(2n,\mathbb{C})$-module with a highest weight λ, such that $L(\lambda)$ is an infinite dimensional module with bounded multiplicities, and let $F(\varpi_1)$ be the defining representation of $\mathfrak{sp}(2n,\mathbb{C})$. In this article, the tensor product $L(\lambda)\otimes F(\varpi_1)$ is explicitly decomposed into irreducible summands. This decomposition may be used in order to define some invariant first order differential operators for metaplectic structures.
Mathematical Subject Classification: Primary: 17B10; Secondary: 17B81, 22E47.
Key words and Phrases: symplectic spinors, harmonic spinors, Kostant's spinors, tensor products, decomposition of tensor products, modules with bounded multiplicities, Kac-Wakimoto formula.

1. Introduction

Let $L(\lambda)$ denote the irreducible highest weight module with a highest weight λ and let us write $F(\lambda)$ instead of $L(\lambda)$, if λ is integral dominant with respect to a choice of a Cartan subalgebra and of a set of positive roots. In this article, we shall study a decomposition of the tensor product $L(\lambda)\otimes F(\varpi_1)$ as a module over complex symplectic Lie algebras, where λ is some nonintegral weight from a suitable set, which will be denoted by $\mathbb{A}$, and ϖ_1 is the highest weight of the defining representation of the complex symplectic Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$.

This study was motivated by author's interest in certain first order invariant differential operators, which are symplectic analogues of orthogonal Dirac-type operators. In general, invariant differential operators are acting between sections of vector bundles associated to some principal fiber bundles via representations of the principal group. The operators, we were interested in, are acting between sections of vector bundles associated to projective contact or symplectic geometries via the so called higher symplectic spinor modules over complex symplectic Lie algebras. *Higher symplectic spinor modules* represent symplectic analogues of spinor

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representations of orthogonal complex Lie algebras $\mathfrak{so}(m, \mathbb{C})$, see Kostant [12]. Projective contact geometries belong to Cartan geometries defined by a contact grading of the tangent bundle and a projective class of partial affine connections, see Krýsl [13]. In physics, these geometries play a role of a phase-space of time dependent Hamiltonian mechanics, while the symplectic geometries are models of the time independent one. To classify invariant differential operators (on the infinitesimal level at least), one needs to decompose the mentioned tensor product $L(\lambda) \otimes F(\varpi_1)$, if the sections take their values in $L(\lambda)$. (See, e.g., Slovák, Souček [15].) One of the invariant differential operators serving as a motivation for our paper appeared already in Kostant [12] and is known as the Kostant Dirac operator. Analytical and geometrical aspects of the Kostant Dirac operator were studied by many authors, see, e.g., Habermann [4], Klein [10] and Kadlčáková [8]. The last author is studying also the so called symplectic twistor and symplectic Rarita-Schwinger operators, which are related to our decomposition as well. Let us mention that for the basic symplectic spinor modules, a kind of globalization is known. These globalizations are called Segal-Shale-Weil representations, see Kashiwara, Vergne [9], where these globalized modules are introduced as representations over the metaplectic group $Mp(2n, \mathbb{R})$. Let us also mention that the study of the corresponding first order differential operators has its application in theoretical physics, namely in the 10 dimensional super string theory, see Green, Hull [3], and in the theory of Dirac-Kähler fields, see, e.g., Reuter [14], where the author of this article found his motivation for this study.

In [1], Britten, Hooper and Lemire and in [2], Britten and Hooper described the decomposition of $L(\lambda_i) \otimes F(\nu)$ for $i = 0, 1$, where ν is a dominant integral weight, $\lambda_0 = -\frac{1}{2}\varpi_n$ and $\lambda_1 = \varpi_{n-1} - \frac{3}{2}\varpi_n$, i.e., λ_i are the highest weights of the so called *basic symplectic spinor modules* $L(\lambda_i)$ (for notation, see bellow). Britten, Hooper, Lemire in [1] and Britten, Hooper in [2] are giving a characterization of all infinite dimensional modules with bounded multiplicities over complex symplectic Lie algebras. The authors of these articles proved that the class of infinite dimensional highest weight modules with bounded multiplicities equals the set of higher symplectic spinor modules, i.e., the set $\{L(\lambda); \lambda \in \mathbb{A}\}$. In this article, we study a problem, which is in a sense complementary to that of Britten, Hooper and Lemire. Namely, we describe the decomposition of the tensor product $L(\lambda) \otimes F(\varpi_1)$ of an arbitrary infinite dimensional module with bounded multiplicities $L(\lambda)$, $\lambda \in \mathbb{A}$, and the defining representation $F(\varpi_1)$ of the complex symplectic Lie algebra. Techniques used to decompose the mentioned tensor product are based on a result on formal characters of tensor products of an irreducible highest weight module and an irreducible finite dimensional module over simple complex Lie algebras, described by Humphreys in [5]. The assumption under which his formula is valid is the same as that one used by Kostant, see [11], for a more general situation. The second ingredient we have used is the famous Kac-Wakimoto formula in Kac, Wakimoto [7], which was published for complex simple Lie algebras in Jantzen [6] earlier, but which is valid for slightly different set of weights.

In the second section of this article, some known results on formal characters of irreducible highest weight modules (Theorem 2.1), decomposition of tensor products (Theorems 2.2, 2.3) and formal character of a tensor product (Theorem 2.4) are presented. The second part contains also Lemma 2.7, in which Theorem 2.1 is adapted to the situation of our interest. The third part of this paper is devoted to the formulation of the decomposition of $L(\lambda) \otimes F(\varpi_1)$ for $\lambda \in \mathbb{A}$ and to its proof (Theorem 3.1).

2. Tensor products and higher symplectic modules

2.1. Tensor products decompositions.

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n and let $(\,,)$ denote the Killing form of $\mathfrak{g}$. Suppose a Cartan subalgebra $\mathfrak{h}$ together with a subset Φ^+ of positive roots of the set Φ of all roots are given. The set of roots determines its $\mathbb{R}$-linear span, denoted by $\mathfrak{h}_0^*$. With help of the Killing form on $\mathfrak{g}$, we can introduce a mapping $\langle\,,\rangle : \mathfrak{h}_0^* \times (\mathfrak{h}_0^* - \{0\}) \to \mathbb{R}$ by the following equation

$$\langle v, w\rangle := 2\frac{(v,w)}{(w,w)},$$

for $v \in \mathfrak{h}_0^*$ and $w \in \mathfrak{h}_0^* - \{0\}$. The half-sum of all positive roots will be denoted by δ, i.e., $\delta := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. Further, let us denote the Weyl group associated to $(\mathfrak{g},\mathfrak{h})$ by $\mathcal{W}$. The determinant of an element $\sigma \in \mathcal{W}$ is denoted by $\epsilon(\sigma)$. If $\lambda \in \mathfrak{h}^*$ then the symbol $\mathcal{W}^\lambda$ is used for a subgroup of the Weyl group $\mathcal{W}$ generated by reflections in planes perpendicular to such simple roots γ, for which $\langle\lambda,\gamma\rangle \in \mathbb{Z}$. Further, let us denote the affine action of a Weyl group element by a dot, thus $\sigma.\lambda := \sigma(\lambda+\delta) - \delta$ is an affine action of an element $\sigma \in \mathcal{W}$ on $\lambda \in \mathfrak{h}^*$. For $\lambda, \mu \in \mathfrak{h}^*$, let us write $\lambda \sim \mu$, if there is an element $\sigma \in \mathcal{W}$ such that $\sigma.\lambda = \mu$. We will call such weights linked to each other. Let us denote the set of positive coroots by R_+ and the set $\{X \in R_+; \lambda(X) \in \mathbb{Z}\}$ for some $\lambda \in \mathfrak{h}^*$ by R_+^λ. Further, denote the basis of $R^\lambda := R_+^\lambda \cup -R_+^\lambda$ by $B^\lambda(\subseteq R_+^\lambda)$.

For a complex simple Lie algebra $\mathfrak{g}$, let $L(\lambda)$ be the irreducible highest weight module over $\mathfrak{g}$ with a highest weight λ and $M(\lambda)$ be the Verma module with a highest weight λ. To stress that λ is integral and dominant for a choice of $(\mathfrak{h},\Phi^+)$, i.e., the corresponding module $L(\lambda)$ is finite dimensional, we will denote $L(\lambda)$ by $F(\lambda)$ or simply by F, if the highest weight is not important or clear from the context. Let $\Pi(\lambda)$ be the set of all weights of the module $L(\lambda)$ and $n(\nu)$ be the multiplicity of weight $\nu \in \Pi(\lambda)$. For a weight $\lambda \in \mathfrak{h}^*$, symbol L_λ denotes the weight space of weight λ of a highest weight module L. Further, let us denote the formal character of a highest weight module L by $ch\ L$. The central character corresponding to a weight λ is denoted by χ_λ, i.e., we have $z.v = \chi_\lambda(z)v$ for each element v of a highest weight module with a highest weight λ and an element $z \in \mathfrak{Z} := Z(\mathfrak{U}(\mathfrak{g}))$ of the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$.

Let L be a highest weight module over a complex semisimple algebra $\mathfrak{g}$. We call L module with bounded multiplicities, if there is $k \in \mathbb{N}$ such that $\dim L_\lambda \leq k$ for all weights λ of the module L. Such minimal k is called degree of L. We call a module with bounded multiplicities completely pointed provided its degree is 1. Let us mention that the basic symplectic spinor modules $L(\lambda_i)$, $i = 0, 1$ (see the Introduction for their definition via fundamental weights) are completely pointed and these are the only ones among infinite dimensional irreducible highest weight modules over the complex symplectic Lie algebra, see Britten, Hooper, Lemire [1].

There is a result on a formal character of an irreducible highest weight module over a complex semisimple algebra. In this theorem, the formal characters of Verma modules $M(\sigma.\lambda)$ for some Weyl group elements σ are related to the formal character of the irreducible module $L(\lambda)$.

Theorem 2.1. *Let $\lambda \in \mathfrak{h}_0^*$ be such that $(\lambda+\delta)\alpha > 0$ for all $\alpha \in B^\lambda$. Then we have*

$$ch\ L(\lambda) = \sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.\lambda).$$

Proof. See Kac, Wakimoto [7], Theorem 1, pp. 4957. ∎

A version of the previous theorem appeared already in Jantzen [6], Theorem 2.23, pp. 70 but for a slightly different set of weights. We will refer to the formula in the preceding theorem as the Kac-Wakimoto formal character formula.

In the next theorem a decomposition of a tensor product of an irreducible highest weight module (possibly of infinite dimension) and a finite dimensional irreducible module into invariant summands is described, for further comments see Humphreys [5], pp. 1 - 64.

Theorem 2.2. *Let F be a finite dimensional module over a complex semisimple Lie algebra $\mathfrak{g}$ and $L(\lambda)$ be an irreducible highest weight module with a highest weight λ over $\mathfrak{g}$, then one has a canonical decomposition $F \otimes L(\lambda) = M^{(1)} \oplus \ldots \oplus M^{(k)}$, where $M^{(i)}$ is the generalized eigenspace corresponding to $\chi_{\lambda+\mu_i}$ and μ_i runs over a subset of the weights of F, so that the indicated central characters are distinct.*

Proof. See Humphreys [5], sect 4.4. and pp. 39. ∎

Let us recall the famous Harish-Chandra theorem, which says that $\chi_\lambda = \chi_\mu$, if and only if $\lambda \sim \mu$. In the next theorem, the generalized eigenspaces are specified more precisely.

Theorem 2.3. *Keep the above notation. Suppose $\mu := \mu_i$ is a weight of F such that for all weights $\nu \neq \mu$ of F, $\lambda+\nu$ and $\lambda+\mu$ are not linked to each other. Then $M := M^{(i)}$ is a direct sum of n copies of $L(\lambda+\mu)$, where $n = dim\ M_{\lambda+\mu}$.*

Proof. See Humphreys [5] sect. 6.3., pp. 40. ∎

In the next theorem, the formal character of the generalized eigenspace is related to formal characters of some Verma modules and to multiplicities of corresponding weights of the finite dimensional module F.

Theorem 2.4. *Keep the above notation and denote by $n(\mu)$ the multiplicity of the weight μ in the irreducible finite dimensional module F. Suppose that for all weights $\nu \neq \mu$ of F, $\lambda+\nu$ and $\lambda+\mu$ are not linked to each other. Further suppose that $(\lambda+\mu+\delta)\alpha >$ for each $\alpha \in B^{\lambda+\mu}$ and each weight μ of F. Then*

$$n(\mu) \sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.(\lambda+\mu)) = n\, ch\, L(\lambda+\mu).$$

Proof. See Humphreys [5] sect. 6.4., pp. 42 and use the Kac-Wakimoto formal character formula in the substitution for $a(w,\lambda)$ from Humphreys. ∎

2.2. The case of $\mathfrak{sp}(2n, \mathbb{C})$ and higher symplectic spinor modules.

In this subsection, we focus our attention to the complex symplectic Lie algebra, i.e., $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})(= C_n)$, and to a distinguished class of infinite dimensional irreducible highest weight modules. For a choice of a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and of a system of positive roots Φ^+ of $\mathfrak{g}$, there is a set of fundamental weights, which will be denoted by $\{\varpi_i\}_{i=1}^n$. Having chosen the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we can define a subset $\{\epsilon_i\}_{i=1}^n$ of $\mathfrak{h}_0^*$, such that $\varpi_i = \sum_{j=1}^i \epsilon_j$, $i = 1, \dots, n$ which is an orthonormal basis of $\mathfrak{h}_0^*$ with respect to the restriction of the Killing form $(\,,)$ to the subspace $\mathfrak{h}_0^* \times \mathfrak{h}_0^*$.

Now, let us describe modules we shall be dealing with.

Definition 2.5. Let us denote the set of weights

$$\{\lambda = \sum_{i=1}^n \lambda_i \varpi_i; \lambda_i \geq 0, i = 1, \dots, n-1, \lambda_n \in \mathbb{Z} + \frac{1}{2}, \lambda_{n-1} + 2\lambda_n + 3 > 0\} \subseteq \mathfrak{h}_0^*$$

by $\mathbb{A}$. We will call the modules $L(\lambda)$ for $\lambda \in \mathbb{A}$ *higher symplectic spinor modules.*

Theorem 2.6. *The following are equivalent:*

1.) $L(\lambda)$ is a higher symplectic spinor module, i.e., $\lambda \in \mathbb{A}$,

2.) $L(\lambda)$ has bounded multiplicities,

3.) $L(\lambda)$ is equivalent to a direct summand of the tensor product $L(-\frac{1}{2}\varpi_n) \otimes F(\nu)$ for some choice of dominant integral weight ν.

Proof. See Britten, Lemire [2], Theorem 2.1 pp. 3417 and Theorem 1.2 pp. 3415. ∎

In the next lemma, Theorem 2.1 is adapted to the situation we are studying.

Lemma 2.7. *Let $\nu \in \Pi(\varpi_1)$ and $\lambda, \lambda + \nu \in \mathbb{A}$, then*

$$ch\ L(\lambda + \nu) = \sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.(\lambda + \nu)).$$

Proof. We must check whether the assumption of Theorem 2.1 is satisfied. At first, we determine the set $R_+^{\lambda+\nu}$ for $\nu \in \Pi(\varpi_1)$ and $\lambda, \lambda + \nu \in \mathbb{A}$. Looking at the definition of the set $R_+^{\lambda+\nu}$, we easily obtain that

$$R_+^{\lambda+\nu} = \{e_i + e_j, 1 \leq i \leq j < n\} \cup \{e_i - e_j, 1 \leq i < j < n\} \cup \{e_k, 1 \leq k < n\},$$

where $\{e_i\}_{i=1}^n$ is the dual basis of $\mathfrak{h}_0$ to the basis $\{\epsilon_i\}_{i=1}^n$. The basis $B^{\lambda+\nu}$ of $R_+^{\lambda+\nu}$ is

$$B^{\lambda+\nu} = \{e_i - e_{i+1}, 1 \leq i \leq n-2\} \cup \{e_{n-1}\}.$$

Secondly, we need to compute $(\lambda + \nu + \delta)\alpha$ for $\alpha \in B^{\lambda+\nu}$. Suppose that $\nu = t\epsilon_p$ for some $p = 1, \dots, n$ and $t \in \{-1, 1\}$.

1.) $A := (\lambda+\nu+\delta)(e_i - e_{i+1}) = [\sum_{r=1}^{n}(\sum_{s=r}^{n}\lambda_s + n - r + 1 + t\delta_{rp})\epsilon_r](e_i - e_{i+1}) = \lambda_i + 1 + t(\delta_{ip} - \delta_{i+1,p})$, $i = 1,\ldots,n-2$. We know that $\lambda+\nu \in \mathbb{A}$, from which it follows that $\lambda_i + t(\delta_{ip} - \delta_{i,p-1}) \geq 0$ for $i = 1,\ldots,n-1$, because $\epsilon_p = \varpi_p - \varpi_{p-1}, p = 1,\ldots,n$, where $\varpi_0 = 0$ and $\delta_{i,-1} := 0$ for $i = 1,\ldots,n$ are to be understood. Thus the condition $A > 0$, we have had to check, is satisfied.

2.) $B := (\lambda+\nu+\delta)(e_{n-1}) = [\sum_{r=1}^{n}(\sum_{s=r}^{n}\lambda_s + n - r + 1 + t\delta_{rp})\epsilon_r](e_{n-1}) = \lambda_{n-1} + \lambda_n + 2 + t\delta_{n-1,p}$. If $\lambda_n > 0$, then the inequality $B > 0$ is evidently satisfied. Now, suppose that $\lambda_n \leq -\frac{1}{2}$. If $p = n-1$, then using the inequality $\lambda_{n-1} + 2\lambda_n + 3 + t \geq 1$ $(\lambda+\nu \in \mathbb{A})$ and $\lambda_n \leq -\frac{1}{2}$, one obtains, that $\lambda_{n-1} + \lambda_n + \frac{3}{2} + t \geq 0$, from which $B > 0$ easily follows. If $p \neq n-1$, then using the inequality $\lambda_{n-1} + 2\lambda_n + 3 \geq 1$ $(\lambda \in \mathbb{A})$ and $\lambda_n \leq -\frac{1}{2}$, one obtains that $\lambda_{n-1} + \lambda_n + \frac{3}{2} \geq 0$, from which $B > 0$ follows.

Thus, we have proved that the assumption of Theorem 2.1 is satisfied and therefore the conclusion of this lemma follows. ■

3. Decomposition of $L(\lambda) \otimes F(\varpi_1)$ for $\lambda \in \mathbb{A}$

Theorem 3.1. *Let $L(\lambda)$ be a higher symplectic spinor module, i.e., $\lambda \in \mathbb{A}$. Then*

$$L(\lambda) \otimes F(\varpi_1) = \bigoplus_{\mu \in \mathbb{A}_\lambda} L(\mu),$$

where $\mathbb{A}_\lambda = \{\lambda+\nu; \nu \in \Pi(\varpi_1)\} \cap \mathbb{A}$. [2]

Proof. **Part I.** We would like to use Theorem 2.3. In this part, we shall verify its assumption. Thus we shall prove that $\lambda+\mu$ and $\lambda+\nu$ are not conjugated by the affine action of an element of the Weyl group $\mathcal{W}$ of the algebra C_n, if $\nu \neq \mu$ are arbitrary weights of $F(\varpi_1)$ and $\lambda \in \mathbb{A}$. Two elements $\phi, \psi \in \mathfrak{h}^*$ are conjugated by the affine action of an element of the Weyl group if and only if $\phi+\delta$ and $\psi+\delta$ are conjugated by an element of the Weyl group, i.e., if and only if $\sigma(\phi+\delta) = \psi+\delta$, for some $\sigma \in \mathcal{W}$.

Let us first prove that $\{\lambda+\nu+\delta, \lambda+\mu+\delta\} \subseteq \overline{W_1} \cup \overline{W_2}$, where

$$W_1 := \{\sum_{i=1}^{n}\beta_i\epsilon_i; \beta_1 > \ldots > \beta_n > 0\},$$

$$W_2 := \{\sum_{i=1}^{n}\beta_i\epsilon_i; \beta_1 > \ldots > \beta_{n-1} > -\beta_n > 0\}$$

are two open neighbor Weyl chambers of C_n, and where $\overline{X}$ denotes the closure of $X \subseteq \mathfrak{h}_0^*$ wr. to the restriction of the Killing form $(,)$ to $\mathfrak{h}_0^* \times \mathfrak{h}_0^*$. An arbitrary weight μ of $F(\varpi_1)$ is of the form $\mu = s\epsilon_p$ for $s \in \{-1, 1\}$ and some $p = 1,\ldots,n$. In the case of C_n, we have $\delta = n\epsilon_1 + (n-1)\epsilon_2 + \ldots + \epsilon_n$. Using the relation

[2] One can easily compute that the (saturated) set $\Pi(\varpi_1)$ of weights of $F(\varpi_1)$ equals $\{\pm\epsilon_i; i = 1,\ldots,n\}$.

$\varpi_j = \sum_{i=1}^{j} \epsilon_i$ $(j = 1, \ldots, n)$, one easily computes that for $\lambda = \sum_{i=1}^{n} \lambda_i \varpi_i$, we have

$$\lambda + \mu + \delta =: \sum_{i=1}^{n} \beta_i \epsilon_i = \sum_{i=1}^{n} [(\sum_{j=i}^{n} \lambda_j) + n - i + 1 + s\delta_{ip}]\epsilon_i.$$

Thus the requirement $\lambda + \mu + \delta \in \overline{W_1}$ reduces to $\lambda_i + 1 \geq s(\delta_{i+1,p} - \delta_{ip})$ which is evidently satisfied for all $i = 1, \ldots, n-1$, see Definition 2.5. For $i = n$, the condition we need to check is $\beta_n \geq 0$ or $\beta_{n-1} \geq -\beta_n \geq 0$. If $\beta_n \geq 0$, we are done. Suppose $\beta_n < 0$, then the remaining condition we need to check is $\beta_{n-1} \geq -\beta_{n-1}$, because $-\beta_n \geq 0$ follows from our assumptions. The inequality $\beta_{n-1} \geq -\beta_{n-1}$ translates into

$$\lambda_{n-1} + 2\lambda_n + 3 + s(\delta_{n-1,p} + \delta_{np}) \geq 0. \tag{1}$$

Condition (1) is satisfied due to the last inequality in Definition 2.5 of higher symplectic spinor modules.

Suppose that there are some weights $\mu \neq \nu$ with $\mu, \nu \in \Pi(\varpi_1)$ for which $\lambda + \mu + \delta$ and $\lambda + \nu + \delta$ are conjugated by an element σ of the Weyl group of C_n, i.e., $\sigma(\lambda + \mu + \delta) = \lambda + \nu + \delta$.

(1) Suppose that $\lambda + \mu + \delta \in W_1$ and $\lambda + \nu + \delta \in W_2$ (or $\lambda + \mu + \delta \in W_2$ and $\lambda + \nu + \delta \in W_1$, which is analogous). The condition $\sigma(\lambda + \mu + \delta) = \lambda + \nu + \delta$ implies $\sigma W_1 = W_2$. It is evident that $\sigma_{\epsilon_n} W_1 = W_2$. The Weyl group acts simply transitively on the set of open (or closed) Weyl chambers. Hence $\sigma = \sigma_{\epsilon_n}$. The weight ϵ_n does not belong to the system of simple roots, but it is evident that we could have written $\sigma_{2\epsilon_n}$ instead of σ_{ϵ_n}. Now, $\sigma_{\epsilon_n}(\lambda + \mu + \delta) = \lambda + \mu + \delta - 2(\epsilon_n, \lambda + \mu + \delta)\epsilon_n = \lambda + \mu + \delta - 2(\lambda_n + s\delta_{np} + 1)\epsilon_n$. This element equals to $\lambda + \nu + \delta$ if and only if $\mu - \nu = 2(\lambda_n + s\delta_{np} + 1)\epsilon_n$ which is impossible due to the structure of the set $\Pi(\varpi_1)$ and the condition $\lambda_n \in \mathbb{Z} + \frac{1}{2}$.

(2) The case $\lambda + \mu + \delta, \lambda + \nu + \delta \in W_i$ and $\sigma(\lambda + \mu + \delta) = \lambda + \nu + \delta$ for $i = 1, 2$ leads to the condition $\sigma = id$, i.e., $\nu = \mu$ - a contradiction.

(3) The remaining case is $\lambda + \mu + \delta, \lambda + \nu + \delta \in \overline{W_1} \cup \overline{W_2} - (W_1 \cup W_2)$, i.e., the considered elements lie on the walls of the two Weyl chambers. (The other cases are impossible: if there is an element lying on a wall of a closed Weyl chamber and the other one is lying in the open Weyl chamber, then they cannot be conjugated.) The inspection of the fact $\lambda + \mu + \delta, \lambda + \nu + \delta \in \overline{W_1} \cup \overline{W_2}$ showed that if these elements lie on the walls of $\overline{W_1}$ and $\overline{W_2}$, then they lie in their interior (i.e., they do not lie on the walls of codimension 2): inequalities in the definition of $\overline{W_1}$ $(\beta_1 \geq \ldots \beta_n \geq 0)$ become equations only once and the same is true for $\overline{W_2}$. Let us define two families of open Weyl chambers

$$Y_r := \{\sum_{i=1}^{n} \beta_i \epsilon_i; \beta_1 > \ldots > \beta_{r-1} > -\beta_r > \beta_{r+1} > \ldots > \beta_n > 0\},$$

$r = 1, \ldots, n-1$ and

$$Y_t' := \{\sum_{i=1}^{n} \beta_i \epsilon_i; \beta_1 > \ldots > \beta_{t-1} > -\beta_t > \beta_{t+1} > \ldots > -\beta_n > 0\},$$

$t = 1, \dots, n-1$.

(3.1) Suppose that $\lambda + \mu + \delta \in \overline{W_1} \cap \overline{Y_r}$ and $\lambda + \nu + \delta \in \overline{W_2} \cap \overline{Y_t'}$ for some $r, t = 1, \dots, n-1$. If we suppose that $\sigma(\lambda + \mu + \delta) = \lambda + \nu + \delta$, then the fact that these elements lie in the interior of the walls implies that $\sigma W_1 = W_2$ or $\sigma W_1 = Y_t'$. The first case leads to a contradiction as we have shown. Using the fact that the Weyl group acts simply transitively, we easily find that $\sigma = \sigma_{\epsilon_t}\sigma_{\epsilon_n}$ in the second case. Let us compute $\sigma_{\epsilon_t}\sigma_{\epsilon_n}(\lambda+\mu+\delta) = \lambda+\mu+\delta - 2(\epsilon_t, \lambda+\mu+\delta)\epsilon_t - 2(\epsilon_n, \lambda+\mu+\delta)\epsilon_n = \lambda + \mu + \delta - 2(\lambda_t + s\delta_{pt} + n - t + 1)\epsilon_t - 2(\lambda_n + s\delta_{pn} + 1)\epsilon_n$. This element equals $\lambda + \nu + \delta$ if and only if $\mu - \nu = 2(\lambda_t + s\delta_{pt} + n - t + 1)\epsilon_t + 2(\lambda_n + s\delta_{pn} + 1)\epsilon_n$. Because of the structure of $\Pi(\varpi_1)$, we obtain: $\mu - \nu \in \{\pm 2\epsilon_t, \pm 2\epsilon_n, \pm\epsilon_t \pm \epsilon_n, \pm\epsilon_t \mp \epsilon_n\}$. The first possibility leads to $0 = \lambda_n + s\delta_{np} + 1$, which is impossible because λ_n is half-integral. The second possibility implies $0 = \lambda_t + s\delta_{tp} + n - t + 1 \geq \lambda_t + n - t > 0$ - a contradiction. The third and fourth possibilities force $\pm 1 = 2(\lambda_t + s\delta_{tp} + n - t + 1)$ - an odd number equals an even one, which is a contradiction.

(3.2) Suppose that $\lambda+\mu+\delta \in \overline{W_1} \cap \overline{Y_r}$ and $\lambda+\nu+\delta \in \overline{W_1} \cap \overline{Y_t}$. In this case, $\sigma W_1 = W_1$ or $\sigma W_1 = Y_t$. The first case leads to a contradiction as we already know. In the second case, one easily finds that $\sigma_{\epsilon_t} W_1 = Y_t$, i.e., using the simplicity of the Weyl group action, this implies $\sigma = \sigma_{\epsilon_t}$. Let us compute $\sigma_{\epsilon_t}(\lambda+\mu+\delta) = \lambda+\mu+\delta - 2(\lambda_t + s\delta_{pt} + n - t + 1)\epsilon_t$. This element equals $\lambda + \nu + \delta$ if and only if $\{\mu, \nu\} = \{\epsilon_t, -\epsilon_t\}$, i.e., $\mu - \nu = \pm 2\epsilon_t$. That means that $1 = \lambda_t + 1 + n - t + 1$ or $-1 = \lambda_t - 1 + n - t + 1$ which are impossible because $\lambda_t \geq 0$ and $t < n$ for $t = 1, \dots, n-1$.

(3.3) The remaining cases are analogous to the previous ones and actually have been done.

Part II. Summarizing part I of the proof, we have proved that the assumption of Theorem 2.3 is satisfied, and therefore for each $\nu_i \in \Pi(\varpi_1)$ we have that the generalized eigenspace $M^{(i)}$ occurring in the canonical decomposition $L(\lambda) \otimes F(\varpi_1) = M^{(1)} \oplus \ldots \oplus M^{(k)}$ can be written as $M^{(i)} = n_i L(\lambda + \nu_i)$ for some nonnegative integer n_i. We should determine the numbers n_i for $i = 1, \dots, k$. To do it, we should use Theorem 2.4. Let us suppose that $\nu_i \in \Pi(\varpi_1)$ is such that $\lambda + \nu_i \in \mathbb{A}$. It follows from the proof of Lemma 2.7 that for such weights, we have $(\lambda + \nu_i + \delta)\alpha > 0$ for each $\alpha \in B^{\lambda+\nu_i}$, i.e., the condition of Theorem 2.4 is satisfied. We may therefore write $n(\nu_i)\sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.(\lambda + \nu_i)) = n_i ch L(\lambda + \nu_i)$. Because we know, that $n(\nu_i) = 1$ for all weights $\nu_i \in \Pi(\varpi_1)$, we get

$$\sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\, M(\sigma.(\lambda + \nu_i)) = n_i ch\, L(\lambda + \nu_i).$$

Using the formal character formula of Kac and Wakimoto from Lemma 2.7, we get $ch\, L(\lambda + \nu_i) = n_i ch\, L(\lambda + \nu_i)$, which implies $n_i = 1$ for such $\nu_i \in \Pi(\varpi_1)$ for which $\lambda + \nu_i \in \mathbb{A}$. From Theorem 2.3, we know that if $L(\mu)$ appears in the decomposition of $L(\lambda) \otimes F(\nu)$, then $\mu = \lambda + \eta$, where $\eta \in \Pi(\varpi_1)$. Still, we have shown that $\mu \in (\lambda + \Pi(\varpi_1)) \cap \mathbb{A} =: \mathbb{A}_\lambda$ occurs in the decomposition, we are interested in, with multiplicity 1. The remaining question is, whether a weight from $(\lambda + \Pi(\varpi_1))\backslash\mathbb{A}$ may occur in the decomposition. But this is not possible, because the highest weight μ of an irreducible summand $L(\mu)$ of the decomposition lies in the set $\mathbb{A}$. To see it, consider an integral dominant weight $\nu \in \mathfrak{h}_0^*$ such that $L(\lambda) \subseteq L(\lambda_0) \otimes F(\nu)$. Such weight ν exists due to Theorem 2.6 (1. $\Rightarrow$ 3.). Using the associativity of a tensor product, we have $L(\mu) \subseteq L(\lambda_0) \otimes (F(\nu) \otimes F(\varpi_1))$. The tensor product $F(\nu) \otimes F(\varpi_1)$ decomposes into a finite direct sum of finite dimensional irreducible $\mathfrak{sp}(2n, \mathbb{C})$-modules, and therefore $L(\mu)$ is a direct summand in a tensor product $L(\lambda_0) \otimes F(\nu')$ for some integral dominant weight ν'. Using Theorem 2.6 (3. $\Rightarrow$ 1.) we get $\mu \in \mathbb{A}$. ∎

Further research could be devoted to an investigation of real higher symplectic spinor representations of real symplectic Lie algebras and to their globalizations.

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Classification of 1st order symplectic spinor operators over contact projective geometries

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Abstract

We give a classification of 1st order invariant differential operators acting between sections of certain bundles associated to Cartan geometries of the so-called metaplectic contact projective type. These bundles are associated via representations, which are derived from the so-called higher symplectic (sometimes also called harmonic or generalized Kostant) spinor modules. Higher symplectic spinor modules are arising from the Segal–Shale–Weil representation of the metaplectic group by tensoring it by finite dimensional modules. We show that for all pairs of the considered bundles, there is at most one 1st order invariant differential operator up to a complex multiple and give an equivalence condition for the existence of such an operator. Contact projective analogues of the well known Dirac, twistor and Rarita–Schwinger operators appearing in Riemannian geometry are special examples of these operators.

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Keywords: Metaplectic contact projective geometry; Symplectic spinors; Segal–Shale–Weil representation; Kostant spinors; First order invariant differential operators

1. Introduction

The operators we would like to classify are 1st order invariant differential operators acting between sections of vector bundles associated to metaplectic contact projective geometries via certain minimal globalizations.

Metaplectic contact projective geometry on an odd dimensional manifold is first a contact geometry, i.e., it is given by a corank one subbundle of the tangent bundle of the manifold which is nonintegrable in the Frobenius sense in each point of the manifold. Second part of the metaplectic contact projective structure on a manifold is given by a class of projectively equivalent contact partial affine connections. Here, partial contact means that the connections are compatible with the contact structure and that they are acting only on the sections of the contact subbundle. These

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connections are called projectively equivalent because they have the same class of unparameterized geodesics going in the contact subbundle direction, see, e.g., D. Fox [9], where you can find a relationship between the contact projective geometries and classical path geometries. The adjective "metaplectic" suggests that in addition to contact projective geometries, the metaplectic contact projective structures include some spin phenomena like the spin structures over Riemannian manifolds. Metaplectic contact projective and contact projective geometries have their description also via Cartan geometries. Contact projective geometries could be modeled on a $(2l+1)$-dimensional projective space $\mathbb{PV}$ of a $(2l+2)$-dimensional real symplectic vector space $\mathbb{V}$, which we suppose to be equipped with a symplectic form ω. Here, the projective space is considered as a homogeneous space G/P, where G is the symplectic Lie group $Sp(\mathbb{V},\omega)$ acting transitively on $\mathbb{PV}$ by the factorization of its defining representation (on $\mathbb{V}$), and P is an isotropy subgroup of this action. In this case, it is easy to see that P is a parabolic subgroup, which turns out to be crucial for our classification. Contact projective geometry, in the sense of É. Cartan, are curved versions $(p:\mathcal{G}\to M,\omega)$ of this homogeneous (also called Klein) model G/P. There exist certain conditions (known as normalization conditions) under which the Cartan's principal bundle approach and the classical one (via the class of connections and the contact subbundle) are equivalent, see, e.g., Čap, Schichl [4] for details. We also remind that contact geometries are an arena for time-dependent Hamiltonian mechanics. Klein model of the metaplectic contact projective geometry consists of two groups $\tilde{G}$ and $\tilde{P}$, where $\tilde{G}$ is the metaplectic group $Mp(\mathbb{V},\omega)$, i.e., a nontrivial double covering of the symplectic group G, and $\tilde{P}$ is the preimage of P by this covering.

Symplectic spinor operators over projective contact geometries are acting between sections of the so-called higher symplectic spinor bundles. These bundles are associated via certain infinite dimensional irreducible admissible representations of the parabolic principal group P. The parabolic group P acts then nontrivially only by its Levi factor G_0, while the action of the unipotent part is trivial. The semisimple part $\mathfrak{g}_0^{ss}$ of the Lie algebra of the Levi part of the parabolic group P is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(2l,\mathbb{R})$. Thus to give an admissible representation of P, we have to specify a representation of $\mathfrak{g}_0^{ss}$. Let us recall that the classification of first order invariant operators was done by Slovák, Souček in [24] (generalizing an approach of Fegan in [8]) for all finite dimensional irreducible representations and general parabolic subgroup P of a semisimple G (almost Hermitian structures are studied in detail). Nevertheless, there are some interesting infinite dimensional representations of the complex symplectic Lie algebra, to which we shall focus our attention. These representations form a class consisting of infinite dimensional modules with bounded multiplicities. Modules with bounded multiplicities are representations, for which there is a nonnegative integer, such that the dimension of each weight space of this module is bounded by it from above. Britten, Hooper and Lemire in [2] and Britten, Hooper in [3] showed that each of these modules appear as direct summands in a tensor product of a finite dimensional $\mathfrak{sp}(2l,\mathbb{C})$-module and the so-called Kostant (or basic) symplectic spinor module $\mathbb{S}_+$ and vice versa. Irreducible representations in this completely reducible tensor product are called higher symplectic, harmonic or generalized Kostant spinors. It is well known, that all finite dimensional modules over complex symplectic Lie algebra appear as irreducible submodules of a tensor power of the defining representation. Thus the infinite dimensional modules with bounded multiplicities are analogous to the spinor–vector representations of complex orthogonal Lie algebras. Namely, each finite dimensional module over orthogonal Lie algebra is an irreducible summand in the tensor product of a basic spinor representation and some power of the defining module (spinor–vector representations), or in the power of the defining representation itself (vector representations). In order to have a complete picture, it remains to show that the basic (or Kostant) spinors are analogous to the orthogonal ones, even though infinite dimensional. The basic symplectic spinor module $\mathbb{S}_+$ was discovered by Bertram Kostant (see [20]), when he was introducing half-forms for metaplectic structures over symplectic manifolds in the context of geometric quantization. While in the orthogonal case spinor representations can be realized using the exterior algebra of a maximal isotropic vector space, the symplectic spinor representations are realized using the symmetric algebra of certain maximal isotropic vector space (called Lagrangian in the symplectic setting). This procedure goes roughly as follows: one takes the Chevalley realization of the symplectic Lie algebra C_l by polynomial coefficients linear differential operators acting on polynomials $\mathbb{C}[z^1,\dots,z^l]$ in l complex variables. The space of polynomials splits into two irreducible summands over the symplectic Lie algebra, namely into the two basic symplectic spinor modules $\mathbb{S}_+$ and $\mathbb{S}_-$. There is a relationship between the modules $\mathbb{S}_+$ and $\mathbb{S}_-$ and the Segal–Shale–Weil or oscillator representation. Namely, the underlying C_l-structure of the Segal–Shale–Weil representation is isomorphic to $\mathbb{S}_+\oplus\mathbb{S}_-$.

In order to classify 1st order invariant differential operators, one needs to understand the structure of the space of P-homomorphisms between the so called 1st jets prolongation P-module of the domain module and the target representation of P, see Section 4. Thus the classification problem translates into an algebraic one. In our case, representation theory teaches us, that it is sometimes sufficient to understand our representation at its infinitesimal level. The only thing one needs in this case, is to understand the infinitesimal version of the 1st jets prolongation module. For our aims, the most important part of the 1st jets prolongation module consists of a tensor product of the defining representation of C_l and a higher symplectic spinor module. In order to describe the space of P-homomorphisms, one needs to decompose the mentioned tensor product into irreducible summands. This was done by Krýsl in [21], where results of Humphreys in [12] and Kac and Wakimoto in [15] were used.

Let us mention that some of these operators are contact analogues of the well-known symplectic Dirac operator, symplectic Rarita–Schwinger and symplectic twistor operator. Analytical properties of these operators were studied by many authors, see, e.g., K. Habermann [11] and A. Klein [18]. These symplectic versions were mentioned also by M.B. Green and C.M. Hull, see [10], in the context of covariant quantization of 10 dimensional super-strings and also in the theory of Dirac–Kähler fields, see Reuter [22], where we found a motivation for our studies of this topic.

In the second section, metaplectic contact projective geometries are defined using the Cartan's approach. Basic properties of higher symplectic spinor modules (Theorem 1) together with a theorem on a decomposition of the tensor product of the defining representation of $\mathfrak{sp}(2l,\mathbb{C})$ and an arbitrary higher symplectic spinor module (Theorem 2) are summarized in Section 3. Section 4 is devoted to the classification result. Theorem 3 and Lemmas 1 and 2 in this section are straightforward generalizations of similar results obtained by Slovák and Souček in [24]. Theorem 4 (in Section 4) is a well-known theorem on the action of a Casimir element on highest weight modules. While in the Section 4.1. we are interested only in the classification at the infinitesimal level (Theorem 5), we present our classification theorem at the globalized level in Section 4.2 (Theorem 6). In the fifth section, three main examples of the 1st order symplectic spinor operators over contact projective structures are introduced.

2. Metaplectic contact projective geometry

The aim of this section is neither to serve as a comprehensive introduction into metaplectic contact projective geometries, nor to list all references related to this subject. We shall only present a definition of metaplectic contact projective geometry by introducing its Klein model, and give only a few references, where one can find links to a broader literature on this topic (contact projective geometries, path geometries etc.).

For a fixed positive integer $l \geqslant 3$, let us consider a real symplectic vector space $(\mathbb{V},\omega)$ of real dimension $2l+2$ together with the defining action of the symplectic Lie group $G := Sp(\mathbb{V},\omega)$. The defining action is transitive on $\mathbb{V}-\{0\}$, and thus it defines a transitive action $G\times\mathbb{PV}\to\mathbb{PV}$ on the projective space $\mathbb{PV}$ of $\mathbb{V}$ by the prescription $(g,[v])\mapsto[gv]$ for $g\in G$ and $v\in\mathbb{V}-\{0\}$. (Here, $[v]$ denotes the one dimensional vector subspace spanned by v.) Let us denote the stabilizer of a point in $\mathbb{PV}$ by P. It is well known that this group is a parabolic subgroup of G, see, e.g., D. Fox [9]. The pair (G,P) is often called Klein pair of contact projective geometry. Let us denote the Lie algebra of P by $\mathfrak{p}$.

Definition 1. Cartan geometry $(p:\mathcal{G}\to M^{2l+1},\omega)$ is called a contact projective geometry of rank l, if it is a Cartan geometry modeled on the Klein geometry of type (G,P) for G and P introduced above.

It is possible to show that each contact projective geometry defines a contact structure on the tangent bundle TM of the base manifold M and a class $[\nabla]$ of contact projectively equivalent partial affine connections ∇ acting on the sections of the contact subbundle (see the Introduction for some remarks). For more details on this topic, see Fox [9]. In Čap, Schichl [4], one can find a treatment on the equivalence problem for contact projective structures. Roughly speaking, the reader can find a proof there, that under certain conditions, there is an isomorphism between the Cartan approach and the classical one (via contact subbundle and a class of connections). Because we would like to include some spin phenomena, let us consider a slightly modified situation. Fix a nontrivial two-fold covering $q:\tilde{G}\to G$ of the symplectic group $G=Sp(\mathbb{V},\omega)$ by the metaplectic group $\tilde{G}=Mp(\mathbb{V},\omega)$, see Kashiwara, Vergne [17]. Let us denote the q-preimage of P by $\tilde{P}$.

Definition 2. Cartan geometry $(p:\tilde{\mathcal{G}}\to M^{2l+1},\omega)$ is called metaplectic contact projective geometry of rank l, if it is a Cartan geometry modeled on the Klein geometry of type $(\tilde{G},\tilde{P})$ with $\tilde{G}$ and $\tilde{P}$ introduced above.

Let us remark, that in Definition 2, we do not demand metaplectic contact projective structure to be connected to a contact projective structure as one demands in the case of spin structures over Riemannian manifolds or in the case of metaplectic structures over manifolds with a symplectic structure.

3. Higher symplectic spinor modules

Let $C_l \simeq \mathfrak{sp}(2l, \mathbb{C})$, $l \geqslant 3$, be the complex symplectic Lie algebra. Consider a Cartan subalgebra $\mathfrak{h}$ of C_l together with a choice of positive roots Φ^+. The set of fundamental weights $\{\varpi_i\}_{i=1}^l$ is then uniquely determined. For later use, we shall need an orthogonal basis (with respect to the form dual to the Killing form of C_l), $\{\epsilon_i\}_{i=1}^l$, for which $\varpi_i = \sum_{j=1}^i \epsilon_j$ for $i = 1, \dots, l$.

For $\lambda \in \mathfrak{h}^*$, let $L(\lambda)$ be the irreducible C_l-module with the highest weight λ. This module is defined uniquely up to a C_l-isomorphism. If λ happens to be integral and dominant (with respect to the choice of $(\mathfrak{h}, \Phi^+)$), i.e., if $L(\lambda)$ is finite dimensional, we shall write $F(\lambda)$ instead of $L(\lambda)$. Let L be an arbitrary (finite or infinite dimensional) weight module over a complex simple Lie algebra. We call L a module with bounded multiplicities, if there is a $k \in \mathbb{N}_0$, such that for each $\mu \in \mathfrak{h}^*$, $\dim L_\mu \leqslant k$, where L_μ is the weight space of weight μ.

Let us introduce the following set of weights

$$A := \left\{ \lambda = \sum_{i=1}^{l} \lambda_i \varpi_i \mid \lambda_i \in \mathbb{N}_0, i = 1, \dots, l-1, \lambda_{l-1} + 2\lambda_l + 3 > 0, \lambda_l \in \mathbb{Z} + \frac{1}{2} \right\}.$$

Definition 3. For a weight $\lambda \in A$, we call the module $L(\lambda)$ higher symplectic spinor module. We shall denote the module $L(-\frac{1}{2}\varpi_l)$ by $\mathbb{S}_+$ and the module $L(\varpi_{l-1} - \frac{3}{2}\varpi_l)$ by $\mathbb{S}_-$. We shall call these two representations basic symplectic spinor modules.

The next theorem says that the class of higher symplectic spinor modules is quite natural and in a sense broad.

Theorem 1. *Let $\lambda \in \mathfrak{h}^*$. Then the following are equivalent:*

1) *$L(\lambda)$ is an infinite dimensional C_l-module with bounded multiplicities*;
2) *$L(\lambda)$ is a direct summand in $\mathbb{S}_+ \otimes F(\nu)$ for some integral dominant $\nu \in \mathfrak{h}^*$*;
3) *$\lambda \in A$.*

Proof. See Britten, Hooper, Lemire [2] and Britten, Lemire [3]. $\square$

In the next theorem, the tensor product of a higher symplectic spinor module and the defining representation $\mathbb{C}^{2l} \simeq F(\varpi_1)$ of the complex symplectic Lie algebra C_l is decomposed into irreducible summands. We shall need this statement in the classification procedure. It gives us an important information on the structure of the 1st jets prolongation module for metaplectic contact projective structures.

Theorem 2. *Let $\lambda \in A$. Then*

$$L(\lambda) \otimes F(\varpi_1) = \bigoplus_{\mu \in A_\lambda} L(\mu),$$

where $A_\lambda := A \cap \{\lambda + \nu \mid \nu \in \Pi(\varpi_1)\}$ and $\Pi(\varpi_1) = \{\pm\epsilon_i \mid i = 1, \dots, l\}$ is the set of weights of the defining representation.

Proof. See Krýsl, [21]. $\square$

Let us remark, that the proof of this theorem is based on the so-called Kac–Wakimoto formal character formula published in [15] (generalizing a statement of Jantzen in [14]) and some results of Humphreys, see [12], in which results of Kostant (from [19]) on tensor products of finite and infinite dimensional modules admitting a central character are specified.

4. Classification of first order invariant operators

In this section, we will be investigating first order invariant differential operators acting between sections of certain vector bundles associated to parabolic geometries $(p : \mathcal{G} \to M, \omega)$, i.e., to Cartan geometries modeled on Klein pairs (G, P), where P is an arbitrary parabolic subgroup of an arbitrary semisimple Lie group G.

We first consider a general real semisimple Lie group G together with its parabolic subgroup P and then we restrict our attention to the metaplectic contact projective case. Let us suppose that the Lie algebra $\mathfrak{g}$ of the group G is equipped with a $|k|$-grading $\mathfrak{g} = \bigoplus_{i=-k}^{k} \mathfrak{g}_i$, i.e., $\mathfrak{g}_1$ generates $\bigoplus_{i=1}^{k} \mathfrak{g}_i$ as a Lie algebra and $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for $i, j \in \{-k, \dots, k\}$.[1] Denote the semisimple part and the center of the reductive Lie algebra $\mathfrak{g}_0 \subset \mathfrak{g}$ (also called Levi factor) by $\mathfrak{g}_0^{ss}$ and $\mathfrak{z}(\mathfrak{g}_0)$, respectively. The subalgebra $\bigoplus_{i=0}^{k} \mathfrak{g}_i$ forms a parabolic subalgebra of $\mathfrak{g}$ and will be denoted by $\mathfrak{p}$. Let us suppose that $\mathfrak{p}$ is isomorphic to the Lie algebra of the fixed parabolic subgroup P of G. The nilpotent part $\bigoplus_{i=1}^{k} \mathfrak{g}_i$ of $\mathfrak{p}$ is usually denoted by $\mathfrak{g}_+$ and the negative $\bigoplus_{i=-k}^{-1} \mathfrak{g}_i$ part of $\mathfrak{g}$ by $\mathfrak{g}_-$. Let us consider Killing forms $(\,,)_{\mathfrak{g}}$ and $(\,,)_{\mathfrak{g}_0^{ss}}$ of $\mathfrak{g}$ and $\mathfrak{g}_0^{ss}$, respectively. Further, fix a basis $\{\xi^i\}_{i=1}^{r}$ of $\mathfrak{g}_+$, such that $\{\xi^i\}_{i=1}^{s}$ is a basis of $\mathfrak{g}_1$ and $\{\xi^i\}_{i=s+1}^{r}$ is a basis of $\bigoplus_{i=2}^{k} \mathfrak{g}_i$. The second basis, we will use, is a basis of $\mathfrak{g}_0^{ss}$, which will be denoted by $\{\eta^i\}_{i=1}^{t}$. The $|k|$-grading of $\mathfrak{g}$ uniquely determines the so-called grading element $Gr \in \mathfrak{z}(\mathfrak{g}_0)$. The defining equation for this element is $[Gr, X] = jX$ for $X \in \mathfrak{g}_j$ and each $j \in \{-k, \dots, k\}$. It is known that for each $|k|$-grading of a real (or complex) semisimple Lie algebra the grading element exists, see, e.g., Yamaguchi [28]. Sometimes, we will denote the grading element Gr by η^{t+1}. The set $\{\eta^i\}_{i=1}^{t+1}$ is then a basis of $\mathfrak{g}_0$. Let us denote the basis of $\mathfrak{g}_-$ dual to $\{\xi^i\}_{i=1}^{r}$ with respect to the Killing form $(\,,)_{\mathfrak{g}}$ by $\{\xi_i\}_{i=1}^{r}$ and the basis of $\mathfrak{g}_0$ dual to the basis $\{\eta^i\}_{i=1}^{t+1}$ with respect to the Killing form $(\,,)_{\mathfrak{g}}$ by $\{\eta_i\}_{i=1}^{t+1}$.

At the beginning, let us consider two complex irreducible representations $(\sigma, \mathbf{E})$ and $(\tau, \mathbf{F})$ of P in the category $\mathcal{R}(P)$, the objects of which are locally convex, Hausdorff vector spaces with a continuous linear action of P, which is admissible, of finite length. Here, admissible action means that the restriction of this action to the Levi subgroup G_0 of P is admissible, see Vogan [27]. The morphisms in the category $\mathcal{R}(P)$ are linear continuous P-equivariant maps between the objects. It is well known that the unipotent part of the parabolic group acts trivially on both $\mathbf{E}$ and $\mathbf{F}$. We shall call $\mathbf{E}$ and $\mathbf{F}$ the domain and the target module, respectively and we shall specify further conditions on these representations later. Generally, for a Lie group G and its admissible representation $\mathbf{E}$, we shall denote the corresponding Harish-Chandra $(\mathfrak{g}, K)$-module (K is maximal compact in G) by E and when we will only be considering the $\mathfrak{g}$-module structure, we shall use the symbol $\mathbb{E}$ for it. Further, we will denote the corresponding actions of an element X from the Lie algebra of G on a vector v simply by $X.v$, and the action of $g \in G$ on a vector v by $g.v$—the considered representation will be clear from a context.

Let us stress that most our proofs are formally almost identical to that ones written by Slovák, Souček in [24], but we formulate them also for infinite dimensional admissible irreducible $\mathbf{E}$ and $\mathbf{F}$, and use the decomposition result in Krýsl [21] when we will be treating the metaplectic contact projective case.

Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry modeled on the Klein pair (G, P). Because $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism for each $u \in \mathcal{G}$ by definition, we can define a vector field $\omega^{-1}(X)$ for each $X \in \mathfrak{g}$ by the equation $\omega_u(\omega^{-1}(X)_u) = X$, the so-called constant vector field. For later use, consider two associated vector bundles $\mathbf{E}M := \mathcal{G} \times_\sigma \mathbf{E}$ and $\mathbf{F}M := \mathcal{G} \times_\tau \mathbf{F}$—the so called domain and target bundle, respectively. To each Cartan geometry, there is an associated derivative ∇^ω defined as follows. For any section $s \in \Gamma(M, \mathbf{E}M)$ considered as $s \in \mathcal{C}^\infty(\mathcal{G}, \mathbf{E})^P$ under the obvious isomorphism, we obtain a mapping $\nabla^\omega s : \mathcal{G} \to \mathfrak{g}_-^* \otimes \mathbf{E}$, defined by the formula

$$\bigl(\nabla^\omega s(u)\bigr)X := \mathcal{L}_{\omega^{-1}(X)} s(u),$$

where $X \in \mathfrak{g}_-$, $u \in \mathcal{G}$ and $\mathcal{L}$ is the Lie derivative. The associated derivative ∇^ω is usually called absolute invariant derivative. The 1st jets prolongation module $J^1\mathbf{E}$ of $\mathbf{E}$ is defined as follows. As a vector space, it is simply the space $\mathbf{E} \oplus (\mathfrak{g}_+ \otimes \mathbf{E})$. To be specific, let us fix the Grothendieck's projective tensor product topology on 1st jets prolongation module, see Treves [26] or/and D. Vogan [27]. The vector space $J^1\mathbf{E}$ comes up with an inherited natural action of the group P, forming the 1st jets prolongation P-module, see Čap, Slovák, Souček [6]. Let us remark that the function $u \mapsto (s(u), \nabla^\omega s(u))$ defines a P-equivariant function on $\mathcal{G}$ with values in $J^1\mathbf{E}$ and thus a section of the first jet prolongation bundle $J^1(\mathbf{E}M)$ of the associated bundle $\mathbf{E}M$. For details, see Čap, Slovák Souček [6].

[1] By definition, $\mathfrak{g}_i = 0$ for $|i| > k$ is to be understood.

By differentiation of the P-action on $J^1\mathbf{E}$, we can obtain a $\mathfrak{p}$-module structure, the so-called infinitesimal 1st jets prolongation $\mathfrak{p}$-module $J^1\mathbb{E}$, which is as a vector space isomorphic to $\mathbb{E} \oplus (\mathfrak{g}_+ \otimes \mathbb{E})$. The $\mathfrak{p}$-representation is then given by the formula

$$R.(v', S \otimes v'') := \left(R.v', S \otimes R.v'' + [R, S] \otimes v'' + \sum_{i=1}^{r} \xi^i \otimes [R, \xi_i]_{\mathfrak{p}}.v'\right) \tag{1}$$

where $R \in \mathfrak{p}$, $S \in \mathfrak{g}_+$, $v', v'' \in \mathbb{E}$ and $[R, \xi_i]_{\mathfrak{p}}$ denotes the projection of $[R, \xi_i]$ to $\mathfrak{p}$. For a derivation of the above formula, see Čap, Slovák, Souček [6] for more details. Obviously, this action does not depend on a choice of the vector space basis $\{\xi^i\}_{i=1}^r$. We will call this action the induced action of $\mathfrak{p}$.

Definition 4. We call a vector space homomorphism $\mathfrak{D} : \Gamma(M, \mathbf{E}M) \to \Gamma(M, \mathbf{F}M)$ first order invariant differential operator, if there is a P-module homomorphism[2] $\mathbf{D} : J^1\mathbf{E} \to \mathbf{F}$, such that $\mathfrak{D}s(u) = \mathbf{D}(s(u), \nabla^\omega s(u))$ for each $u \in \mathcal{G}$ and each section $s \in \Gamma(M, \mathbf{E}M)$ (considered as a P-equivariant $\mathbf{E}$-valued smooth function on $\mathcal{G}$).

Let us remark, that this definition could be generalized for an arbitrary order. The corresponding operators are called strongly invariant. There exist also operators which are invariant in a broader sense (see Čap, Slovák, Souček [5]) and not strongly invariant.

We shall denote the vector space of first order invariant differential operators by $\mathrm{Diff}(\mathbf{E}M, \mathbf{F}M)^1_{(p:\mathcal{G}\to M)}$. It is clear that $\mathrm{Diff}(\mathbf{E}M, \mathbf{F}M)^1_{(p:\mathcal{G}\to M)} \simeq \mathrm{Hom}_P(J^1\mathbf{E}, \mathbf{F})$ as complex vector spaces. Let us denote the restricted 1st jets prolongation P-module, i.e., the quotient P-module

$$\big[\mathbf{E} \oplus (\mathfrak{g}_+ \otimes \mathbf{E})\big] \Big/ \left[\{0\} \oplus \left(\bigoplus_{i=2}^{k} \mathfrak{g}_i \otimes \mathbf{E}\right)\right],$$

by $J^1_R\mathbf{E}$. According to our notation, the meanings of $J^1_R E$ and $J^1_R\mathbb{E}$ are also fixed. Now, let us introduce a linear mapping $\Psi : \mathfrak{g}_1 \otimes \mathbb{E} \to \mathfrak{g}_1 \otimes \mathbb{E}$ given by the following formula

$$\Psi(X \otimes v) := \sum_{i=1}^{s} \xi^i \otimes [X, \xi_i].v.$$

Obviously, mapping Ψ does not depend on a choice of the basis $\{\xi^i\}_{i=1}^s$.

First, let us derive the following

Theorem 3. *Let $\mathbb{E}$ and $\mathbb{F}$ be two $\mathfrak{p}$-modules such that the nilpotent part $\mathfrak{g}_+$ acts trivially on them. If $D \in \mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E}, \mathbb{F})$ is a $\mathfrak{p}$-homomorphism, then D vanishes on the image of Ψ and D factors through the restricted jets, i.e., $D(0, Z \otimes v'') = 0$ for each $v'' \in \mathbb{E}$ and $Z \in \bigoplus_{i=2}^k \mathfrak{g}_i$. Conversely, suppose $D \in \mathrm{Hom}_{\mathfrak{g}_0}(J^1\mathbb{E}, \mathbb{F})$ is a $\mathfrak{g}_0$-homomorphism, D factors through the restricted jets, and D vanishes on the image of Ψ, then D is a $\mathfrak{p}$-module homomorphism.*

Proof. Let $D \in \mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E}, \mathbb{F})$ be a $\mathfrak{p}$-homomorphism. Take an element $\tilde{v} \in \mathfrak{g}_+.J^1\mathbb{E}$. Then $D(\tilde{v}) = D(X.v)$ for some $X \in \mathfrak{g}_+$ and $v \in \mathbb{E}$. Using the fact, that D is a $\mathfrak{p}$-homomorphism, we can write $D(\tilde{v}) = X.D(v) = 0$, because the nilpotent algebra $\mathfrak{g}_+$ acts trivially on the module $\mathbb{F}$. Thus D vanishes on the image of $\mathfrak{g}_+$ on $J^1\mathbb{E}$.

Now, we would like to prove, that D factors through $J^1_R\mathbb{E}$. Take an arbitrary element $Z \in \bigoplus_{i=2}^k \mathfrak{g}_i$ and $v'' \in \mathbb{E}$. Because $\mathfrak{g}$ is a $|k|$-graded algebra, there are $n \in \mathbb{N}$ and $X_i, Y_i \in \mathfrak{g}_+$ for $i = 1, \ldots, n$, such that $Z = \sum_{i=1}^n [X_i, Y_i]$. It is easy to compute that $\sum_{i=1}^n X_i.(0, Y_i \otimes v'') = (0, \sum_{i=1}^n Y_i \otimes X_i.v'' + [X_i, Y_i].v'' + 0) = (0, \sum_{i=1}^n [X_i, Y_i] \otimes v'') = (0, Z \otimes v'')$. Thus we may write $D(0, Z \otimes v'') = D(\sum_{i=1}^n X_i.(0, Y_i \otimes v'')) = 0$, because D acts trivially on $\mathfrak{g}_+.J^1\mathbb{E}$, as we have already proved.

Second, we shall prove that D vanishes on the image of Ψ. Substituting $v'' = 0$ into formula (1) for the induced action, we get that $X.(v', 0) = (X.v', \sum_{i=1}^r \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v')$ for $v' \in \mathbb{E}$ and $X \in \mathfrak{g}_1$. Assuming that the nilpotent subalgebra $\mathfrak{g}_+$ acts trivially on $\mathbb{E}$, one obtains $X.(v', 0) = (0, \sum_{i=1}^r \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v') = (0, \sum_{i=1}^s \xi^i \otimes$

[2] By a P-module homomorphism, we mean a morphism in $\mathcal{R}(P)$.

$[X, \xi_i]_{\mathfrak{p}}.v' + \sum_{i=s+1}^{r} \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v')$. The last summand is zero, because $[X, \xi_i]_{\mathfrak{p}} = 0$ for $i > s$. Thus we have $X.(v', 0) = (0, \sum_{i=1}^{s} \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v')$. Because D vanishes on the image of the action of $\mathfrak{g}_+$ on $J^1\mathbb{E}$, we know that $0 = D(X.(v', 0)) = D(0, \sum_{i=1}^{s} \xi^i \otimes [X, \xi_i]_{\mathfrak{p}}.v')$. Since one can omit the restriction of the Lie bracket in the last term to the subalgebra $\mathfrak{p}$ (we are considering ξ_i only for $i = 1, \dots, s$), D vanishes on the image of Ψ.

Now, we would like to prove the opposite direction. Hence suppose, a $\mathfrak{g}_0$-homomorphism D is given. Let us take an element $S \in \mathfrak{g}_+$ (for $S \in \mathfrak{g}_0$ it is clear) and an arbitrary element $\tilde{v} = (v', Y \otimes v'') \in J^1\mathbb{E}$. Thus $D(S.\tilde{v}) = D(S.(v', Y \otimes v'')) = D(S.v', Y \otimes S.v'' + [S, Y] \otimes v'' + \sum_{i=1}^{r} \xi^i \otimes [S, \xi_i]_{\mathfrak{p}}.v') = D(0, \sum_{i=1}^{r} \xi^i \otimes [S, \xi_i]_{\mathfrak{p}}.v') = 0 = S.D(\tilde{v})$, where we have used that the action of $\mathfrak{g}_+$ is trivial on $\mathbb{E}$, D factors through the restricted jets, vanishes on the image of Ψ, and the fact that $\mathfrak{g}_+$ acts trivially on $\mathbb{F}$. $\square$

Now, we derive the following

Lemma 1. *For the mapping Ψ, we have*

$$\Psi(X \otimes v) = \sum_{j=1}^{t+1} [\eta_j, X] \otimes \eta^j.v$$

for each $X \in \mathfrak{g}_1$ and $v \in \mathbb{E}$.

Proof. Take an element $X \in \mathfrak{g}_1$. Using the invariance of the Killing form $(,)_{\mathfrak{g}}$, expressed by

$$[X, \xi_i] = \sum_{i=1}^{t+1} \big(\eta_i, [X, \xi_i]\big)_{\mathfrak{g}} \eta^i = \sum_{i=1}^{t+1} \big([\eta_i, X], \xi_i\big)_{\mathfrak{g}} \eta^i,$$

we compute the value $\Psi(X \otimes v)$ as

$$\begin{aligned}
\Psi(X \otimes v) &= \sum_{i=1}^{s} \xi^i \otimes [X, \xi_i].v \\
&= \sum_{i=1}^{s} \xi^i \otimes \sum_{j}^{t+1} \big(\eta_j, [X, \xi_i]\big)_{\mathfrak{g}} \eta^j.v \\
&= \sum_{i=1}^{s} \xi^i \otimes \sum_{j=1}^{t+1} \big([\eta_j, X], \xi_i\big)_{\mathfrak{g}} \eta^j.v \\
&= \sum_{i=1}^{s} \sum_{j=1}^{t+1} \big([\eta_j, X], \xi_i\big)_{\mathfrak{g}} \xi^i \otimes \eta^j.v \\
&= \sum_{j=1}^{t+1} [\eta_j, X] \otimes \eta^j.v. \qquad \square
\end{aligned}$$

For any real Lie algebra $\mathfrak{g}$, let us denote its complexification over reals by $\mathfrak{g}^{\mathbb{C}}$, i.e., $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathfrak{h}$ be a (complex) Cartan subalgebra of $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$. For each $\lambda, \mu, \alpha \in \mathfrak{h}^*$, we define a complex number

$$c_{\lambda\alpha}^{\mu} = \frac{1}{2}\big[(\lambda, \lambda + 2\delta)_{\mathfrak{g}_0^{ss}} + (\alpha, \alpha + 2\delta)_{\mathfrak{g}_0^{ss}} - (\mu, \mu + 2\delta)_{\mathfrak{g}_0^{ss}}\big],$$

where δ denotes the sum of fundamental weights with respect to a choice of positive roots.[3]

[3] We are denoting the Killing form on $\mathfrak{g}_0^{ss}$ as well as the dual form on $(\mathfrak{g}_0^{ss})^*$ by the same symbol $(,)_{\mathfrak{g}_0^{ss}}$. We shall also not distinguish between the Killing form of a real algebra and that one of the complexification of this algebra. We hope that this will cause no confusion.

From now on, we shall suppose that the semisimple part of the Levi factor of P is actually simple and the center of the Levi factor is one dimensional. These assumptions are rather technical and introduced only in order to simplify formulations of our statements. Until yet, we have demanded the considered modules to be admissible irreducible P-modules. In particular, we have used the fact that the unipotent part of P acts trivially on them. From now on, we will suppose in addition that the modules $\mathbb{E}$ and $\mathbb{F}$ are irreducible highest weight modules over the complexification $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}_0^{ss}$ of the semisimple part of the Levi factor G_0 of P. Further we shall suppose, that the grading element acts by a complex multiple on each of the modules $\mathbb{E}$ and $\mathbb{F}$. We call a pair $(\lambda, c) \in \mathfrak{h}^* \times \mathbb{C}$ a highest weight of a representation $\mathbb{E}$ over the reductive Lie algebra $(\mathfrak{g}_0)^{\mathbb{C}}$, if the restriction of the representation of $\mathfrak{g}_0$ on $\mathbb{E}$ to the simple part $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$ has highest weight λ and the grading element Gr acts by a complex number c. The complex number c is often called generalized conformal weight of the $\mathfrak{p}$-module $\mathbb{E}$.

Recall a well-known theorem on the action of the universal Casimir element on highest weight modules.

Theorem 4. *Let $\mathbb{E}$ be a highest weight module over the simple complex Lie algebra $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$ with a highest weight $\lambda \in \mathfrak{h}^*$ and $C \in \mathfrak{U}((\mathfrak{g}_0^{ss})^{\mathbb{C}})$ be the universal Casimir element of $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$. Then*

$$C.v = (\lambda, \lambda + 2\delta)_{\mathfrak{g}_0^{ss}} v,$$

where $v \in \mathbb{E}$.

Proof. See, e.g., Humphreys [13]. □

Before we state the next lemma, let us do some comments on the relationship between the Killing forms $(\,,\,)_{\mathfrak{g}_0^{ss}}$ and $(\,,\,)_{\mathfrak{g}}$. It is well known that the restriction of $(\,,\,)_{\mathfrak{g}}$ to $\mathfrak{g}_0^{ss}$ is a nondegenerate and obviously an invariant bilinear form, and therefore there is a constant $\kappa \in \mathbb{C}^{\times}$, such that for $X, Y \in \mathfrak{g}_0^{ss}$ we have $(X, Y)_{\mathfrak{g}_0^{ss}} = \kappa (X, Y)_{\mathfrak{g}}$—due to the uniqueness of invariant nondegenerate forms up to a nonzero complex multiple. The bases $\{\eta^i\}_{i=1}^t$ and $\{\eta_i\}_{i=1}^t$ of $\mathfrak{g}_0^{ss}$ are not dual with respect to the Killing form $(\,,\,)_{\mathfrak{g}_0^{ss}}$ in general. For further purposes, we can consider these bases being also bases of the appropriate complexified Lie algebras. According to the relationship between the Killing forms in question, we know that $\{\eta^i\}_{i=1}^t$ and $\{\kappa^{-1}\eta_i\}_{i=1}^t$ are dual with respect to $(\,,\,)_{\mathfrak{g}_0^{ss}}$. We would like to compute $(\sum_{i=1}^t \eta^i \eta_i).v$. Due to Theorem 4, we can write $(\sum_{i=1}^t \eta^i \kappa^{-1} \eta_i).v = (\lambda, \lambda + 2\delta)_{\mathfrak{g}_0^{ss}} v$, if $v \in L(\lambda)$. Therefore $(\sum_{i=1}^t \eta^i \eta_i).v = \kappa(\lambda, \lambda + 2\delta)_{\mathfrak{g}_0^{ss}} v$. Let us denote $(Gr, Gr)_{\mathfrak{g}} =: \rho^{-1}$, i.e., $\eta^{t+1} = Gr$ whereas $\eta_{t+1} = \rho Gr$. Thus if Gr acts by a complex number c, we have that the action of $\eta^{t+1}\eta_{t+1}$ is by ρc^2. We will use these computations in the proof of the following

Lemma 2. *Suppose $\mathbb{E}$ is an irreducible $\mathfrak{p}^{\mathbb{C}}$-module, the action of $(\mathfrak{g}_+)^{\mathbb{C}}$ being trivial and the highest weight of $\mathbb{E}$ over $(\mathfrak{g}_0)^{\mathbb{C}}$ is $(\lambda, c) \in \mathfrak{h}^* \times \mathbb{C}$. Let us further suppose that $\mathbb{E} \otimes (\mathfrak{g}_1)^{\mathbb{C}}$ decomposes into a finite direct sum $\mathbb{E} \otimes \mathfrak{g}_1 = \bigoplus_{\mu} \mathbb{E}^{\mu}$ of irreducible $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-modules, where $\mathbb{E}^{\mu}$ is an irreducible $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-module with a highest weight μ. Let us fix a set of projections π_{μ} onto the irreducible summands in $\mathbb{E} \otimes (\mathfrak{g}_1)^{\mathbb{C}}$. Assume further that $(\mathfrak{g}_1)^{\mathbb{C}}$ is an irreducible $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-module with a highest weight α. Then*

$$\Psi = \sum_{\mu} (\rho c - \kappa c_{\lambda\alpha}^{\mu}) \pi_{\mu}. \tag{2}$$

Proof. Let us do the following computation with "Casimir" operators $\sum_{i=1}^{t+1} \eta^i \eta_i \in \mathfrak{U}(\mathfrak{g}_0)$. For $X \in \mathfrak{g}_1$ and $v \in \mathbb{E}$, we have:

$$\sum_{i=1}^{t+1} (\eta^i \eta_i).(X \otimes v) = \sum_{i=1}^{t+1} (\eta^i \eta_i).X \otimes v + X \otimes \sum_{i=1}^{t+1} (\eta^i \eta_i).v + 2\Psi(X \otimes v), \tag{3}$$

where we have used Lemma 1. Now, we would like to compute the first two terms of the R.H.S. of the last written equation using the universal Casimir element of $\mathfrak{g}_0^{ss}$, see Theorem 4.

$$\sum_{i=1}^{t+1} (\eta^i \eta_i).X \otimes v = \kappa (\alpha, \alpha + 2\delta)_{\mathfrak{g}_0^{ss}} X \otimes v + \rho X \otimes v, \tag{4}$$

$$X \otimes \sum_{i=1}^{t+1} (\eta^i \eta_i).v = \kappa(\lambda, \lambda + 2\delta)_{\mathfrak{g}_0^{ss}} X \otimes v + \rho c^2 X \otimes v. \tag{5}$$

Let us compute the L.H.S. of (3)

$$\sum_{i=1}^{t+1} (\eta^i \eta_i).(X \otimes v) = \sum_{\mu} \kappa(\mu, \mu + 2\delta)_{\mathfrak{g}_0^{ss}} \pi_\mu(X \otimes v) + \sum_{\mu} \pi_\mu[\rho X \otimes v + 2\rho c X \otimes v + \rho c^2 X \otimes v]. \tag{6}$$

Substituting Eqs. (4), (5) and (6) into Eq. (3) we obtain

$$\sum_{\mu} \kappa(\mu, \mu + 2\delta)_{\mathfrak{g}_0^{ss}} \pi_\mu(X \otimes v) + 2\sum_{\mu} \rho c \pi_\mu(X \otimes v) + \sum_{\mu} \rho c^2 \pi_\mu(X \otimes v) + \rho X \otimes v$$
$$= 2\Psi(X \otimes v) + \kappa(\alpha, \alpha + 2\delta)_{\mathfrak{g}_0^{ss}} X \otimes v + \rho X \otimes v + \kappa(\lambda, \lambda + 2\delta)_{\mathfrak{g}_0^{ss}} X \otimes v + \rho c^2 X \otimes v.$$

As a result we obtain

$$\Psi(X \otimes v) = \sum_{\mu} \big(\rho c - \kappa c_{\lambda\alpha}^{\mu}\big) \pi_\mu(X \otimes v). \qquad \square$$

4.1. Infinitesimal level classification

Let $(\mathbb{V}, \omega)$ be a real symplectic vector space of dimension $2l + 2$, $l \geqslant 3$. In this subsection, we shall focus our attention to the specific case of symplectic Lie algebra $\mathfrak{sp}(\mathbb{V}, \omega) \simeq \mathfrak{sp}(2l + 2, \mathbb{R})$ and its parabolic subalgebra $\mathfrak{p}$ introduced in Section 2. We shall be investigating the vector space $\mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E}, \mathbb{F})$ for suitable $\mathfrak{p}$-modules $\mathbb{E}, \mathbb{F}$, i.e., classify the first order invariant differential operator at the infinitesimal level. For a moment, we shall consider a complex setting.

The complex symplectic Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(2l + 2, \mathbb{C})$ possesses a $|2|$-grading,

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-2}^{\mathbb{C}} \oplus \mathfrak{g}_{-1}^{\mathbb{C}} \oplus \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_1^{\mathbb{C}} \oplus \mathfrak{g}_2^{\mathbb{C}},$$

such that $\mathfrak{g}_2^{\mathbb{C}} \simeq \mathbb{C}$, $\mathfrak{g}_1^{\mathbb{C}} \simeq \mathbb{C}^{2l}$, $\mathfrak{g}_0^{\mathbb{C}} = (\mathfrak{g}_0^{ss})^{\mathbb{C}} \oplus (\mathfrak{z}(\mathfrak{g}_0))^{\mathbb{C}} \simeq \mathfrak{sp}(2l, \mathbb{C}) \oplus \mathbb{C}$. This splitting could be displayed as follows. Choose a basis B of $\mathbb{V}$ such that ω, expressed in coordinates with respect to B, is given by $\omega((z^1, \ldots, z^{2l+2}), (w^1, \ldots, w^{2l+2})) = w^1 z^{2l+2} + \cdots + w^{l+1} z^{l+2} - w^{l+2} z^{l+1} - \cdots - w^{2l+2} z^1$. For $A \in \mathfrak{sp}(2l + 2, \mathbb{C})$ we have:

$$A = \left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right)$$

with respect to B. As one can easily compute, the parabolic subalgebra $\mathfrak{p}^{\mathbb{C}} = (\mathfrak{g}_0)^{\mathbb{C}} \oplus (\mathfrak{g}_1)^{\mathbb{C}} \oplus (\mathfrak{g}_2)^{\mathbb{C}}$ is a complexification of the Lie algebra of the group P introduced in Section 2, where we have defined the metaplectic contact projective geometry. Before we state the next theorem, we should compute the coefficients ρ and κ for the case $\mathfrak{g} = \mathfrak{sp}(2l + 2, \mathbb{C})$ considered with the grading given above. One can easily realize, that

$$Gr = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0_{2l} & 0 \\ \hline 0 & 0 & -1 \end{array}\right)$$

is the grading element, and that $(Gr, Gr)_{\mathfrak{g}} = 4l + 8$. Computing the square-norm of an element of $\mathfrak{g}_0^{ss}$ via $(\,,\,)_{\mathfrak{g}}$ and $(\,,\,)_{\mathfrak{g}_0^{ss}}$, one obtains for the ratio $\kappa = \frac{l+1}{l+2}$. Further, let us introduce a bilinear form $\langle\,,\,\rangle$ on $\mathfrak{h}^*$, in which the orthogonal basis $\{\epsilon_i\}_{i=1}^{l}$ is orthonormal. The relation between the Killing form $(\,,\,)_{\mathfrak{g}_0^{ss}}$ and $\langle\,,\,\rangle$ is given by $(X, Y)_{\mathfrak{g}_0^{ss}} = \frac{1}{4l+4}\langle X, Y\rangle$ for $X, Y \in \mathfrak{h}^*$. For each $\lambda, \mu, \alpha \in \mathfrak{h}^*$, let us define a complex number

$$\tilde{c}_{\lambda\alpha}^{\mu} = \frac{1}{2}\big(\langle\mu, \mu + 2\delta\rangle - \langle\lambda, \lambda + 2\delta\rangle - \langle\alpha, \alpha + 2\delta\rangle\big).$$

Substituting the computed values of ρ and κ and the relation between $(\,,\,)_{\mathfrak{g}_0^{ss}}$ and $\langle\,,\,\rangle$ into formula (2), we obtain a prescription for mapping Ψ (in the metaplectic contact projective case)

$$\Psi = \frac{1}{4l + 8} \sum_{\mu} (c - \tilde{c}_{\lambda\alpha}^{\mu}) \pi_\mu.$$

Theorem 5. *For $(\lambda, c), (\mu, d) \in A \times \mathbb{C}$, let $\mathbb{E}$ and $\mathbb{F}$ be two $\mathfrak{p}^{\mathbb{C}}$-modules such that $\mathbb{E}$ and $\mathbb{F}$ are irreducible if considered as $(\mathfrak{g}_0)^{\mathbb{C}}$-modules with highest weight (λ, c) and (μ, d), respectively, and let $(\mathfrak{g}_+)^{\mathbb{C}}$ has a trivial action on each of these modules. Further, suppose $\lambda \neq \mu$. Then*

$$\mathrm{Hom}_{\mathfrak{p}^{\mathbb{C}}}\big(J^1\mathbb{E}, \mathbb{F}\big) \simeq \begin{cases} \mathbb{C}, & \text{if } \mu \in A_\lambda \text{ and } d - 1 = c = \tilde{c}^{\mu}_{\lambda\varpi_1}, \\ 0 & \text{in other cases.} \end{cases}$$

Proof. Let us start with the second part of the statement, i.e., $\mu \notin A_\lambda$ or $c \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$ or $d - 1 \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$, and consider an element $T \in \mathrm{Hom}_{\mathfrak{p}^{\mathbb{C}}}(J^1\mathbb{E}, \mathbb{F})$. Then $T \in \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1\mathbb{E}, \mathbb{F})$. Because T is a $\mathfrak{p}^{\mathbb{C}}$-homomorphism, we have that $T \in \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E}, \mathbb{F})$ due to Theorem 3 (used in the complexified setting). We also know that

$$\mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E}, \mathbb{F}) = \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(\mathbb{E}, \mathbb{F}) \oplus \bigoplus_{\nu \in A_\lambda} \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}\big(L(\nu), L(\mu)\big)$$

due to Theorem 2. If we suppose $\mu \notin A_\lambda$ and $\lambda \neq \mu$, then due to Theorems 2.6.5, 2.6.6 in Dixmier [7], each member of the direct sum is zero. Now suppose that $\mu \in A_\lambda$. Thus $c \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$ or $d - 1 \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$. First suppose that $c \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$. Using Theorem 2 and the cited theorems of Dixmier, we see that $\mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E}, \mathbb{F}) \simeq \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(\mathbb{E}, \mathbb{F}) \oplus \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(L(\mu), L(\mu)) \simeq \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(L(\mu), L(\mu))$, because the decomposition of $(\mathfrak{g}_1)^{\mathbb{C}} \otimes \mathbb{E}$ is multiplicity-free and $\lambda \neq \mu$. Thus we can consider T to be a $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-intertwining operator acting on the irreducible highest weight module $L(\mu)$. We have two possibilities: $T : L(\mu) \to L(\mu)$ is either zero and we are done, or $\mathrm{Ker}\, T = \{0\}$. We will suppose the latter possibility. Take a nonzero element $0 \neq v \in L(\mu)$. Using the formula $\Psi = (4l + 8)^{-1} \sum_\nu (c - \tilde{c}^{\nu}_{\lambda\varpi_1})\pi_\nu$, we obtain under the assumption $c \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$ that $\Psi(v) = (4l + 8)^{-1}(c - \tilde{c}^{\mu}_{\lambda\alpha})v \neq 0$. Because $\mathrm{Ker}\, T = \{0\}$, we have that $T\Psi(v) \neq 0$ and thus, according to Theorem 3, T it is not a $\mathfrak{p}^{\mathbb{C}}$-module homomorphism because it does not vanish on the image of Ψ. Secondly, consider the case $d \neq \tilde{c}^{\mu}_{\lambda\alpha} + 1$. We can make the following easy computation. $d(S_1 \otimes v'') = Gr.(S_1 \otimes v'') = [Gr, S_1] \otimes v'' + S_1 \otimes Gr.v'' = (1 + c)S_1 \otimes v''$ for $S_1 \in (\mathfrak{g}_1)^{\mathbb{C}}$ and $v'' \in \mathbb{E}$. Thus $c = d - 1$ and we are obtaining the case $c \neq \tilde{c}^{\mu}_{\lambda\varpi_1}$, which was already handled.

Now, consider the case $\mu \in A_\lambda$, $c = \tilde{c}^{\mu}_{\lambda\alpha}$ and $d - 1 = \tilde{c}^{\mu}_{\lambda\alpha}$ and take a $T \in \mathrm{Hom}_{\mathfrak{p}^{\mathbb{C}}}(J^1\mathbb{E}, \mathbb{F})$. As in the previous case, this implies $T \in \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E}, \mathbb{F})$. Decomposing $J^1_R\mathbb{E} = L(\lambda) \oplus (F(\varpi_1) \otimes L(\lambda))$ into irreducible modules and substituting this decomposition into $\mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E}, \mathbb{F})$, we obtain a direct sum

$$\mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(\mathbb{E}, \mathbb{F}) \oplus \bigoplus_{\nu \in A_\lambda} \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}\big(L(\nu), L(\mu)\big).$$

According to our assumptions $\mu \in A_\lambda$ and $\lambda \neq \mu$, and due to the structure of the set A_λ, we know that the direct sum simplifies into a space isomorphic to $\mathbb{C}$ (using the above cited theorem of Dixmier once more). Thus we know that $\mathrm{Hom}_{\mathfrak{p}^{\mathbb{C}}}(J^1\mathbb{E}, \mathbb{F}) \subseteq \mathrm{Hom}_{(\mathfrak{g}_0^{ss})^{\mathbb{C}}}(J^1_R\mathbb{E}, \mathbb{F}) \simeq \mathbb{C}$. To obtain an equality in the previous inclusion, consider the one dimensional vector space of $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-homomorphisms $\{w\widetilde{\pi_\mu} | w \in \mathbb{C}\}$, where $\widetilde{\pi_\mu}$ is a trivial extension of the projection $(\mathfrak{g}_1)^{\mathbb{C}} \otimes \mathbb{E} \to L(\mu)$. The elements of this vector space are clearly $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-homomorphisms, which vanish on the image of Ψ, if $c = \tilde{c}^{\mu}_{\lambda\alpha}$, and they factorize through the restricted jets. What remains is to show that for each $w \in \mathbb{C}$, mappings $w\widetilde{\pi_\mu}$ are not only $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-homomorphisms, but also $(\mathfrak{g}_0)^{\mathbb{C}}$-homomorphisms. Notice that it is sufficient to test the condition only on $(\mathfrak{g}_1)^{\mathbb{C}} \otimes \mathbb{E}$ because $Gr \in (\mathfrak{g}_0)^{\mathbb{C}}$, and $\widetilde{\pi_\mu}$ is the trivial extension, see formula (1). For $S_1 \in (\mathfrak{g}_1)^{\mathbb{C}}$ and $v'' \in \mathbb{E}$, we have $Gr.\widetilde{\pi_\mu}(S_1 \otimes v'') = d\widetilde{\pi_\mu}(S_1 \otimes v'')$ by definition. Now, let us evaluate $\widetilde{\pi_\mu}Gr.(S_1 \otimes v'') = \widetilde{\pi_\mu}([Gr, S_1] \otimes v'' + S_1 \otimes Gr.v'') = \widetilde{\pi_\mu}(S_1 \otimes v'' + cS_1 \otimes v'') = (1 + c)\widetilde{\pi_\mu}(S_1 \otimes v'') = d\widetilde{\pi_\mu}(S_1 \otimes v'') = Gr.\widetilde{\pi_\mu}(S_1 \otimes v'')$, thus $\widetilde{\pi_\mu}$ commutes with the action of Gr. Therefore $\widetilde{\pi_\mu}$ is a $(\mathfrak{g}_0)^{\mathbb{C}}$-homomorphism and the statement follows using Theorem 3. □

Let us remark, that for $\lambda = \mu$, the space of homomorphisms is also one dimensional. But this case leads to zeroth order operators, which are not interesting from the point of view of our classification. Let us derive an easy corollary of the above theorem.

Corollary 1. *The preceding theorem remains true for a real form $\mathfrak{f}$ of $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$, if one considers complex representations and complex linear homomorphisms. In particular, it remains true for the split real form $\mathfrak{f} = \mathfrak{g}_0^{ss} \simeq \mathfrak{sp}(2l, \mathbb{R})$.*

Proof. First, observe that the decomposition of $F(\varpi_1) \otimes L(\lambda)$ remains the same also over $\mathfrak{f}$. For it, let us take an irreducible summand $\mathbb{M}$ in the decomposition and suppose there is a proper nontrivial complex submodule $\mathbb{M}'$ of $\mathbb{M}$. For $v \in \mathbb{M}'$ and $X + iY \in \mathfrak{f} + i\mathfrak{f}$, we get that $(X + iY).v = X.v + iY.v$. Using the fact that $\mathbb{M}'$ is closed under complex number multiplication and $X.v, Y.v \in \mathbb{M}'$, we would obtain that $\mathbb{M}'$ is $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-invariant, which is a contradiction.

Second, we would like to prove that each $\mathfrak{f}$-invariant complex linear endomorphism of an irreducible module, say $\mathbb{F}$, is a scalar. It is easy to observe, that such an endomorphism is actually $(\mathfrak{g}_0^{ss})^{\mathbb{C}}$-endomorphism, i.e., the theorem of Dixmier used in the proof of the previous theorem, could be applied and the corollary follows. □

4.2. Globalized level classification

In this subsection, we shall extend the results obtained in the previous one to the group level. We will do it using some basic facts on globalization techniques.

Let $(\mathbb{V}, \omega)$ be a real symplectic vector space of real dimension $2l + 2$, $l \geqslant 3$, $G = Sp(\mathbb{V}, \omega)$ and P as described in Section 2. First, we introduce the groups, we shall be considering. Let G_+, G_0, G_0^{ss}, K be the unipotent part, the Levi factor, the semisimple part of P and the maximal compact subgroup of G, respectively. Recall that we have fixed a nontrivial 2-fold covering $q : \tilde{G} \to G$ of the symplectic group G by the metaplectic group $\tilde{G} = Mp(\mathbb{V}, \omega)$. Let us denote the respective q-preimages by $\tilde{G}_+, \tilde{G}_0, \tilde{G}_0^{ss}, \tilde{K}$. Further, let us denote the maximal compact subgroup of the semisimple part G_0^{ss} of the Levi factor by K_0^{ss} and its q-preimage by $\tilde{K}_0^{ss}$. We have

$$\tilde{K}_0^{ss} \simeq \widetilde{U(l)} = \{(u, z) \in U(l) \times \mathbb{C}^\times \mid \det u = z^2\},$$

which is obviously connected, see Tirao, Vogan and Wolf [25].

Second, let us introduce a class of $\tilde{P}$-modules we shall be dealing with. In Kashiwara, Vergne [17], the so called metaplectic (or Segal–Shale–Weil or oscillator) representation over $\tilde{G}_0^{ss}$ is introduced. Let $\mathbf{S}_+$ be the irreducible submodule of the Segal–Shale–Weil representation consisting of even functions. Let us take the underlying $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module and denote it by S_+. The $\mathfrak{g}_0^{ss}$-module structure of this representation coincides with the irreducible highest weight module structure of $\mathbb{S}_+$, which was introduced in Section 3. For a choice of a weight $\lambda \in A$, we know that there exists a dominant integral weight ν (with respect to choices made in Section 3), such that $\mathbb{L} := L(\lambda) \subseteq \mathbb{S}_+ \otimes F(\nu)$. Because $\mathbb{S}_+ \otimes F(\nu)$ decomposes without multiplicities, we have an identification of $L(\lambda)$ with its isomorphic module in $\mathbb{S}_+ \otimes F(\nu)$. Now we would like to make $\mathbb{L}$ a $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module. Using a result of Baldoni [1], this could be done as follows. Because S_+ and $F(\nu)$ are $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-modules, their tensor product is a $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module as well. Using the fact that $\tilde{K}_0^{ss} = \widetilde{U(l)}$ is connected, we are obtaining a $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module structure on each irreducible summand in $\mathbb{S}_+ \otimes F(\nu)$, in particular on $\mathbb{L}$. Denote the resulting $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module by L. Using globalization results of Kashiwara and Schmid in [16], there exists a minimal globalization for this $(\mathfrak{g}_0^{ss}, \tilde{K}_0^{ss})$-module, which will be denoted by $\mathbf{L} =: \mathbf{L}(\lambda)$. (For this topic, see also Vogan [27] and Schmid [23].) Thus $\mathbf{L}(\lambda)$ is a complex $\tilde{G}_0^{ss}$-module. Further, we need to specify the action of the center of $\tilde{G}_0$ and that one of the unipotent part $\tilde{G}_+$. For each $(\lambda, c) \in A \times \mathbb{C}$ we suppose, that the unipotent $\tilde{G}_+$ acts trivially on $\mathbf{L}(\lambda)$ and the grading element Gr in the Lie algebra of the center of the Levi factor $\tilde{G}_0$ acts by multiplication by a complex number $c \in \mathbb{C}$. Since the center is isomorphic to $\mathbb{R}^\times$ we need to specify the action of, e.g., $-1 \in \mathbb{R}^\times$. This action should be any $\gamma \in \mathbb{R}$ satisfying $\gamma^2 = 1$. So we have obtained a $\tilde{P}$-module structure on $\mathbf{L}(\lambda)$ which we will refer to as $\mathbf{L}(\lambda, c)_\gamma$. Let us remark, that defining the action of $\tilde{G}_+$ to be trivial, is actually no restriction, when one considers only irreducible admissible $\tilde{P}$-modules. We shall call the corresponding associated bundles higher symplectic bundles and the corresponding 1st order invariant differential operators symplectic spinor operators, stressing the fact that the representations of $\tilde{P}$ we are considering are coming from higher symplectic spinor modules.

Theorem 6. *Let $(\lambda, c, \gamma), (\mu, d, \gamma') \in A \times \mathbb{C} \times \mathbb{Z}_2$,[4] $\lambda \neq \mu$ and $(p : \tilde{\mathcal{G}} \to M^{2l+1}, \omega)$ be a metaplectic contact projective geometry of rank l. Consider the $\tilde{P}$-modules $\mathbf{E} := \mathbf{L}(\lambda, c)_\gamma$ and $\mathbf{F} := \mathbf{L}(\mu, d)_{\gamma'}$. Then for the vector space of*

[4] The group $\mathbb{Z}_2$ is considered as multiplicative, i.e., $\mathbb{Z}_2 = \{-1, 1\}$.

invariant differential operators up to a zeroth order we have

$$\mathrm{Diff}(\mathbf{E}M,\mathbf{F}M)^1_{(p:\tilde{\mathcal{G}}\to M^{2l+1},\omega)} \simeq \begin{cases} \mathbb{C} & \text{if } \mu\in A_\lambda,\ d-1=c=\tilde{c}^{\mu}_{\lambda\varpi_1} \text{ and } \gamma=\gamma', \\ 0 & \text{in other cases.}\end{cases}$$

Proof. According to the definition of first order invariant differential operators between sections of associated vector bundles over Cartan geometries, the vector space $\mathrm{Diff}(\mathbf{E}M,\mathbf{F}M)^1_{(p:\tilde{\mathcal{G}}\to M,\omega)}$ is isomorphic to the space $\mathrm{Hom}_{\tilde{P}}(J^1\mathbf{E},\mathbf{F})$.

From the definition of the minimal globalization, it follows that it gives a natural bijection between Hom's of respective categories: the Harish-Chandra category of $(\mathfrak{p},\widetilde{K\cap G_0})$-modules and the category of admissible $\tilde{P}$-modules, see Kashiwara, Schmid [16]. Thus we have $\mathrm{Hom}_{\tilde{P}}(J^1\mathbf{E},\mathbf{F})\simeq\mathrm{Hom}_{(\mathfrak{p},\widetilde{K\cap G_0})}(J^1E,F)$. Because the identity component $(\widetilde{K\cap G_0})_1$ is connected by definition, we can write $\mathrm{Hom}_{(\mathfrak{p},(\widetilde{K\cap G_0})_1)}(J^1E,F)\simeq\mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E},\mathbb{F})$, see W. Baldoni [1]. It remains to show that each $\mathfrak{p}$-module homomorphism is actually a $(\mathfrak{p},(\widetilde{K\cap G_0})_{-1})$-module homomorphism, where $(\widetilde{K\cap G_0})_{-1}$ denotes the component of the group $\widetilde{K\cap G_0}$ to which -1 belongs. Let us parameterize the elements of the (-1)-component of $(\widetilde{K\cap G_0})\simeq\widetilde{U(l)}\times\mathbb{Z}_2$ by pairs $(k,-1)$, $k\in\widetilde{U(l)}$, and denote the appropriate $\tilde{P}$-representation on E by ρ. We can easily check that for $(v',S\otimes v'')\in J^1\mathbf{E}$, we have $(k,-1).(v',S\otimes v'')=(\rho(k,-1)v',Ad(k,-1)S\otimes\rho(k,-1)v'')=(\gamma\rho(k,1)v',Ad(k,1)S\otimes\gamma\rho(k,1)v'')=\gamma(k,1).(v',S\otimes v'')$. Further for a $\mathfrak{p}$-homomorphism $T\in\mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E},\mathbb{F})$, we can write $T(k,-1).(v',S\otimes v'')=\gamma T(k,1).(v',S\otimes v'')=\gamma(k,1).T(v',S\otimes v'')=\gamma\gamma'(k,-1).T(v',S\otimes v'')$. Thus we have also $\mathrm{Hom}_{(\mathfrak{p},\widetilde{K\cap G_0})}(J^1E,F)\simeq\mathrm{Hom}_{\mathfrak{p}}(J^1\mathbb{E},\mathbb{F})$ if $\gamma=\gamma'$. The Hom at the right hand side was determined in Corollary 1. In the case $\gamma\neq\gamma'$, we have that $T=0$ and the proof is finished. □

5. Examples: contact projective Dirac, twistor and Rarita–Schwinger operators

In this section, we shall introduce three main examples of contact projective analogues of Dirac, twistor and Rarita–Schwinger operators known from Riemannian and partly from symplectic geometry. In each of the next paragraphs, we suppose that a metaplectic contact projective geometry $(p:\tilde{\mathcal{G}}\to M^{2l+1},\omega)$ of rank l is fixed.

Contact projective Dirac operator. For $\lambda=-\frac{1}{2}\varpi_l$, we have $A_\lambda=\{\varpi_1-\frac{1}{2}\varpi_l,\varpi_{l-1}-\frac{3}{2}\varpi_l\}$ according to Theorem 2. Take $\mu=\varpi_{l-1}-\frac{3}{2}\varpi_l\in A_\lambda$. Using $\delta=l\epsilon_1+(l-1)\epsilon_2+\cdots+\epsilon_l$, we obtain that $\tilde{c}^{\mu}_{\lambda\alpha}=\frac{1+2l}{2}$. Thus for conformal weight $c=\frac{1+2l}{2}$ and $\gamma\in\mathbb{Z}_2$ there is an invariant differential operator $\mathfrak{D}^{\frac{1}{2}}:\Gamma(M,\mathbf{L}(\lambda,\frac{1+2l}{2})_\gamma M)\to\Gamma(M^{2l+1},\mathbf{L}(\mu,\frac{3+2l}{2})_\gamma M)$. This operator could be called contact projective Dirac operator because of the analogy with the orthogonal case.

Contact projective twistor operator. Taking the same $\lambda=-\frac{1}{2}\varpi_l$ as in the previous example and $\mu=\varpi_1-\frac{1}{2}\varpi_l$, we obtain $c=\frac{1}{2}$ and the corresponding operator $\mathfrak{T}:\Gamma(M,\mathbf{L}(\lambda,\frac{1}{2})_\gamma M)\to\Gamma(M,\mathbf{L}(\mu,\frac{3}{2})_\gamma M)$ $(\gamma\in\mathbb{Z}_2)$ is called contact projective twistor operator also due to the analogy with the orthogonal case.

Contact projective Rarita–Schwinger operator. Here, take $\lambda=\varpi_1-\frac{1}{2}\varpi_l$. $A_\lambda=\{\varpi_2-\frac{1}{2}\varpi_l,2\varpi_1-\frac{1}{2}\varpi_l,-\frac{1}{2}\varpi_l,\varpi_1+\varpi_{l-1}-\frac{3}{2}\varpi_l\}$. For $\mu=\varpi_1+\varpi_{l-1}-\frac{3}{2}\varpi_l$, we obtain $c=\frac{1+2l}{2}$, and we shall call this operator contact projective Rarita–Schwinger operator, $\mathfrak{D}^{\frac{3}{2}}:\Gamma(M,\mathbf{L}(\lambda,\frac{1+2l}{2})_\gamma M)\to\Gamma(M,\mathbf{L}(\mu,\frac{3+2l}{2})_\gamma M)$, where again $\gamma\in\mathbb{Z}_2$.

Remark. It may be interesting to mention, that computing formally the conformal weights using a Lepowsky generalization of a result of Bernstein–Gelfand–Gelfand on homomorphism of nontrue Verma-modules, one gets exactly the same weights, although Lepowsky is considering only Verma modules induced by finite dimensional representations.

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Structure of the curvature tensor on symplectic spinors

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ABSTRACT

We study symplectic manifolds (M^{2l}, ω) equipped with a symplectic torsion-free affine (also called Fedosov) connection ∇ and admitting a metaplectic structure. Let $\mathcal{S}$ be the so-called symplectic spinor bundle over M and let R^S be the curvature field of the symplectic spinor covariant derivative ∇^S associated to the Fedosov connection ∇. It is known that the space of symplectic spinor valued exterior differential 2-forms, $\Gamma(M, \bigwedge^2 T^*M \otimes \mathcal{S})$, decomposes into three invariant subspaces with respect to the structure group, which is the metaplectic group $Mp(2l, \mathbb{R})$ in this case. For a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, we compute explicitly the projections of $R^S\phi \in \Gamma(M, \bigwedge^2 T^*M \otimes \mathcal{S})$ onto the three mentioned invariant subspaces in terms of the symplectic Ricci and symplectic Weyl curvature tensor fields of the connection ∇. Using this decomposition, we derive a complex of first order differential operators provided the Weyl curvature tensor field of the Fedosov connection is trivial.

1. Introduction

In the paper, we shall study the action of the curvature tensor field on symplectic spinors over a symplectic manifold (M^{2l}, ω) admitting a metaplectic structure and equipped with a symplectic torsion-free affine connection ∇. Such connections are usually called Fedosov connections. It is well known that in the case of $l > 1$, the curvature tensor field of the connection ∇ decomposes into two parts, namely into the symplectic Weyl and the symplectic Ricci curvature tensor field. In the case $l = 1$, only the symplectic Ricci curvature tensor field appears. See [1] for details.

Now, let us say a few words about the metaplectic structure. In the symplectic case, there exists (in a parallel to the Riemannian case) a non-trivial two-fold covering of the symplectic group $Sp(2l, \mathbb{R})$, the so-called metaplectic group. We shall denote it by $Mp(2l, \mathbb{R})$. A metaplectic structure on a symplectic manifold (M^{2l}, ω) is a notion parallel to the notion of a spin structure on a Riemannian manifold. In particular, one of its parts is a principal $Mp(2l, \mathbb{R})$-bundle. For a symplectic manifold admitting a metaplectic structure, one can construct the so-called *symplectic spinor bundle* $\mathcal{S}$, introduced by B. Kostant in 1974. The symplectic spinor bundle $\mathcal{S}$ is the vector bundle associated to the metaplectic structure on M (more precisely to the mentioned principal $Mp(2l, \mathbb{R})$-bundle) via the so-called Segal–Shale–Weil representation of the metaplectic group $Mp(2l, \mathbb{R})$. See [2] for details.

The Segal–Shale–Weil representation is an infinite dimensional unitary representation of the metaplectic group $Mp(2l, \mathbb{R})$ on the space of all complex valued square Lebesgue integrable functions $\mathbf{L}^2(\mathbb{R}^l)$. Because of the infinite dimension, the Segal–Shale–Weil representation is not so easy to handle. It is known, see, e.g., [3], that the infinitesimal structure of the underlying Harish–Chandra module of this representation is equivalent to the space $\mathbb{C}[x^1, \ldots, x^l]$ of polynomials

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in l variables, on which the symplectic Lie algebra $\mathfrak{sp}(2l, \mathbb{R})$ acts via the so-called Chevalley homomorphism,[1] see [4]. Thus, the underlying vector space of the infinitesimal structure of the Segal–Shale–Weil representation can be viewed as the complexified *symmetric* algebra $\left(\bigoplus_{i=0}^{\infty} \odot^i \mathbb{R}^l\right) \otimes_{\mathbb{R}} \mathbb{C}$ of the Lagrangian subspace $\mathbb{R}^l$ of the canonical symplectic vector space $\mathbb{R}^{2l} \simeq \mathbb{R}^l \oplus \mathbb{R}^l$. This shows that the situation is completely parallel to the complex orthogonal case and the spinor representation, which can be realized as the *exterior* algebra of a maximal isotropic subspace. An interested reader is referred to [5,3] and also to [4] for more details. For some technical reasons, we shall be using the so-called minimal globalization of the underlying Harish–Chandra $(\mathfrak{g}, K)$-module of the Segal–Shale–Weil representation, which we will call *metaplectic representation* and denote it by $\mathbf{S}$ (the elements of $\mathbf{S}$ will be called *symplectic spinors*). This representation, as well as the Segal–Shale–Weil one, decomposes into two irreducible subrepresentations. In the case of the module $\mathbf{S}$, we shall denote them by $\mathbf{S}_+$ and $\mathbf{S}_-$.

For any symplectic connection ∇ on a symplectic manifold (M, ω) admitting a metaplectic structure, we can form the associated covariant derivative ∇^S acting on the sections of the symplectic spinor bundle $\mathcal{S}$. The curvature tensor field $R^S : \Gamma(M, \mathcal{S}) \to \Gamma(M, \bigwedge^2 TM^* \otimes \mathcal{S})$ of the associated covariant derivative ∇^S is defined by the classical formula. The tensor field R^S decomposes also into two parts, one of which depends on the symplectic Ricci and the remaining one on the symplectic Weyl tensor field. It is known (cf. [6]) that the space of the symplectic spinor valued exterior 2-forms, $\bigwedge^2 \mathbb{R}^{2l} \otimes \mathbf{S}_\pm$, decomposes into three irreducible summands with respect to the natural action of $Mp(2l, \mathbb{R})$ on this space. We shall briefly describe the decomposition in this paper. Let us denote the mentioned three summands of the decomposition of $\bigwedge^2 \mathbb{R}^{2l} \otimes \mathbf{S}_\pm$ by $\mathbf{E}^{20}_\pm$, $\mathbf{E}^{21}_\pm$ and $\mathbf{E}^{22}_\pm$ and the corresponding vector bundles associated to the chosen metaplectic structure via the mentioned modules by $\mathcal{E}^{20}_\pm$, $\mathcal{E}^{21}_\pm$ and $\mathcal{E}^{22}_\pm$, respectively. We define $\mathcal{E}^{2j} := \mathcal{E}^{2j}_+ \oplus \mathcal{E}^{2j}_-$ for $j = 0, 1, 2$.

In the paper, we shall prove that the part of R^S corresponding to the symplectic Ricci tensor field maps a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$ into $\Gamma(M, \mathcal{E}^{20} \oplus \mathcal{E}^{21})$ and that one corresponding to the symplectic Weyl tensor field maps a symplectic spinor field into $\Gamma(M, \mathcal{E}^{21} \oplus \mathcal{E}^{22})$. Parallel and similar conclusions were done in the Riemannian case, see [7].

For an arbitrary symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, the projections of $R^S\phi$ to the invariant subspaces $\Gamma(M, \mathcal{E}^{2j})$ $(j = 0, 1, 2)$ are computed explicitly. More precisely, we have described a structure of the action of the curvature tensor field R^S on the space of symplectic spinor fields in terms of the invariant parts of the curvature of the underlying affine connection ∇. In what follows, this result will be called the decomposition result. Although this result seems to be rather abstract or technical, knowing the decomposition of $R^S\phi$ makes it possible to derive several conclusions for certain invariant differential operators, which are defined with help of the Fedosov connection.

This is the case of the application that we shall mention. Let us briefly describe its context. In 1994, Habermann introduced a symplectic analogue of the Riemannian Dirac operator known from Riemannian geometry, the so-called symplectic Dirac operator. The symplectic Dirac operator was introduced with help of the so-called symplectic Clifford multiplication, see [8]. It is possible to define the same operator using the de Rham sequence tensored (twisted) by symplectic spinor fields as one usually does in the Riemannian spin geometry to get a definition of the Riemannian Dirac, twistor and Rarita–Schwinger operator and their further higher spin analogues. Not only the symplectic Dirac operator but also symplectic analogues of the Riemannian twistor operators can be defined using the de Rham sequence twisted by symplectic spinor fields. We will call these symplectic versions symplectic twistor operators and denote the first two of them by T_0 and T_1. Under the assumption the symplectic Weyl tensor W of the Fedosov connection is trivial, we prove the existence of a complex consisting of the two mentioned symplectic twistor operators T_0 and T_1. One of the advantages of knowing the decomposition result is a complete avoidance of possibly lengthy computations in coordinates when proving that T_0 and T_1 form a complex (provided $W = 0$). One can say that the coordinate computations were absorbed into the proof of the decomposition result. Though finding the complex seems to be a rather particular result, there is a strong hope of deriving a longer complex under the same assumption.

The reader interested in applications of symplectic spinors in physics is referred to [9], where they are used in the context of 10 dimensional superstring theory. In [10], symplectic spinors are used in the theory of the so-called Dirac–Kähler fields.

In the second section, some basic facts on the metaplectic representation and higher symplectic spinors are recalled. In Section 3, basic properties of symplectic torsion-free, i.e., Fedosov, connections and their curvature tensor fields are mentioned. In Corollary 11 (Section 4), the action of the curvature tensor field R^S of the associated symplectic spinor covariant derivative ∇^S acting on the space of symplectic spinor fields (the decomposition result) is described. In this section, the mentioned complex consisting of the two symplectic twistor operators is presented (Theorem 12).

2. Metaplectic representation, higher symplectic spinors and basic notation

We start with a summary of notions from representation theory that we shall need in this paper. From the point of view of this article, these notions are of rather a technical character. Let G be a reductive Lie group in the sense of Vogan (see [11]), $\mathfrak{g}$ be the Lie algebra of G and K be a maximal compact subgroup of G. Typical examples of reductive groups are finite covers of semisimple Lie subgroups of the general linear group of a finite dimensional vector space. Let $\mathcal{R}(G)$ be the category the object

[1] The Chevalley homomorphism realizes the complex symplectic Lie algebra as a Lie subalgebra of the algebra of polynomial coefficients differential operators acting on $\mathbb{C}[x^1, \ldots, x^l]$.

of which are complete, locally convex, Hausdorff topological spaces with continuous linear G-action, such that the resulting representation is admissible and of finite length; the morphisms are continuous G-equivariant linear maps between the objects. Let $\mathcal{HC}(\mathfrak{g}, K)$ be the category of Harish–Chandra $(\mathfrak{g}, K)$-modules and let us consider the forgetful Harish–Chandra functor $HC : \mathcal{R}(G) \to \mathcal{HC}(\mathfrak{g}, K)$. It is well known that there exists an adjoint functor $mg : \mathcal{HC}(\mathfrak{g}, K) \to \mathcal{R}(G)$ to the Harish–Chandra functor HC. This functor is usually called the minimal globalization functor and its existence is a deep result in representation theory. For details and for the existence of the minimal globalization functor mg, see [12,11].

For a representation $\mathbf{E} \in \mathcal{R}(G)$ of G, we shall denote the corresponding Harish–Chandra $(\mathfrak{g}, K)$-module $HC(\mathbf{E})$ by E. When we will only be considering its $\mathfrak{g}$-module structure, we shall use the symbol $\mathbb{E}$ for it.

Now, suppose that $\mathfrak{g}^{\mathbb{C}}$ is a simple Lie algebra, K is connected and two complex $(\mathfrak{g}, K)$-modules $E, F \in \mathcal{HC}(\mathfrak{g}, K)$ are given such that both $\mathbb{E}$ and $\mathbb{F}$ are irreducible highest weight $\mathfrak{g}^{\mathbb{C}}$-modules. Because mg is an adjoint functor to the functor HC, we have $\mathrm{Hom}_G(mg(E), mg(F)) \simeq \mathrm{Hom}_{(\mathfrak{g},K)}(E, F)$. It is well known that the category of $(\mathfrak{g}, K)$-modules is a full subcategory of the category of $\mathfrak{g}$-modules provided K is connected. Due to that, we have $\mathrm{Hom}_{(\mathfrak{g},K)}(E, F) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathbb{E}, \mathbb{F})$. Because $\mathbb{E}$ and $\mathbb{F}$ are complex irreducible highest weight modules over $\mathfrak{g}^{\mathbb{C}}$, Dixmier's version of the Schur lemma implies $\dim \mathrm{Hom}_{\mathfrak{g}}(\mathbb{E}, \mathbb{F}) = 1$ iff $\mathbb{E} \simeq \mathbb{F}$ (see [13], Theorem 2.6.5 and Theorem 2.6.6). Summing up, we have $\dim \mathrm{Hom}_G(mg(E), mg(F)) = 1$ iff $\mathbb{E} \simeq \mathbb{F}$. For brevity, we will refer to this simple statement as the *globalized Schur lemma*.

Further, if $(p : \mathcal{G} \to M, G)$ is a principal G-bundle, we shall denote the vector bundle associated to this principal bundle via a representation $\sigma : G \to \mathrm{Aut}(\mathbf{W})$ of G on $\mathbf{W}$ by $\mathcal{W}$, i.e., $\mathcal{W} = \mathcal{G} \times_\sigma \mathbf{W}$. Let us also mention that we shall often use the Einstein summation convention for repeated indices (lower and upper) without mentioning it explicitly.

Now, we shall focus our attention to the studied case, i.e., to the symplectic one. To fix a notation, let us recall some notions from the symplectic linear algebra. Let us consider a real symplectic vector space $(\mathbb{V}, \omega_0)$ of dimension $2l$, i.e., $\mathbb{V}$ is a $2l$ dimensional real vector space and ω_0 is a non-degenerate antisymmetric bilinear form on $\mathbb{V}$. Let us choose two Lagrangian subspaces[2] $\mathbb{L}, \mathbb{L}' \subseteq \mathbb{V}$ such that $\mathbb{L} \oplus \mathbb{L}' = \mathbb{V}$. It follows that $\dim(\mathbb{L}) = \dim(\mathbb{L}') = l$. Throughout this article, we shall use a symplectic basis $\{e_i\}_{i=1}^{2l}$ of $\mathbb{V}$ chosen in such a way that $\{e_i\}_{i=1}^{l}$ and $\{e_i\}_{i=l+1}^{2l}$ are respective bases of $\mathbb{L}$ and $\mathbb{L}'$. Because the definition of a symplectic basis is not unique, let us fix one which shall be used in this text. A basis $\{e_i\}_{i=1}^{2l}$ of $\mathbb{V}$ is called a symplectic basis of $(\mathbb{V}, \omega_0)$ if $\omega_{ij} := \omega_0(e_i, e_j)$ satisfies $\omega_{ij} = 1$ if and only if $i \leq l$ and $j = i + l$; $\omega_{ij} = -1$ if and only if $i > l$ and $j = i - l$ and finally, $\omega_{ij} = 0$ in other cases. Let $\{\epsilon^i\}_{i=1}^{2l}$ be the basis of $\mathbb{V}^*$ dual to the basis $\{e_i\}_{i=1}^{2l}$. For $i, j = 1, \ldots, 2l$, we define ω^{ij} by $\sum_{k=1}^{2l} \omega_{ik}\omega^{jk} = \delta_i^j$, for $i, j = 1, \ldots, 2l$. Notice that not only $\omega_{ij} = -\omega_{ji}$, but also $\omega^{ij} = -\omega^{ji}$, $i, j = 1, \ldots, 2l$.

Let us denote the symplectic group of $(\mathbb{V}, \omega_0)$ by G, i.e., $G := Sp(\mathbb{V}, \omega_0) \simeq Sp(2l, \mathbb{R})$. Because the maximal compact subgroup K of G is isomorphic to the unitary group $K \simeq U(l)$ which is of homotopy type $\mathbb{Z}$, there exists (up to an isomorphism) a unique nontrivial two-fold covering $\tilde{G}$ of G. See, e.g., [14] for details. This two-fold covering is called a metaplectic group of $(\mathbb{V}, \omega_0)$ and it is denoted by $Mp(\mathbb{V}, \omega_0)$ in this text. In the considered case, we have $\tilde{G} \simeq Mp(2l, \mathbb{R})$. Let us remark that $Mp(\mathbb{V}, \omega_0)$ is reductive in the sense of Vogan. For later use, let us reserve the symbol λ for the mentioned covering. Thus $\lambda : \tilde{G} \to G$ is a fixed member of the isomorphism class of all nontrivial 2:1 coverings of G. Because $\lambda : \tilde{G} \to G$ is a homomorphism of Lie groups and G is a subgroup of the general linear group $GL(\mathbb{V})$ of $\mathbb{V}$, the mapping λ is also a representation of the metaplectic group $\tilde{G}$ on the vector space $\mathbb{V}$. Let us define $\tilde{K} := \lambda^{-1}(K)$. Then $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$. One can easily see that $\tilde{K} \simeq \widetilde{U(l)} := \{(g, z) \in U(l) \times \mathbb{C}^\times \,|\, \det(g) = z^2\}$ and thus $\tilde{K}$ is connected. The Lie algebra of the metaplectic group $\tilde{G}$ is isomorphic to the Lie algebra $\mathfrak{g}$ of G and we will identify them. One has $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}, \omega_0) \simeq \mathfrak{sp}(2l, \mathbb{R})$.

From now on, we shall restrict ourselves to the case $l \geq 2$ without mentioning it explicitly. The case $l = 1$ should be handled separately (though analogously) because the shape of the root system of $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R})$ is different from that one of the root system of $\mathfrak{sp}(2l, \mathbb{R})$ for $l > 1$. As usual, we shall denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}^{\mathbb{C}}$. Obviously, $\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sp}(2l, \mathbb{C})$. Let us choose a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ and an ordering on the set of roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. If $\mathbb{E}$ is an irreducible highest weight $\mathfrak{g}^{\mathbb{C}}$-module with a highest weight λ, we shall denote it by the symbol $L(\lambda)$. Let us denote the fundamental weight basis of $\mathfrak{g}^{\mathbb{C}}$ with respect to the above choices by $\{\varpi_i\}_{i=1}^{l}$.

2.1. *Metaplectic representation and symplectic spinors*

There exists a distinguished infinite dimensional unitary representation of the metaplectic group $\tilde{G}$ which does not descend to a representation of the symplectic group G. This representation, called *Segal–Shale–Weil*,[3] plays a fundamental role in geometric quantization of Hamiltonian mechanics, see, e.g., [15], and in the theory of modular forms and theta correspondence, see, e.g., [16]. We shall not give a definition of this representation in this text and refer the interested reader to [5] or [14]. We only mention some of its properties which we shall need.

The Segal–Shale–Weil representation, which we shall denote by U here, is a complex infinite dimensional unitary representation of $\tilde{G}$ on the space of complex valued square Lebesgue integrable functions defined on the Lagrangian subspace

[2] Maximal isotropic with respect to ω_0.

[3] The names oscillator or metaplectic representation are also used in the literature. We shall use the name Segal–Shale–Weil in this text, and reserve the name metaplectic for certain representation arising from the Segal–Shale–Weil one.

$\mathbb{L}$, i.e.,

$$U : \tilde{G} \to \mathcal{U}(\mathbf{L^2}(\mathbb{L})),$$

where $\mathcal{U}(\mathbf{W})$ denotes the group of unitary operators on a Hilbert space $\mathbf{W}$. In order to be precise, let us refer to the space $\mathbf{L^2}(\mathbb{L})$ as the Segal–Shale–Weil module. It is known that the Segal–Shale–Weil module belongs to the category $\mathcal{R}(\tilde{G})$. (See [3] for details and the Segal–Shale–Weil representation in general.) It is easy to see that this representation splits into two irreducible modules $\mathbf{L^2}(\mathbb{L}) \simeq \mathbf{L^2}(\mathbb{L})_+ \oplus \mathbf{L^2}(\mathbb{L})_-$. The first module consists of even and the second one of odd complex valued square Lebesgue integrable functions on the Lagrangian subspace $\mathbb{L}$. Let us remark that one of the typical constructions of the Segal–Shale–Weil representation is based on the so-called Schrödinger representation of the Heisenberg group of $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega_0)$ and a use of the Stone-von Neumann theorem.

For technical reasons, we shall need the minimal globalization of the underlying $(\mathfrak{g}, \tilde{K})$-module $HC(\mathbf{L^2}(\mathbb{L}))$ of the introduced Segal–Shale–Weil module. We shall call this minimal globalization *metaplectic representation* and denote it by meta, i.e.,

$$\mathrm{meta} : \tilde{G} \to \mathrm{Aut}(mg(HC(\mathbf{L^2}(\mathbb{L})))),$$

where mg is the minimal globalization functor (see this section and the references therein). For our convenience, let us denote the module $mg(HC(\mathbf{L^2}(\mathbb{L})))$ by $\mathbf{S}$. Similarly we define $\mathbf{S}_+$ and $\mathbf{S}_-$ to be the minimal globalizations of the underlying Harish–Chandra modules of the modules $\mathbf{L^2}(\mathbb{L})_+$ and $\mathbf{L^2}(\mathbb{L})_-$ introduced above. Accordingly to $\mathbf{L^2}(\mathbb{L}) \simeq \mathbf{L^2}(\mathbb{L})_+ \oplus \mathbf{L^2}(\mathbb{L})_-$, we have $\mathbf{S} \simeq \mathbf{S}_+ \oplus \mathbf{S}_-$. We shall call the $Mp(\mathbb{V}, \omega_0)$-module $\mathbf{S}$ the *symplectic spinor module* and its elements *symplectic spinors*. For the name "spinor", see [2] or the Introduction.

A further notion related to the symplectic vector space $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega_0)$ is the so-called symplectic Clifford multiplication of elements of $\mathbf{S}$ by vectors from $\mathbb{V}$. For a symplectic spinor $f \in \mathbf{S}$, we define

$$(e_i.f)(x) := \imath x^i f(x),^{4}$$

$$(e_{i+l}.f)(x) := \frac{\partial f}{\partial x^i}(x), \quad x = \sum_{i=1}^{l} x^i e_i \in \mathbb{L},\ i = 1, \ldots, l.$$

Extending this multiplication $\mathbb{R}$-linearly, we get the mentioned symplectic Clifford multiplication. Let us remark that the multiplication and the differentiation make sense for any $f \in \mathbf{S}$ because of an interpretation of the minimal globalization. (See [11] for details.) Let us notice that in the physical literature, the symplectic Clifford multiplication is usually called the Schrödinger quantization prescription.

The following lemma is an easy consequence of the definition of the symplectic Clifford multiplication.

Lemma 1. *For $v, w \in \mathbb{V}$ and $s \in \mathbf{S}$, we have*

$$v.w.s - w.v.s = -\imath\omega_0(v, w)s.$$

Proof. See [14], pp. 11. □

Sometimes, we shall write $v.w.s$ instead of $v.(w.s)$ for $v, w \in \mathbb{V}$ and a symplectic spinor $s \in \mathbf{S}$ and similarly for a higher number of multiplying elements. Instead of $e_i.e_j.s$, we shall write $e_{ij}.s$ simply and similarly for expressions with higher numbers of multiplying elements, e.g., $e_{ijk}.s$ abbreviates $e_i.e_j.e_k.s$.

2.2. Higher symplectic spinors

In this subsection, we shall present a result on a decomposition of the tensor product of the symplectic spinor module $\mathbf{S}$ with exterior forms of degree one and two into irreducible $\tilde{G}$-modules, $\tilde{G}$ being the metaplectic group $Mp(\mathbb{V}, \omega_0)$. Let $\lambda^* : \tilde{G} \to GL(\mathbb{V}^*)$ be the representation of $\tilde{G}$ dual to the representation $\lambda : \tilde{G} \to G$. Recall that λ is the chosen two-fold covering of the symplectic group. Further let us reserve the symbol ρ for the mentioned tensor product representation of $\tilde{G}$, i.e.,

$$\rho : \tilde{G} \to \mathrm{Aut}\left(\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}\right)$$

and

$$\rho(g)(\alpha \otimes s) := \lambda(g)^{*\wedge r}\alpha \otimes \mathrm{meta}(g)s$$

for $g \in \tilde{G}, \alpha \in \bigwedge^r \mathbb{V}^*, s \in \mathbf{S}, r = 0, \ldots, 2l$ and extended linearly. For definiteness, let us equip the tensor product $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ with the so-called Grothendieck tensor product topology. See [11,17] for details on this topological structure. In a parallel to the Riemannian case, we shall call the elements of $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ *higher symplectic spinors*.

In the next theorem, the modules of the exterior 1-forms and 2-forms with values in the module $\mathbf{S}$ of symplectic spinors are decomposed into irreducible summands.

[4] The symbol $\imath$ denotes the imaginary unit, $\imath = \sqrt{-1}$.

Theorem 2. *For $\frac{1}{2}\dim(\mathbb{V}) =: l > 2$, the following isomorphisms*

$$\mathbb{V}^* \otimes \mathbf{S} \simeq \mathbf{E}^{10}_{\pm} \oplus \mathbf{E}^{11}_{\pm} \quad \textit{and}$$

$$\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}_{\pm} \simeq \mathbf{E}^{20}_{\pm} \oplus \mathbf{E}^{21}_{\pm} \oplus \mathbf{E}^{22}_{\pm}$$

hold. For $j_1 = 0, 1$ and $j_2 = 0, 1, 2$ the modules $\mathbf{E}^{1j_1}_{\pm}$ and $\mathbf{E}^{2j_2}$ are uniquely determined by the conditions that first, they are submodules of the corresponding tensor products and second,

$$\mathbb{E}^{10}_{+} \simeq \mathbb{E}^{20}_{-} \simeq \mathbb{S}_{-} \simeq L\left(\varpi_{l-1} - \frac{3}{2}\varpi_l\right), \qquad \mathbb{E}^{10}_{-} \simeq \mathbb{E}^{20}_{+} \simeq \mathbb{S}_{+} \simeq L\left(-\frac{1}{2}\varpi_l\right),$$

$$\mathbb{E}^{11}_{+} \simeq \mathbb{E}^{21}_{-} \simeq L\left(\varpi_1 - \frac{1}{2}\varpi_l\right), \qquad \mathbb{E}^{11}_{-} \simeq \mathbb{E}^{21}_{+} \simeq L\left(\varpi_1 + \varpi_{l-1} - \frac{3}{2}\varpi_l\right),$$

$$\mathbb{E}^{22}_{+} \simeq L\left(\varpi_2 - \frac{1}{2}\varpi_l\right) \quad \textit{and} \quad \mathbb{E}^{22}_{-} \simeq L\left(\varpi_2 + \varpi_{l-1} - \frac{3}{2}\varpi_l\right).$$

Proof. See [18] or [19]. □

Remark. In this paper, the multiplicity freeness of the previous two decompositions will be used substantially. One can show that the decompositions are multiplicity-free also in the case $l = 2$. (One only has to modify the prescription for the highest weights of the summands in the decompositions. See [19] for this case.) Let us also mention that Theorem 2 is a simple consequence of a theorem of [4].

Let us set $\mathbf{E}^{ij_i} := \mathbf{E}^{ij_i}_{+} \oplus \mathbf{E}^{ij_i}_{-}$, for $i = 1, 2, j_1 = 0, 1$ and $j_2 = 0, 1, 2$. For the mentioned i, j_i, let us consider the projections $p^{ij_i} : \bigwedge^i \mathbb{V}^* \otimes \mathbf{S} \to \mathbf{E}_{ij_i}$. The definition of p^{ij_i} is correct because of the multiplicity freeness of the decomposition of the appropriate tensor products. In this paper, we shall need some explicit formulas for these projections. In order to find them, let us introduce the following mappings.

For $r = 0, \ldots, 2l$ and $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, we set

$$X : \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S}, \qquad X(\alpha \otimes s) := -\sum_{i=1}^{2l} \epsilon^i \wedge \alpha \otimes e_i.s;$$

$$Y : \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r-1} \mathbb{V}^* \otimes \mathbf{S}, \qquad Y(\alpha \otimes s) := \sum_{i,j=1}^{2l} \omega^{ij} \iota_{e_i}\alpha \otimes e_j.s \quad \text{and}$$

$$H : \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}, \qquad H := \{X, Y\} = XY + YX.$$

Because we would like to use these operators in a geometric setting, we shall make use of the following lemma.

Lemma 3. *The homomorphisms X, Y, H are $\tilde{G}$-equivariant with respect to the representation ρ of $\tilde{G}$.*

Proof. This can be verified by a direct computation. See [18] or [19] for a proof. □

In the next lemma, the values of H on the degree homogeneous components of $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ are computed.

Lemma 4. *Let $(\mathbb{V}, \omega_0)$ be a $2l$ dimensional symplectic vector space. Then for $r = 0, \ldots, 2l$, we have*

$$H_{|\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}} = \imath(r - l)\mathrm{Id}_{|\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}}.$$

Proof. This can be verified by a direct computation as well. See [18] or [19] for a proof. □

In the next lemma, the projections $p^{2j}, j = 0, 1, 2$, are computed explicitly with help of the operators X and Y.

Lemma 5. *For $l > 1$, the following identities hold on $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}$.*

$$p^{20} = \frac{1}{l} X^2 Y^2, \tag{1}$$

$$p^{21} = \frac{\imath}{l-1}\left(XY - \frac{\imath}{l} X^2 Y^2\right) \quad \textit{and} \tag{2}$$

$$p^{22} = \mathrm{Id}_{|\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}} - \frac{\imath}{l-1} XY - \frac{1}{l-1} X^2 Y^2. \tag{3}$$

Proof. 1. From the definition of Y, the fact that it is $\tilde{G}$-equivariant (Lemma 3) and Theorem 2, we know that Y^2 maps $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}_+$ into $\mathbf{S}_+$ and $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}_-$ into $\mathbf{S}_-$. Because X^2 is $\tilde{G}$-equivariant (Lemma 3), it maps $\mathbf{S}_\pm$ into a submodule of $\bigwedge^2 \mathbb{V}^* \otimes (\mathbf{S}_+ \oplus \mathbf{S}_-)$ which is a (possibly empty) direct sum of submodules isomorphic to $\mathbf{S}_\pm$. Regarding the multiplicity-free decomposition structure of $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}_\pm$ (Theorem 2), we see that $p' := X^2Y^2$ maps $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}_\pm$ into $\mathbf{E}^{20}_\pm$. Computing the value of p' on the element $\psi := \omega_{ij}\epsilon^i \wedge \epsilon^j \otimes s$ for an $s \in \mathbf{S}$, we find that $p'\psi = l\psi$. Using the globalized Schur lemma (see the beginning of Section 2), we obtain that necessarily $p^{20} = \frac{1}{l}X^2Y^2$.

2. As in the first item, it is easy to see that $p'' := XY(\mathrm{Id}_{|\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}} - \frac{1}{l}X^2Y^2)$ maps $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}$ into $\mathbf{E}^{21}$. Let us consider a symmetric 2-vector $\sigma \in \odot^2 \mathbb{V}$ and denote its (i,j)-th component with respect to the basis $\{e_i\}_{i=1}^{2l}$ by σ^{ij}. Computing the value $p''\psi$ for $\psi := \sigma^{ij}\epsilon^k \wedge \epsilon^l \omega_{jl} \otimes e_{ik}.s - \frac{1}{2l}\sigma^{ij}\omega_{km}\epsilon^k \wedge \epsilon^m \otimes e_{ij}.s$, $s \in \mathbf{S}$, we get $p''\psi = \imath(1-l)\psi$. Using the globalized Schur lemma again, we have $p^{21} = \frac{\imath}{l-1}p''$. Using the defining identity $H = XY + YX$ and Lemma 4, we get the formula for p^{21} written in the statement of the lemma.

3. The third equation follows from the fact $p^{20} + p^{21} + p^{22} = \mathrm{Id}_{|\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}}$ and the preceding two items. □

3. Symplectic curvature tensor field

After we have finished the 'algebraic' part of this paper, we shall recall some results of Vaisman [1] and of Gelfand, Retakh and Shubin [20]. Let (M, ω) be a symplectic manifold and ∇ be a symplectic torsion-free affine connection. By symplectic and torsion-free, we mean $\nabla\omega = 0$ and $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$ for all $X, Y \in \mathfrak{X}(M)$, respectively. Such connections are sometimes called Fedosov connections and were used, e.g., in the so-called Fedosov quantization. See [21] for this use. Let us remark that the Fedosov connection is not unique, in contrast to the case of Riemannian manifolds and Riemannian connections. The triple (M, ω, ∇) will be called a *Fedosov manifold*.

To fix our notation, let us recall the classical definition of the curvature tensor R^∇ of the connection ∇. Let

$$R^\nabla(X, Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for $X, Y, Z \in \mathfrak{X}(M)$. Let us choose a local symplectic frame $(U, \{e_i\}_{i=1}^{2l})$, where U is an open subset of M. Whenever a local symplectic frame will be chosen, we denote its dual coframe by $(U, \{\epsilon^i\}_{i=1}^{2l})$. We have $\epsilon^j(e_i) = \delta_i^j$ for $i, j = 1, \ldots, 2l$. We shall often write expressions for sections of vector bundles which are valid only locally, although the sections are global. (For instance in the case when the expressions will contain local frames.) We shall not mention this restriction of validness explicitly further in the text.

We shall use the following convention. For $i, j, k, l = 1, \ldots, 2l$, we set

$$R_{ijkl} := \omega(R^\nabla(e_k, e_l)e_j, e_i). \tag{4}$$

Let us remark that the convention is different from that one used in [14]. We shall often write expressions in which indices i, j, k or l etc. occur. We will implicitly mean i, j, k or l are running from 1 to the dimension of the manifold M without mentioning it explicitly.

Obviously, one has

$$R_{ijkl} = -R_{ijlk} \quad \text{and} \tag{5}$$

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad \text{(1st Bianchi identity)}. \tag{6}$$

One can also prove the identity

$$R_{ijkl} = R_{jikl}. \tag{7}$$

See [20] for the proof.

For a symplectic manifold with a Fedosov connection, one has also the following simple consequence of the first Bianchi identity:

$$R_{ijkl} + R_{jkli} + R_{klij} + R_{lijk} = 0 \quad \text{(extended 1st Bianchi identity)}. \tag{8}$$

From the symplectic curvature tensor field R^∇, we can build the symplectic Ricci curvature tensor field σ^∇ defined by the classical formula

$$\sigma^\nabla(X, Y) := \mathrm{Tr}(V \mapsto R^\nabla(V, X)Y)$$

for each $X, Y \in \mathfrak{X}(M)$ (the variable V denotes a vector field on M). For the chosen frame and $i, j = 1, \ldots, 2l$, we define

$$\sigma_{ij} := \sigma^\nabla(e_i, e_j).$$

Further, let us define

$$\widetilde{\sigma}^{\nabla}_{ijkl} := \frac{1}{2(l+1)}(\omega_{il}\sigma_{jk} - \omega_{ik}\sigma_{jl} + \omega_{jl}\sigma_{ik} - \omega_{jk}\sigma_{il} + 2\sigma_{ij}\omega_{kl}),$$
$$\widetilde{\sigma}^{\nabla}(X, Y, Z, V) := \widetilde{\sigma}_{ijkl}X^iY^jZ^kV^l \quad \text{and}$$
$$W^{\nabla} := R^{\nabla} - \widetilde{\sigma}^{\nabla}$$

for local vector fields $X = X^ie_i$, $Y = Y^je_j$, $Z = Z^ke_k$ and $V = V^le_l$. We will call the tensor field W^{∇} the symplectic Weyl curvature tensor field. These tensor fields were introduced in [1] already. We shall sometimes drop the index ∇ in the previous expressions. Thus we shall often write W, σ and $\widetilde{\sigma}$ instead of W^{∇}, σ^{∇} and $\widetilde{\sigma}^{\nabla}$, respectively.

Remark. As in the Riemannian geometry, we would like to raise and lower indices. Because the symplectic form ω is antisymmetric, we should be more careful in this case. For coordinates $K_{ab\ldots c\ldots d}{}^{rs\ldots t\ldots u}$ of a tensor field on the considered symplectic manifold (M, ω), we denote the expression $\omega^{ic}K_{ab\ldots c\ldots d}{}^{rs\ldots t}$ by $K_{ab\ldots}{}^{i}{}_{\ldots d}{}^{rs\ldots t}$ and $K_{ab\ldots c}{}^{rs\ldots t\ldots u}\omega_{ti}$ by $K_{ab\ldots c}{}^{rs\ldots}{}_{i}{}^{\ldots u}$ (similarly for other types of tensor fields).

In the next lemma, a symmetry of σ and an equivalent definition of σ are stated.

Lemma 6. *The symplectic Ricci curvature tensor field σ is symmetric and*

$$R^{ijkl}\omega_{kl} = 2\sigma^{ij}.$$

Proof. The proof follows from the definition of the symplectic Ricci curvature tensor field and Eq. (7). See [1] for a proof. □

Remark. In [1], one can find a proof of a statement saying that the space of tensors $R \in \mathbb{V}^{\otimes 4}$ ($\dim \mathbb{V} = 2l$) satisfying the relations (5), (6) and (7) is an $Sp(\mathbb{V}, \omega_0)$-irreducible module if $l = 1$ and decomposes into a direct sum of two irreducible $Sp(\mathbb{V}, \omega_0)$-submodules if $l > 1$.

In the next lemma, two properties of the symplectic Weyl tensor field are described.

Lemma 7. *The symplectic Weyl curvature tensor field is totally trace-free, i.e.,*

$$W^{ijkl}\omega_{ij} = W^{ijkl}\omega_{ik} = W^{ijkl}\omega_{il} = W^{ijkl}\omega_{jk} = W^{ijkl}\omega_{jl} = W^{ijkl}\omega_{kl} = 0$$

and the following equation

$$W_{ijkl} + W_{lijk} + W_{klij} + W_{jkli} = 0 \quad (\textit{extended 1st Bianchi identity for} W) \tag{9}$$

holds.

Proof. The proof is straightforward and can be done just using the definitions of the symplectic Weyl curvature tensor field W, the tensor field $\tilde{\sigma}$ and Lemma 6. □

4. Metaplectic structure and the curvature tensor acting on symplectic spinor fields

Let us start describing the geometric structure with the help of which the action of the symplectic curvature tensor field on symplectic spinors, and the symplectic twistor operators are defined. This structure, called metaplectic, is a precise symplectic analogue of the notion of a spin structure in the Riemannian geometry.

For a symplectic manifold (M^{2l}, ω) of dimension $2l$, let us denote the bundle of symplectic repères in TM by $\mathcal{P}$ and the foot-point projection of $\mathcal{P}$ onto M by p. Thus $(p : \mathcal{P} \to M, G)$, where $G \simeq Sp(2l, \mathbb{R})$, is a principal G-bundle over M. As in Section 2, let $\lambda : \tilde{G} \to G$ be a member of the isomorphism class of the non-trivial two-fold coverings of the symplectic group G. In particular, $\tilde{G} \simeq Mp(2l, \mathbb{R})$. Further, let us consider a principal $\tilde{G}$-bundle $(q : \mathcal{Q} \to M, \tilde{G})$ over the symplectic manifold (M, ω). We call a pair $(\mathcal{Q}, \Lambda)$ a metaplectic structure if $\Lambda : \mathcal{Q} \to \mathcal{P}$ is a surjective bundle homomorphism over the identity on M and if the following diagram,

$$\begin{tikzcd}
\mathcal{Q} \times \tilde{G} \arrow[r] \arrow[dd, "\Lambda\times\lambda"] & \mathcal{Q} \arrow[dd, "\Lambda"] \arrow[dr, "q"] & \\
 & & M \\
\mathcal{P} \times G \arrow[r] & \mathcal{P} \arrow[ur, "p"] &
\end{tikzcd}$$

with the horizontal arrows being respective actions of the displayed groups, commutes. See [14,2] for details on metaplectic structures. Let us only remark that typical examples of symplectic manifolds admitting a metaplectic structure are cotangent bundles of orientable manifolds (phase spaces), Calabi–Yau manifolds and complex projective spaces $\mathbb{CP}^{2k+1}$, $k \in \mathbb{N}_0$.

Let us denote the vector bundle associated to the introduced principal $\tilde{G}$-bundle $(q : \mathcal{Q} \to M, \tilde{G})$ via the representation ρ acting on $\mathbf{S}$ by $\mathcal{S}$ and call this associated vector bundle a *symplectic spinor bundle*. Thus, we have $\mathcal{S} = \mathcal{Q} \times_\rho \mathbf{S}$. (Recall that the representation ρ was introduced in Section 2.) The sections $\phi \in \Gamma(M, \mathcal{S})$, will be called *symplectic spinor fields*. Further for $i = 1, 2$ and $j_1 = 0, 1$ and $j_2 = 0, 1, 2$, we define the associated vector bundles $\mathcal{E}^{ij_i}$ by the prescription: $\mathcal{E}^{ij_i} := \mathcal{Q} \times_\rho \mathbf{E}^{ij_i}$.

Because the projections $p^{10}, p^{11}, p^{20}, p^{21}$ and p^{22} and the operators X, Y and H are $\tilde{G}$-equivariant (Lemma 3), they lift to operators acting on sections of the corresponding associated vector bundles. We shall use the same symbols as for the previously defined operators as for their "lifts" to the associated vector bundle structure.

4.1. Curvature tensor on symplectic spinor fields

Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure $(\mathcal{Q}, \Lambda)$. The (symplectic) connection ∇ determines the associated principal bundle connection Z on the G-bundle $(p : \mathcal{P} \to M, G)$. This connection lifts to a principal bundle connection on the principal bundle $(q : \mathcal{Q} \to M, \tilde{G})$ and defines the associated covariant derivative on the symplectic bundle $\mathcal{S}$, which we shall denote by ∇^S and call it a *symplectic spinor covariant derivative*. The curvature field R^S on the symplectic spinor bundle is given by the classical formula

$$R^S := d^{\nabla^S} \nabla^S,$$

where d^{∇^S} is the associated exterior covariant derivative.

In the next lemma, the action of R^S on the space of symplectic spinors is described using the symplectic curvature tensor field R only.

Lemma 8. *Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Then for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, we have*

$$R^S\phi = \frac{\imath}{2} R^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_i.e_j.\phi.$$

Proof. See [14] pp. 42. □

Let us define the operators σ^S and W^S by the formulas

$$\sigma^S\phi := \frac{\imath}{2}\sigma^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_i.e_j.\phi \quad \text{and}$$
$$W^S\phi := \frac{\imath}{2}W^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_i.e_j.\phi,$$

where $\phi \in \Gamma(M, \mathcal{S})$ is a symplectic spinor field.

Theorem 9. *Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Then for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, we have*

$$\sigma^S\phi \in \Gamma(M, \mathcal{E}^{20} \oplus \mathcal{E}^{21}).$$

Proof. Using the definition of $\tilde{\sigma}$ and Lemma 1 repeatedly we have for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$,

$$\begin{aligned}
\frac{4(l+1)}{\imath}\sigma^S\phi &= 2(l+1)\tilde{\sigma}^{ij}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi \\
&= (\omega^i{}_l\sigma^j{}_k - \omega^i{}_k\sigma^j{}_l + \omega^j{}_l\sigma^i{}_k - \omega^j{}_k\sigma^i{}_l + 2\sigma^{ij}\omega_{kl})\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi \\
&= (-\sigma^j{}_k\epsilon^k \wedge \epsilon^i + \sigma^j{}_k\epsilon^i \wedge \epsilon^k - \sigma^i{}_k\epsilon^k \wedge \epsilon^j + \sigma^i{}_k\epsilon^j \wedge \epsilon^k + 2\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l) \otimes e_{ij}.\phi \\
&= 2\sigma^j{}_k\epsilon^i \wedge \epsilon^k \otimes e_{ij}.\phi - 2\sigma^i{}_k\epsilon^k \wedge \epsilon^j \otimes e_{ij}.\phi + 2\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi \\
&= 2\sigma^j{}_k\epsilon^i \wedge \epsilon^k \otimes (e_{ij}. + e_{ji}.)\phi + 2\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi \\
&= 2\sigma^{jl}\omega_{lk}\epsilon^i \wedge \epsilon^k \otimes (e_{ij}. + e_{ji}.)\phi + 2\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi \\
&= 4\sigma^{jl}\omega_{lk}\epsilon^i \wedge \epsilon^k \otimes e_{ij}.\phi + 2\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi.
\end{aligned}$$

It is straightforward but tedious to verify the next identities:

$$X^2Y^2(2\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi) = 2l\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi,$$
$$X^2Y^2(4\sigma^{jl}\omega_{lk}\epsilon^i \wedge \epsilon^k \otimes e_{ij}.\phi) = 2\sigma^{ik}\omega_{jm}\epsilon^j \wedge \epsilon^m \otimes e_{ik}.\phi,$$
$$XY(2\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi) = 2\imath\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi \quad \text{and}$$
$$XY(4\sigma^{jl}\omega_{lk}\epsilon^i \wedge \epsilon^k \otimes e_{ij}.\phi) = 4\imath(1-l)\sigma^{jl}\omega_{lk}\epsilon^m \wedge \epsilon^k \otimes e_{mj}.\phi + 2\imath\sigma^{jl}\omega_{mi}\epsilon^m \wedge \epsilon^i \otimes e_{lj}.\phi.$$

Using the formulas (1) and (2), we get:

$$p^{20}\sigma^S\phi = \frac{\imath}{2l}\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi \quad \text{and} \tag{10}$$

$$p^{21}\sigma^S\phi = \frac{\imath}{l+1}\sigma^{ij}\epsilon^k \wedge \epsilon^l \otimes \left(\omega_{il}e_{kj}. - \frac{1}{2l}\omega_{kl}e_{ij}.\right)\phi. \tag{11}$$

Adding these two formulas and comparing them with the result of the computation of $\frac{4(l+1)}{\imath}\sigma^S\phi$, we get $(p^{20}+p^{21})\sigma^S\phi = \sigma^S\phi$. Now, the statement follows. □

Theorem 10. *Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Then for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, we have*

$$W^S\phi \in \Gamma(M, \mathcal{E}^{21} \oplus \mathcal{E}^{22}).$$

Proof. Let us compute $Y^2W^S\phi$ for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$.

$$\begin{aligned}
\frac{2}{\imath}Y^2W^S\phi &= Y(\omega^{nm}W^{ij}{}_{kl}\iota_{e_n}(\epsilon^k \wedge \epsilon^l) \otimes e_{mij}.\phi)\\
&= Y(\omega^{nm}W^{ij}{}_{kl}(\delta^k_n\epsilon^l - \delta^l_n\epsilon^k) \otimes e_{mij}.\phi)\\
&= Y(\omega^{nm}(W^{ij}{}_{nl}\epsilon^l - W^{ij}{}_{kn}\epsilon^k) \otimes e_{mij}.\phi)\\
&= 2\omega^{nm}Y(W^{ij}{}_{nl}\epsilon^l \otimes e_{mij}.\phi)\\
&= 2\omega^{pk}\omega^{nm}W^{ij}{}_{nl}\iota_{e_p}\epsilon_l \otimes e_{kmij}.\phi\\
&= 2\omega^{pk}\omega^{nm}W^{ij}{}_{np}e_{kmij}.\phi = 2W^{ijkl}e_{lkij}.\phi.
\end{aligned}$$

Now, let us use the extended first Bianchi identity for the symplectic Weyl curvature tensor field, Eq. (9), i.e.,

$$W^{ijkl} + W^{jkli} + W^{klij} + W^{lijk} = 0.$$

Multiplying this identity by the operator $e_{lkij}.$, using the relation $e_{ij}. - e_{ji}. = -\imath\omega_{ij}$ (Lemma 1) and the fact that the symplectic Weyl tensor field is totally trace free (Lemma 7), we get the following chain of equations.

$$W^{ijkl}e_{lkij}. + W^{jkli}e_{lkij}. + W^{klij}e_{lkij}. + W^{lijk}e_{lkij}. = 0,$$
$$W^{ijkl}e_{lkij}. + W^{jkli}e_l(e_{ik}. - \imath\omega_{ki})e_j. + W^{klij}(e_{likj}. - \imath\omega_{ki}e_{lj}.) + W^{lijk}(e_{klij}. - \imath\omega_{lk}e_{ij}.) = 0,$$
$$W^{ijkl}e_{lkij}. + W^{jkli}(e_{ilkj}. - \imath\omega_{li}e_{kj}.) + W^{klij}(e_{ilkj}. - \imath\omega_{li}e_{kj}.) + W^{likj}(e_{klji}. - \imath\omega_{ij}e_{kl}.) = 0,$$
$$W^{ijkl}e_{lkij}. + W^{jkli}(e_{iljk}. - \imath\omega_{kj}e_{il}.) + W^{klij}(e_{iklj}. - \imath\omega_{lk}e_{ij}.)W^{lijk}(e_{kjli}. - \imath\omega_{lj}e_{ki}.) = 0 \quad \text{and}$$
$$3W^{ijkl}e_{lkij}. + W^{klij}(e_{ikjl}. - \imath\omega_{lj}e_{ik}.) = 0.$$

Continuing in a similar way, we get $4W^{ijkl}e_{lkij}. = 0$. Summing up, we have $Y^2W^S = 0$. Using the relation (1) for p^{20}, we have $p^{20}W^S\phi = 0$. Hence the statement follows. □

Let us consider a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$. By a straightforward way, we get $XYW^S\phi = 2W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}.\phi$. Using this result, Theorem 10, the definition of W^S and the relations (2) and (3) for p^{21} and p^{22}, respectively, we get

$$p^{21}W^S\phi = \frac{1}{l-1}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}.\phi \quad \text{and} \tag{12}$$

$$p^{22}W^S\phi = \frac{\imath}{2}W^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi - \frac{1}{l-1}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}.\phi. \tag{13}$$

Summing up the preceding two theorems, we can formulate the decomposition result in the following

Corollary 11. *In the situation described in the formulation of Theorem 10, we have for a symplectic spinor field* $\phi \in \Gamma(M, \mathcal{S})$

$$p^{20}R^S\phi = \frac{\imath}{2l}\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi,$$

$$p^{21}R^S\phi = \frac{\imath}{l+1}\sigma^{ij}\epsilon^k \wedge \epsilon^l \otimes \left(\omega_{il}e_{kj}. - \frac{1}{2l}\omega_{kl}e_{ij}.\right)\phi + \frac{1}{l-1}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}.\phi \quad \text{and}$$

$$p^{22}R^S\phi = \frac{\imath}{2}W^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi - \frac{1}{l-1}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}.\phi.$$

Proof. The equations in the formulation of the corollary follow from Eqs. (10)–(13) and the definitions of σ^S and W^S. $\square$

Now, let us turn our attention to the mentioned application of the decomposition result (Corollary 11). Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure $(\mathcal{Q}, \Lambda)$. Then we have the associated bundles $\mathcal{E}^{ij_i} \to M$ ($i = 1, 2$, $j_1 = 0, 1$ and $j_2 = 0, 1, 2$) and the symplectic spinor covariant derivative ∇^S as well as the associated exterior covariant derivative d^{∇^S} at our disposal. Let us introduce the following first order $Mp(2l, \mathbb{R})$-invariant differential operators:

$$T_0 : \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{E}^{11}), \qquad T_0 := p^{11}\nabla^S \quad \text{and}$$
$$T_1 : \Gamma(M, \mathcal{E}^{11}) \to \Gamma(M, \mathcal{E}^{22}), \qquad T_1 := p^{22}d^{\nabla^S}_{|\Gamma(M,\mathcal{E}^{11})}.$$

We shall call these operators *symplectic twistor operators*. These definitions are symplectic counterparts of the definitions of twistor operators in Riemannian spin-geometry.

Using Corollary 11, we get

Theorem 12. *Let* (M, ω, ∇) *be a Fedosov manifold admitting a metaplectic structure. Suppose the symplectic Weyl tensor field* $W = 0$. *Then*

$$0 \longrightarrow \Gamma(M, \mathcal{S}) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^{11}) \xrightarrow{T_1} \Gamma(M, \mathcal{E}^{22})$$

is a complex of first order differential operators.

Proof. Let us suppose $W = 0$. Then $p^{22}R^S = 0$ (due to Corollary 11). Using the definition of R^S, we have $0 = p^{22}R^S = p^{22}(d^{\nabla^S}\nabla^S) = p^{22}d^{\nabla^S}(p^{11} + p^{10})\nabla^S = p^{22}d^{\nabla^S}p^{11}\nabla^S + p^{22}d^{\nabla^S}p^{10}\nabla^S$. According to Krýsl [6], $p^{22}d^{\nabla^S}p^{10} = 0$. Thus we have $0 = p^{22}d^{\nabla^S}p^{11}\nabla^S = T_1T_0$, which gives the statement. $\square$

Remark. In [6], the $Mp(2l, \mathbb{R})$-module $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ was decomposed into irreducible summands. Let us denote these irreducible summands by $\mathbf{E}^{ij}$ (the specification of the indices i, j can be found in the mentioned article or in [19]). Similarly as above, we can introduce the projections $p^{ij} : \bigwedge^i \mathbb{V}^* \otimes \mathbf{S} \to \mathbf{E}^{ij}$. In the mentioned article, we proved that $p^{i+1,j}d^{\nabla^S}_{|\Gamma(M,\mathcal{E}^{ik})} = 0$ for all appropriate specified i, k and $j > k + 1$ or $j < k - 1$. In the proof of the preceding theorem, we used this information in the case of $i = 1$, $k = 0$ and $j = 2$.

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Complex of twistor operators in symplectic spin geometry

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Abstract For a symplectic manifold admitting a metaplectic structure (a symplectic analogue of the Riemannian spin structure), we construct a sequence consisting of differential operators using a symplectic torsion-free affine connection. All but one of these operators are of first order. The first order ones are symplectic analogues of the twistor operators known from Riemannian spin geometry. We prove that under the condition the symplectic Weyl curvature tensor field of the symplectic connection vanishes, the mentioned sequence forms a complex. This gives rise to a new complex for the so called Ricci type symplectic manifolds, which admit a metaplectic structure.

Keywords Fedosov manifolds · Metaplectic structures · Symplectic spinors · Kostant spinors · Segal-Shale-Weil representation · Complexes of differential operators

Mathematics Subject Classification (2000) 53C07 · 53D05 · 58J10

1 Introduction

In this paper, we shall introduce a sequence of differential operators acting on symplectic spinor valued exterior differential forms defined over a symplectic manifold (M, ω) admitting the so called metaplectic structure. To define these operators, we

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make use of a symplectic torsion-free affine connection ∇ on (M, ω). Under certain condition on the curvature of the connection ∇, described below, we prove that the mentioned sequence forms a complex.

Let us say a few words about the metaplectic structure. The symplectic group $Sp(2l, \mathbb{R})$ admits a non-trivial two-fold covering, the so called metaplectic group, which we shall denote by $Mp(2l, \mathbb{R})$. Let $\mathfrak{g}$ be the Lie algebra of $Mp(2l, \mathbb{R})$. A metaplectic structure on a symplectic manifold (M^{2l}, ω) is a notion parallel to a spin structure on a Riemannian manifold. In particular, one of its part is a principal $Mp(2l, \mathbb{R})$-bundle $(q : \mathcal{Q} \to M, Mp(2l, \mathbb{R}))$.

For a symplectic manifold admitting a metaplectic structure, one can construct the so called symplectic spinor bundle $\mathcal{S} \to M$, introduced by Bertram Kostant in 1974. The symplectic spinor bundle $\mathcal{S}$ is the vector bundle associated to the principal metaplectic bundle $(q : \mathcal{Q} \to M, Mp(2l, \mathbb{R}))$ on M via the so called Segal-Shale-Weil representation of the metaplectic group $Mp(2l, \mathbb{R})$. See Kostant [12] for details.

The Segal-Shale-Weil representation is an infinite dimensional unitary representation of the metaplectic group $Mp(2l, \mathbb{R})$ on the space of all complex valued square Lebesgue integrable functions $\mathbf{L^2}(\mathbb{R}^l)$. Because of the infinite dimension, the Segal-Shale-Weil representation is not so easy to handle. It is known, see, e.g., Kashiwara and Vergne [11], that the $\mathfrak{g}^{\mathbb{C}}$-module structure of the underlying Harish-Chandra module of this representation is equivalent to the space $\mathbb{C}[x^1, \ldots, x^l]$ of polynomials in l variables, on which the Lie algebra $\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sp}(2l, \mathbb{C})$ acts via the so called Chevalley homomorphism.[1] Thus, the infinitesimal structure of the Segal-Shale-Weil representation can be viewed as the complexified *symmetric* algebra $(\bigoplus_{i=0}^{\infty} \odot^i \mathbb{R}^l) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}[x^1, \ldots, x^l]$ of the Lagrangian subspace $(\mathbb{R}^l, 0)$ of the canonical symplectic vector space $\mathbb{R}^{2l} \simeq (\mathbb{R}^l, 0) \oplus (0, \mathbb{R}^l)$. This shows that the situation is "supersymmetric" to the complex orthogonal case, where the spinor representation can be realized as the *exterior* algebra of a maximal isotropic subspace. An interested reader is referred to Weil [22], Kashiwara and Vergne [11] and also to Britten et al. [1] for details. For some technical reasons, we shall be using the so called minimal globalization of the underlying Harish-Chandra module of the Segal-Shale-Weil representation, which we will call *metaplectic representation* and denote it by $\mathbf{S}$. The elements of $\mathbf{S}$ are usually called symplectic spinors.

Now, let us consider a symplectic manifold (M, ω) together with a symplectic torsion-free affine connection ∇ on it. Such connections are usually called Fedosov connections. Because the Fedosov connection is not unique for a choice of (M, ω) (in the contrary to Riemannian geometry), it seems natural to add the connection to the studied symplectic structure and investigate the triples (M, ω, ∇) consisting of a symplectic manifold (M, ω) and a Fedosov connection ∇. Such triples are usually called Fedosov manifolds and they were used in the deformation quantization. See, e.g., Fedosov [4]. Let us recall that in Vaisman [20], the space of the so called symplectic curvature tensors was decomposed wr. to $Sp(2l, \mathbb{R})$. For $l = 1$, the module of the symplectic curvature tensors is irreducible, while for $l \geq 2$, it decomposes into

[1] The Chevalley homomorphism is a Lie algebra monomorhism of the complex symplectic Lie algebra $\mathfrak{sp}(2l, \mathbb{C})$ into the Lie algebra of the associative algebra of polynomial coefficients differential operators acting on $\mathbb{C}[x^1, \ldots, x^l]$. See, e.g., Britten et al. [1].

two irreducible submodules. These modules are usually called symplectic Ricci and symplectic Weyl modules, respectively. This decomposition translates to differential geometry level giving rise to the symplectic Ricci and symplectic Weyl curvature tensor fields, which add up to the curvature tensor field of ∇. See Vaisman [20] and also Gelfand et al. [6] for a comprehensive treatment on Fedosov manifolds.

Further, let us suppose that a Fedosov manifold (M, ω, ∇) admits a metaplectic structure and denote the corresponding principle bundle by $(q : \mathcal{Q} \to M^{2l}, Mp(2l, \mathbb{R}))$. Let $\mathcal{S} \to M$ be the symplectic spinor bundle associated to $q : \mathcal{Q} \to M$ and let us consider the space $\Omega^\bullet(M, \mathcal{S})$ of exterior differential forms with values in $\mathcal{S}$, i.e., $\Omega^\bullet(M, \mathcal{S}) := \Gamma(M, \mathcal{Q} \times_\rho (\bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S}))$, where ρ is the obvious tensor product representation of $Mp(2l, \mathbb{R})$ on $\bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S}$. In Krýsl [15], the $Mp(2l, \mathbb{R})$-module $\bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S}$ was decomposed into irreducible submodules. The elements of $\bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S}$ are specific examples of the so called higher symplectic spinors. For $i = 0, \ldots, 2l$, let us denote the so called Cartan component (certain explicitly given submodule) of the tensor product $\bigwedge^i(\mathbb{R}^{2l})^* \otimes \mathbf{S}$ by $\mathbf{E}^{im_i}$. (For $i = 0, \ldots, 2l$, the numbers m_i will be specified in the text.) For $i = 0, \ldots, 2l - 1$, we introduce an operator T_i acting between the sections of the vector bundle $\mathcal{E}^{im_i}$ associated to $\mathbf{E}^{im_i}$ and the sections of the vector bundle $\mathcal{E}^{i+1,m_{i+1}}$ associated to $\mathbf{E}^{i+1,m_{i+1}}$. In a parallel to the Riemannian case, we shall call these operators symplectic twistor operators. See, e.g., Friedrich [5] for a study of the Riemannian twistor operators and Sommen and Souček [18] for a study of the de Rham complex tensored by (orthogonal) spinor fields and a description of the Riemannian twistor operators appearing there. The symplectic twistor operators $T_i, i = 0, \ldots, 2l - 1$, are first order differential operators and they are defined using the symplectic torsion-free affine connection ∇ as follows. First, the connection ∇ induces a covariant derivative ∇^S on the bundle $\mathcal{S} \to M$ in the usual way. Second, the covariant derivative ∇^S determines the associated exterior covariant derivative, which we denote by d^{∇^S}. For $i = 0, \ldots, 2l - 1$, we define the symplectic twistor operator T_i as the restriction of d^{∇^S} to $\Gamma(M, \mathcal{E}^{im_i})$ composed with the projection to $\Gamma(M, \mathcal{E}^{i+1,m_{i+1}})$.

Because we would like to derive a condition under which $T_{i+1}T_i = 0$, $i = 0, \ldots, 2l - 1$, we should focus our attention to the curvature tensor $R^{\Omega^\bullet(M,\mathcal{S})} := d^{\nabla^S} d^{\nabla^S}$ of d^{∇^S} acting on the space $\Omega^\bullet(M, \mathcal{S})$. The curvature $R^{\Omega^\bullet(M,\mathcal{S})}$ depends only on the curvature of the symplectic connection ∇, which consists of the symplectic Ricci and symplectic Weyl curvature tensor fields as we have already mentioned. In the paper, we will analyze the action of the symplectic Ricci curvature tensor field on symplectic spinor valued exterior differential forms and especially on $\Gamma(M, \mathcal{E}^{im_i})$, $i = 0, \ldots, 2l-2$. We shall prove that the symplectic Ricci curvature tensor field when restricted to $\Gamma(M, \mathcal{E}^{im_i})$ maps this submodule into at most three $Mp(2l, \mathbb{R})$-submodules sitting in symplectic spinor valued forms of degree $i + 2$, $i = 0, \ldots, 2l - 2$. These submodules will be explicitly described. The crucial method used to derive this result, was a computation based description of the (anti-)commutators of operators from which one may construct the Ricci curvature tensor field.

This will help us to prove that $T_{i+1}T_i = 0$ $(i = 0, \ldots, l - 2)$ and $T_{i+1}T_i = 0$ $(i = l, \ldots, 2l - 2)$ assuming the symplectic Weyl curvature tensor field vanishes. In this way, we will obtain two complexes. Unfortunately, it is questionable under which condition $T_l T_{l-1} = 0$. This will influence the way, how we construct one complex of the two complexes introduced above. Let us notice that similar complex was investigated in Severa [17] in the case of spheres equipped with the conformal structure of their round metrics.

The reader interested in applications of the symplectic spinor fields in theoretical physics is referred to Green and Hull [7], where the symplectic spinors are used in the context of 10 dimensional super string theory. In Reuter [16], symplectic spinors are used in the theory of the so called Dirac-Kähler fields.

Let us describe the structure of the paper briefly. In the second section, some basic facts on the metaplectic representation and higher symplectic spinors are recalled. In this section, we also introduce several mappings acting on the graded space $\bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S}$, derive the (anti-)commutation relations between them and determine a superset of the image of two of them (Lemma 4), which are components of an infinitesimal version of the symplectic Ricci curvature tensor field. In the Sect. 3, basic properties of torsion-free symplectic connections and their curvature tensor field are recalled and the metaplectic structure is introduced. In Sect. 3.1., the theorem on the complex consisting of the symplectic twistor operators is presented and proved.

2 Metaplectic representation, higher symplectic spinors and basic notation

To fix a notation, let us recall some notions from symplectic linear algebra. Let us consider a real symplectic vector space $(\mathbb{V}, \omega)$ of dimension $2l$, i.e., $\mathbb{V}$ is a $2l$ dimensional real vector space and ω is a non-degenerate antisymmetric bilinear form on $\mathbb{V}$. Let us choose two Lagrangian subspaces[2] $\mathbb{L}, \mathbb{L}' \subseteq \mathbb{V}$ such that $\mathbb{L} \oplus \mathbb{L}' = \mathbb{V}$. It follows that $\dim(\mathbb{L}) = \dim(\mathbb{L}') = l$. Throughout this article, we shall use a symplectic basis $\{e_i\}_{i=1}^{2l}$ of $\mathbb{V}$ chosen in such a way that $\{e_i\}_{i=1}^{l}$ and $\{e_i\}_{i=l+1}^{2l}$ are respective bases of $\mathbb{L}$ and $\mathbb{L}'$. Because the definition of a symplectic basis is not unique, let us fix one which shall be used in this text. A basis $\{e_i\}_{i=1}^{2l}$ of $\mathbb{V}$ is called symplectic basis of $(\mathbb{V}, \omega)$ if $\omega_{ij} := \omega(e_i, e_j)$ satisfies $\omega_{ij} = 1$ if and only if $i \leq l$ and $j = i + l$; $\omega_{ij} = -1$ if and only if $i > l$ and $j = i - l$ and finally, $\omega_{ij} = 0$ in other cases. Let $\{\epsilon^i\}_{i=1}^{2l}$ be the basis of $\mathbb{V}^*$ dual to the basis $\{e_i\}_{i=1}^{2l}$. For $i, j = 1, \ldots, 2l$, we define ω^{ij} by $\sum_{k=1}^{2l} \omega_{ik}\omega^{jk} = \delta_i^j$, for $i, j = 1, \ldots, 2l$. Notice that not only $\omega_{ij} = -\omega_{ji}$, but also $\omega^{ij} = -\omega^{ji}$, $i, j = 1, \ldots, 2l$.

As in the orthogonal case, we would like to rise and lower indices. Because the symplectic form ω is antisymmetric, we should be more careful in this case. For coordinates $K_{ab\ldots c\ldots d}{}^{rs\ldots t\ldots u}$ of a tensor K over $\mathbb{V}$, we denote the expression $\omega^{ic} K_{ab\ldots c\ldots d}{}^{rs\ldots t}$ by $K_{ab\ldots}{}^{i}{}_{\ldots d}{}^{rs\ldots t}$ and $K_{ab\ldots c}{}^{rs\ldots t\ldots u}\omega_{ti}$ by $K_{ab\ldots c}{}^{rs\ldots}{}_{i}{}^{\ldots u}$ and similarly for other types of tensors and also in the geometric setting when we will be considering tensor fields over a symplectic manifold (M, ω).

Let us denote the symplectic group of $(\mathbb{V}, \omega)$ by G, i.e., $G := Sp(\mathbb{V}, \omega) \simeq Sp(2l, \mathbb{R})$. Because the maximal compact subgroup K of G is isomorphic to the unitary group $K \simeq U(l)$ which is of homotopy type $\mathbb{Z}$, there exists a nontrivial two-fold

[2] That is, maximal isotropic wr. to ω.

covering $\tilde{G}$ of G. See, e.g., Habermann and Habermann [9] for details. This two-fold covering is called metaplectic group of $(\mathbb{V}, \omega)$ and it is denoted by $Mp(\mathbb{V}, \omega)$. Let us remark that $Mp(\mathbb{V}, \omega)$ is reductive in the sense of Vogan [21]. In the considered case, we have $\tilde{G} \simeq Mp(2l, \mathbb{R})$. For a later use, let us reserve the symbol λ for the mentioned covering. Thus $\lambda : \tilde{G} \to G$ is a fixed member of the isomorphism class of all nontrivial $2:1$ covering homomorphisms of G. Because $\lambda : \tilde{G} \to G$ is a homomorphism of Lie groups and G is a subgroup of the general linear group $GL(\mathbb{V})$ of $\mathbb{V}$, the mapping λ is also a representation of the metaplectic group $\tilde{G}$ on the vector space $\mathbb{V}$. Let us define $\tilde{K} := \lambda^{-1}(K)$. Obviously, $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$. Further, one can easily see that $\tilde{K} \simeq \widetilde{U(l)} := \{(g, z) \in U(l) \times \mathbb{C}^{\times} | \det(g) = z^2\}$ and thus in particular, $\tilde{K}$ is connected. The Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ is isomorphic to the Lie algebra $\mathfrak{g}$ of G and we will identify them. One has $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}, \omega) \simeq \mathfrak{sp}(2l, \mathbb{R})$.

Now let us recall some notions from representation theory of reductive groups which we shall need in this paper. From the point of view of this article, these notions are rather of a technical character. Let $\mathcal{R}(\tilde{G})$ be the category the object of which are complete, locally convex, Hausdorff topological spaces with a continuous linear $\tilde{G}$-action, such that the resulting representation is admissible and of finite length; the morphisms are continuous $\tilde{G}$-equivariant linear maps between the objects. Let $\mathcal{HC}(\mathfrak{g}, \tilde{K})$ be the category of Harish-Chandra $(\mathfrak{g}, \tilde{K})$-modules and let us consider the forgetful Harish-Chandra functor $HC : \mathcal{R}(\tilde{G}) \to \mathcal{HC}(\mathfrak{g}, \tilde{K})$. It is well known that there exists an adjoint functor $mg : \mathcal{HC}(\mathfrak{g}, \tilde{K}) \to \mathcal{R}(\tilde{G})$ to the Harish-Chandra functor HC. This functor is usually called the minimal globalization functor and its existence is a deep result in representation theory. For details and for the existence of the minimal globalization functor mg, see Kashiwara and Schmid [10] or Vogan [21].

From now on, we shall restrict ourselves to the case $l \geq 2$ not always mentioning it explicitly. The case $l = 1$ should be handled separately (though analogously) because the shape of the root system of $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R})$ is different from that one of of the root system of $\mathfrak{sp}(2l, \mathbb{R})$ for $l \geq 2$. As usual, we shall denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}^{\mathbb{C}}$. Obviously, $\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sp}(2l, \mathbb{C})$.

Further, for any Lie group G and a principal G-bundle $(p : \mathcal{P} \to M, G)$ over a manifold M, we shall denote the vector bundle associated to this principal bundle via a representation $\sigma : G \to \mathrm{Aut}(\mathbf{W})$ of G on $\mathbf{W}$ by $\mathcal{W}$, i.e., $\mathcal{W} = \mathcal{G} \times_{\sigma} \mathbf{W}$. Let us also mention that we shall often use the Einstein summation convention for repeated indices (lower and upper) without mentioning it explicitly.

2.1 Metaplectic representation and symplectic spinors

There exists a distinguished faithful infinite dimensional unitary representation of the metaplectic group $\tilde{G}$ which does not descend to a representation of the symplectic group G. This representation, called *Segal-Shale-Weil*,[3] plays an important role in geometric quantization of Hamiltonian mechanics, see, e.g., Woodhouse [23].

[3] The names oscillator or metaplectic representation are also used in the literature. We shall use the name Segal-Shale-Weil in this text, and reserve the name metaplectic for certain representation arising from the Segal-Shale-Weil one.

We shall not give a definition of this representation here and refer the interested reader to Weil [22], Habermann and Habermann [9] or Kashiwara and Vergne [11].

The Segal-Shale-Weil representation, which we shall denote by U, is a complex infinite dimensional unitary representation of $\tilde{G}$ on the space of complex valued square Lebesgue integrable functions defined on the Lagrangian subspace $\mathbb{L}$, i.e., a homomorphism

$$U : \tilde{G} \to \mathcal{U}\left(\mathbf{L^2}(\mathbb{L})\right),$$

where $\mathcal{U}(\mathbf{W})$ denotes the group of unitary operators on a Hilbert space $\mathbf{W}$. In order to be precise, let us refer to the space $\mathbf{L^2}(\mathbb{L})$ as to the Segal-Shale-Weil module. It is known that the Segal-Shale-Weil module belongs to the category $\mathcal{R}(\tilde{G})$. See, e.g., Kashiwara and Vergne [11]. It is easy to see that the Segal-Shale-Weil representation splits into two irreducible $Mp(2l, \mathbb{R})$-submodules $\mathbf{L^2}(\mathbb{L}) \simeq \mathbf{L^2}(\mathbb{L})_+ \oplus \mathbf{L^2}(\mathbb{L})_-$. The first module consists of even and the second one of odd complex valued square Lebesgue integrable functions on the Lagrangian subspace $\mathbb{L}$. Let us remark that a typical construction of the Segal-Shale-Weil representation is based on the so called Schrödinger representation of the Heisenberg group of $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega)$ and a use of the Stone-von Neumann theorem.

For technical reasons, we shall need the minimal globalization of the underlying Harish-Chandra $(\mathfrak{g}, \tilde{K})$-module $HC(\mathbf{L^2}(\mathbb{L}))$ of the introduced Segal-Shale-Weil module. We shall call this minimal globalization *metaplectic representation* and denote it by $meta$, i.e.,

$$meta : \tilde{G} \to \mathrm{Aut}\left(mg\left(HC\left(\mathbf{L^2}(\mathbb{L})\right)\right)\right),$$

where mg is the minimal globalization functor (see this section and the references therein). For our convenience, let us denote the module $mg(HC(\mathbf{L^2}(\mathbb{L})))$ by $\mathbf{S}$. Similarly we define $\mathbf{S}_+$ and $\mathbf{S}_-$ to be the minimal globalizations of the underlying Harish-Chandra $(\mathfrak{g}, \tilde{K})$-modules of the modules $\mathbf{L^2}(\mathbb{L})_+$ and $\mathbf{L^2}(\mathbb{L})_-$. Accordingly to $\mathbf{L^2}(\mathbb{L}) \simeq \mathbf{L^2}(\mathbb{L})_+ \oplus \mathbf{L^2}(\mathbb{L})_-$, we have $\mathbf{S} \simeq \mathbf{S}_+ \oplus \mathbf{S}_-$. We shall call the $Mp(\mathbb{V}, \omega)$-module $\mathbf{S}$ the symplectic spinor module and its elements *symplectic spinors*. For the name "spinor", see Kostant [12] or Sect. 1.

Further notion, related to the symplectic vector space split into the two chosen Lagrangian subspaces $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega)$, is the so called symplectic Clifford multiplication of elements of $\mathbf{S}$ by vectors from $\mathbb{V}$. For $i = 1, \ldots, l$ and a symplectic spinor $f \in \mathbf{S}$, we define

$$(e_i.f)(x) := \imath x^i f(x) \quad \text{and}$$
$$(e_{i+l}.f)(x) := \frac{\partial f}{\partial x^i}(x),$$

where $x = \sum_{i=1}^{l} x^i e_i \in \mathbb{L}$ and $\imath = \sqrt{-1}$ denotes the imaginary unit. Extending this multiplication $\mathbb{R}$-linearly, we get the mentioned symplectic Clifford multiplication. Let us mention that the multiplication and the differentiation make sense for any $f \in \mathbf{S}$ because of the "analytic" interpretation of the minimal globalization. (See Vogan [21] for details.) Let us remark that in the physical literature, the symplectic Clifford multiplication is usually called Schrödinger quantization prescription.

The following lemma is an easy consequence of the definition of the symplectic Clifford multiplication.

Lemma 1 *For $v, w \in \mathbb{V}$ and $s \in \mathbf{S}$, we have*

$$v.(w.s) - w.(v.s) = -\imath\omega(v, w)s.$$

Proof See Habermann and Habermann [9, pp. 11]. □

We shall often write $v.w.s$ instead of $v.(w.s)$ for $v, w \in \mathbb{V}$ and a symplectic spinor $s \in \mathbf{S}$ and similarly for higher number of multiplying elements. Further instead of $e_i.e_j.s$, we shall write $e_{ij}.s$ simply and similarly for expressions with higher number of multiplying elements, e.g., $e_{ijk}.s$ abbreviates $e_i.e_j.e_k.s$.

2.2 Higher symplectic spinors

In this subsection, we shall present a result on a decomposition of the tensor product of the metaplectic representation $meta : \tilde{G} \to \mathrm{Aut}(\mathbf{S})$ with the wedge power of the representation $\lambda^* : \tilde{G} \to GL(\mathbb{V}^*)$ of $\tilde{G}$ (dual to the representation λ) into irreducible summands. Let us reserve the symbol ρ for the mentioned tensor product representation of $\tilde{G}$, i.e.,

$$\rho : \tilde{G} \to \mathrm{Aut}\left(\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}\right)$$
$$\rho\,(g)\,(\alpha \otimes s) := \lambda\,(g)^{*\wedge r}\,\alpha \otimes meta\,(g)\,s$$

for $r = 0, \ldots, 2l$, $g \in \tilde{G}$, $\alpha \in \bigwedge^r \mathbb{V}^*$, $s \in \mathbf{S}$, and extend this defining formula linearly. For definiteness, let us equip the tensor product $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ with the so called Grothendieck tensor product topology. See Vogan [21] and Treves [19] for details on this topological structure. In a parallel to the Riemannian case, we shall call the elements of $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ higher symplectic spinors.

Let us introduce the following subsets of the set of pairs of non-negative integers. We define

$$\begin{aligned}
\Xi &:= \{(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0 | i = 0, \ldots, l;\ j = 0, \ldots, i\} \\
&\quad \cup \{(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0 | i = l + 1, \ldots, 2l;\ j = 0, \ldots, 2l - i\}, \\
\Xi_+ &:= \Xi - \{(i, i) | i = 0, \ldots, l\} \quad \text{and} \\
\Xi_- &:= \Xi - \{(i, 2l - i) | i = l, \ldots, 2l\}.
\end{aligned}$$

For each $(i, j) \in \Xi$, a $\mathfrak{g}^{\mathbb{C}}$-module $\mathbb{E}^{ij}_{\pm}$ was introduced in Krýsl [15]. These modules are irreducible infinite dimensional highest modules over $\mathfrak{sp}(\mathbb{V}, \omega)^{\mathbb{C}}$ and they are

$$\begin{tikzcd}
\mathbf{E}^{0,0} \arrow[r] \arrow[dr] & \mathbf{E}^{1,0} \arrow[r] \arrow[dr] & \mathbf{E}^{2,0} \arrow[r] \arrow[dr] & \mathbf{E}^{3,0} \arrow[r] \arrow[dr] & \mathbf{E}^{4,0} \arrow[r] \arrow[dr] & \mathbf{E}^{5,0} \arrow[r] & \mathbf{E}^{6,0} \\
& \mathbf{E}^{1,1} \arrow[r] \arrow[ur] \arrow[dr] & \mathbf{E}^{2,1} \arrow[r] \arrow[ur] \arrow[dr] & \mathbf{E}^{3,1} \arrow[r] \arrow[ur] \arrow[dr] & \mathbf{E}^{4,1} \arrow[r] \arrow[ur] & \mathbf{E}^{5,1} \arrow[ur] & \\
& & \mathbf{E}^{2,2} \arrow[r] \arrow[ur] \arrow[dr] & \mathbf{E}^{3,2} \arrow[r] \arrow[ur] & \mathbf{E}^{4,2} \arrow[ur] & & \\
& & & \mathbf{E}^{3,3} \arrow[ur] & & &
\end{tikzcd}$$

Fig. 1 Decomposition structure of $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ for $l = 3$

described via their highest weights in the mentioned article explicitly. In Theorem 2, the module of symplectic spinor valued exterior forms $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ is decomposed into irreducible submodules.

Theorem 2 *For $l \geq 2$, the following decomposition into irreducible $Mp(\mathbb{V}, \omega)$-submodules*

$$\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}_\pm \simeq \bigoplus_{j,(i,j)\in\Xi} \mathbf{E}^{ij}_\pm, \quad i = 0, \dots, 2l, \quad \textit{holds.}$$

The modules $\mathbf{E}^{ij}_\pm$ are determined, as objects in the category $\mathcal{R}(\tilde{G})$, by the fact that first they are submodules of the corresponding tensor product and second the $\mathfrak{g}^\mathbb{C}$-structure of $HC(\mathbf{E}^{ij}_\pm)$ is isomorphic to $\mathbb{E}^{ij}_\pm$.

Proof See Krýsl [13,15]. □

At the Fig. 1, the decomposition in the case $l = 3$ is displayed. In the ith column, when counted from zero, the summands of $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}$, $i = 0, \dots, 6$, are written. The meaning of the arrows at the figure will be explained later.

Remark Let us mention that for any $(i, j), (i, k) \in \Xi$, $j \neq k$, we have $\mathbb{E}^{ij}_\pm \not\simeq \mathbb{E}^{ik}_\pm$ (as $\mathfrak{g}^\mathbb{C}$-modules) for all combinations of $\pm$ on the left hand as well as on the right hand side. Using this fact, we see that for $i = 0, \dots, 2l$ the $\tilde{G}$-modules $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}_\pm$ are multiplicity free. Moreover for $(i, j), (k, j) \in \Xi$, we have $\mathbb{E}^{ij}_\pm \simeq \mathbb{E}^{kj}_\mp$. These facts will be crucial in this paper.

For our convenience, let us set $\mathbf{E}^{ij}_\pm := \{0\}$ for $(i, j) \in \mathbb{Z} \times \mathbb{Z} - \Xi$ and $\mathbf{E}^{ij} := \mathbf{E}^{ij}_+ \oplus \mathbf{E}^{ij}_-$.

Now, we shall introduce four operators which help us to describe the action of the symplectic Ricci curvature tensor field acting on symplectic spinor valued exterior differential forms. For $r = 0, \ldots, 2l$, $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$ and $\sigma \in \odot^2 \mathbb{V}^*$, we set

$$X : \bigwedge^{r} \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S}, \quad X(\alpha \otimes s) := \sum_{i=1}^{2l} \epsilon^i \wedge \alpha \otimes e_i.s,$$

$$Y : \bigwedge^{r} \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r-1} \mathbb{V}^* \otimes \mathbf{S}, \quad Y(\alpha \otimes s) := \sum_{i,j=1}^{2l} \omega^{ij} \iota_{e_i} \alpha \otimes e_j.s,$$

$$\Sigma^\sigma : \bigwedge^{r} \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S}, \quad \Sigma^\sigma(\alpha \otimes s) := \sum_{i,j=1}^{2l} \sigma^i{}_j \epsilon^j \wedge \alpha \otimes e_i.s \text{ and}$$

$$\Theta^\sigma : \bigwedge^{r} \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r} \mathbb{V}^* \otimes \mathbf{S}, \quad \Theta^\sigma(\alpha \otimes s) := \sum_{i,j=1}^{2l} \alpha \otimes \sigma^{ij} e_{ij}.s$$

and extend it linearly. Here $\sigma_{ij} := \sigma(e_i, e_j)$, $i, j = 1, \ldots, 2l$, and the contraction of an exterior form $\alpha \in \bigwedge^\bullet \mathbb{V}^*$ by a vector $v \in \mathbb{V}$ is denoted by $\iota_v \alpha$.

Remark (1) One easily finds out that the operators are independent of the choice of a symplectic basis $\{e_i\}_{i=1}^{2l}$. The operators X and Y were used to prove the Howe correspondence for $Mp(\mathbb{V}, \omega)$ acting on $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ via the representation ρ. See Krýsl [13] for details.

(2) The symmetric tensor σ is an infinitesimal version of a part of the curvature of a Fedosov connection. This part is called symplectic Ricci curvature tensor field and will be introduced below. The operators Σ^σ and Θ^σ will help us to describe the action of the symplectic Ricci curvature tensor field acting on symplectic spinor valued exterior differential forms.

In what follows, we shall write $\iota_{e_{ij}} \alpha$ instead of $\iota_{e_i} \iota_{e_j} \alpha$, $i, j = 1, \ldots, 2l$, and similarly for higher number of elements contracting a form $\alpha \in \bigwedge^\bullet \mathbb{V}^*$.

Using the Lemma 1, it is easy to compute that

$$X^2(\alpha \otimes s) = -\frac{\imath}{2} \omega_{ij} \epsilon^i \wedge \epsilon^j \wedge \alpha \otimes s \quad \text{and} \quad Y^2(\alpha \otimes s) = \frac{\imath}{2} \omega^{ij} \iota_{e_{ij}} \alpha \otimes s \qquad (1)$$

for any element $\alpha \otimes s \in \bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$.

In order to be able to use the operators X and Y in a geometric setting and some further reasons, we shall need the following

Lemma 3 (1) *The operators X, Y are $\tilde{G}$-equivariant wr. to the representation ρ of $\tilde{G}$.*

(2) *For $(i, j) \in \Xi_-$, the operator X is an isomorphism if restricted to $\mathbf{E}^{ij}$. For $(i, j) \in \Xi_+$, the operator Y is an isomorphism if restricted to $\mathbf{E}^{ij}$.*

Proof For the $\tilde{G}$-equivariance of X and Y, see Krýsl [14]. The fact that the mentioned restrictions are isomorphisms is proved in Krýsl [13]. □

In Lemma 4, four relations are proved which will be used later in order to determine a superset of the image of a restriction of the symplectic Ricci curvature tensor field acting on symplectic spinor valued exterior differential forms. Without these (anti-)commutation relations many of the computations presented below would became very difficult to manage because of their increasing length. Often, we shall write Σ and Θ simply instead of the more explicit Σ^σ and Θ^σ. The symmetric tensor σ is assumed to be chosen. The symbol $\{,\}$ denotes the anticommutator on $\mathrm{End}(\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S})$ viewed as an associative algebra.

Lemma 4 *The following relations*

$$\{\Sigma, X\} = 0, \tag{2}$$
$$\left[\{\Sigma, Y\}, Y^2\right] = 0, \tag{3}$$
$$[X, \Theta] = 2\imath\Sigma \quad \textit{and} \tag{4}$$
$$\left[\Theta, Y^2\right] = 0 \tag{5}$$

hold on $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$.

Proof We shall prove these identities for $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, $r = 0, \ldots, 2l$ only. The statement then follows by linearity of the considered operators.

(1) Let us compute

$$\begin{aligned}(X\Sigma + \Sigma X)(\alpha \otimes s) &= X(\sigma^i{}_j\epsilon^j \wedge \alpha \otimes e_i.s) + \Sigma(\epsilon^i \wedge \alpha \otimes e_i.s)\\ &= \sigma^i{}_j\epsilon^k \wedge \epsilon^j \wedge \alpha \otimes e_{ki}.s + \sigma^j{}_k\epsilon^k \wedge \epsilon^i \wedge \alpha \otimes e_{ji}.s\\ &= \sigma^i{}_k\epsilon^j \wedge \epsilon^k \wedge \alpha \otimes e_{ji}.s + \sigma^i{}_k\epsilon^k \wedge \epsilon^j \wedge \alpha \otimes e_{ij}.s\\ &= \sigma^i{}_k\epsilon^j \wedge \epsilon^k \wedge \alpha \otimes (e_{ji}. - e_{ij}.)s\\ &= -\imath\sigma^i{}_k\omega_{ji}\epsilon^j \wedge \epsilon^k \wedge \alpha \otimes s\\ &= \imath\sigma_{jk}\epsilon^j \wedge \epsilon^k \wedge \alpha \otimes s\\ &= 0,\end{aligned}$$

where we have renumbered indices, used the Lemma 1 and the fact that σ is symmetric. In what follows, we shall use similar procedures without mentioning them explicitly.

(2) Let us compute

$$\begin{aligned}P(\alpha \otimes s) &:= \{\Sigma, Y\}(\alpha \otimes s)\\ &= Y(\sigma^i{}_j\epsilon^j \wedge \alpha \otimes e_i.s) + \Sigma(\omega^{ij}\iota_{e_i}\alpha \otimes e_j.s)\\ &= \sigma^i{}_j\omega^{kl}\iota_{e_k}(\epsilon^j \wedge \alpha) \otimes e_{li}.s + \omega^{ij}\sigma^k{}_l\epsilon^l \wedge \iota_{e_i}\alpha \otimes e_{kj}.s\\ &= \sigma^i{}_j\omega^{kl}(\delta^j_k\alpha - \epsilon^j \wedge \iota_{e_k}\alpha) \otimes e_{li}.s + \omega^{ij}\sigma^k{}_l\epsilon^l \wedge \iota_{e_i}\alpha \otimes e_{kj}.s\\ &= -\sigma^{il}\alpha \otimes e_{li}.s - \sigma^i{}_j\omega^{kl}\epsilon^j \wedge \iota_{e_k}\alpha \otimes e_{li}.s + \omega^{ij}\sigma^k{}_l\epsilon^l \wedge \iota_{e_i}\alpha \otimes e_{kj}.s\\ &= -\sigma^{il}\alpha \otimes e_{li}.s - \sigma^k{}_l\omega^{ij}\epsilon^l \wedge \iota_{e_i}\alpha \otimes e_{jk}.s + \omega^{ij}\sigma^k{}_l\epsilon^l \wedge \iota_{e_i}\alpha \otimes e_{kj}.s\end{aligned}$$

$$= -\sigma^{il}\alpha \otimes e_{li}.s - \imath\omega^{ij}\omega_{kj}\sigma^{k}{}_{l}\epsilon^{l} \wedge \iota_{e_i}\alpha \otimes s$$
$$= -\sigma^{ij}\alpha \otimes e_{ji}.s - \imath\sigma^{i}{}_{j}\epsilon^{j} \wedge \iota_{e_i}\alpha \otimes s.$$

Now, we use the derived prescription for P and the Eq. (1) to compute

$$\begin{aligned}
\left[P, 2\imath Y^2\right](\alpha \otimes s) &= 2\imath PY^2(\alpha \otimes s) - 2\imath Y^2 P(\alpha \otimes s)\\
&= -P\left(\omega^{ij}\iota_{e_{ij}}\alpha\otimes s\right) - 2\imath Y^2\left(-\sigma^{ij}\alpha\otimes e_{ji}.s - \imath\sigma^{i}{}_{j}\epsilon^{j}\wedge\iota_{e_i}\alpha\otimes s\right)\\
&= \omega^{ij}\sigma^{kl}\iota_{e_{ij}}\alpha \otimes e_{lk}.s + \imath\omega^{ij}\sigma^{k}{}_{l}\epsilon^{l} \wedge \iota_{e_{kij}}\alpha \otimes s\\
&\quad - \sigma^{ij}\omega^{kl}\iota_{e_{kl}}\alpha \otimes e_{ij}.s - \imath\sigma^{i}{}_{j}\omega^{kl}\iota_{e_{kl}}(\epsilon^{j} \wedge \iota_{e_i}\alpha) \otimes s\\
&= \omega^{ij}\sigma^{kl}\iota_{e_{ij}}\alpha \otimes e_{kl}.s + \imath\omega^{ij}\sigma^{k}{}_{l}\epsilon^{l} \wedge \iota_{e_{kij}}\alpha \otimes s\\
&\quad - \sigma^{ij}\omega^{kl}\iota_{e_{kl}}\alpha \otimes e_{ji}.s - \imath\omega^{kl}\sigma^{i}{}_{j}(\delta^{j}_{l}\iota_{e_{ki}}\alpha - \delta^{j}_{k}\iota_{e_{li}}\alpha\\
&\quad + \epsilon^{j} \wedge \iota_{e_{kli}}\alpha) \otimes s\\
&= \omega^{ij}\sigma^{kl}\iota_{e_{ij}}\alpha \otimes e_{kl}.s + \imath\omega^{ij}\sigma^{k}{}_{l}\epsilon^{l} \wedge \iota_{e_{kij}}\alpha \otimes s\\
&\quad - \sigma^{kl}\omega^{ij}\iota_{e_{ij}}\alpha \otimes e_{kl}.s - \imath\omega^{ij}\sigma^{k}{}_{l}\epsilon^{l} \wedge \iota_{e_{ijk}}\alpha \otimes s\\
&= 0.
\end{aligned}$$

(3) Due to the definition of Θ, we have

$$\begin{aligned}
[X, \Theta](\alpha \otimes s) &= \epsilon^{k} \wedge \alpha \otimes \sigma^{ij}e_{kij}.s - \epsilon^{i} \wedge \alpha \otimes \sigma^{jk}e_{jki}.s\\
&= \epsilon^{k} \wedge \alpha \otimes \sigma^{ij}e_{kij}.s - \epsilon^{k} \wedge \alpha \otimes \sigma^{ij}e_{ijk}.s\\
&= \sigma^{ij}\epsilon^{k} \wedge \alpha \otimes (e_{ikj}.s - \imath\omega_{ki}e_{j}.s - e_{ijk}.s)\\
&= \sigma^{ij}\epsilon^{k} \wedge \alpha \otimes (e_{ijk}.s - \imath\omega_{kj}e_{i}.s - \imath\omega_{ki}e_{j}.s - e_{ijk}.s)\\
&= 2\imath\Sigma(\alpha \otimes s).
\end{aligned}$$

(4) This relation follows easily from the definition of Θ and the relation (1). □

Remark It may be interesting to find a representation theoretical aspects of the previous relations regarding the fact that the symmetric bilinear form $\sigma \in \odot^2\mathbb{V}^*$ when supposed to be $Sp(\mathbb{V}, \omega)$-equivariant, became zero.

In Proposition 5, a superset of the image of Σ and Θ restricted to $\mathbf{E}^{ij}$, for $(i, j) \in \Xi$, is determined.

Proposition 5 *For* $(i, j) \in \Xi$, *we have*

$$\Sigma_{|\mathbf{E}^{ij}} : \mathbf{E}^{ij} \to \mathbf{E}^{i+1,j-1} \oplus \mathbf{E}^{i+1,j} \oplus \mathbf{E}^{i+1,j+1} \quad \textit{and}$$
$$\Theta_{|\mathbf{E}^{ij}} : \mathbf{E}^{ij} \to \mathbf{E}^{i,j-1} \oplus \mathbf{E}^{i,j} \oplus \mathbf{E}^{i,j+1}.$$

Proof (1) For $i = 0, \dots, l$, let us choose an element $\psi = \alpha \otimes s \in \mathbf{E}^{ii}$. Using the relation (3), we have $0 = [P, Y^2]\psi = (PY^2 - Y^2P)\psi = (\Sigma Y^3 + Y\Sigma Y^2 - Y^2\Sigma Y - Y^3\Sigma)\psi$. Because Y is $\tilde{G}$-equivariant (Lemma 3 item 1), decreasing

the form degree of ψ by one and there is no summand isomorphic to $\mathbf{E}^{ii}_{+}$ or $\mathbf{E}^{ii}_{-}$ in $\bigwedge^{i-1}\mathbb{V}^* \otimes \mathbf{S}$ (see the Remark below the Theorem 2), $Y\psi = 0$. Using this equation, we see that the first three summands in the above expression for $[P, Y^2]$ are zero. Therefore we have $0 = Y^3\Sigma\psi$. Because Y is injective on $\mathbf{E}^{ij}$ for $(i, j) \in \Xi_+$ (Lemma 3 item 2), we see that $\Sigma\psi \in \mathbf{E}^{i+1,i-1} \oplus \mathbf{E}^{i+1,i} \oplus \mathbf{E}^{i+1,i+1}$. (It is recommendable to have a look at the Fig. 1.)
Now, let us consider a general $(i, j) \in \Xi$ and $\psi \in \mathbf{E}^{ij}$. Let us take an element $\psi' \in \mathbf{E}^{jj}$ such that $\psi = X^{(i-j)}\psi'$. This element exists because according to Lemma 3 item 2, the operator X is an isomorphism when restricted to $\mathbf{E}^{ij}$ for $(i, j) \in \Xi_-$. Because of the relation (2), we have $\Sigma\psi = \Sigma X^{(i-j)}\psi' = \pm X^{(i-j)}\Sigma\psi'$. From the previous item, we know that $\Sigma\psi' \in \mathbf{E}^{j+1,j-1} \oplus \mathbf{E}^{j+1,j} \oplus \mathbf{E}^{j+1,j+1}$. Because X is $\tilde{G}$-equivariant (Lemma 3 item 1) and the only summands in $\bigwedge^{i+1}\mathbb{V}^* \otimes \mathbf{S}$ isomorphic to $\mathbf{E}^{j+1,j-1} \oplus \mathbf{E}^{j+1,j} \oplus \mathbf{E}^{j+1,j+1}$ are those described in the formulation of this proposition (see the Remark below the Theorem 2), the statement follows.

(2) For $i = 0, \ldots, l$, let us consider an element $\psi = \alpha \otimes s \in \mathbf{E}^{ii}$. Using the relation (5), we have $0 = [\Theta, Y^2]\psi = \Theta Y^2\psi - Y^2\Theta\psi$. Using similar reasoning to that one in the first item, we get $Y\psi = 0$. Using the expression for $[\Theta, Y^2]$ above, we get $Y^2\Theta\psi = 0$ and consequently, $\Theta\psi \in \mathbf{E}^{ii} \oplus \mathbf{E}^{i,i-1}$. Now, let us suppose $\psi \in \mathbf{E}^{ij}$ for $(i, j) \in \Xi$. There exists an element $\psi' \in \mathbf{E}^{jj}$ such that $\psi = X^{(i-j)}\psi'$ (Lemma 3 item 2). Using the relations (4) and (2), we have $\Theta\psi = \Theta X^{(i-j)}\psi' = X^{(i-j)}\Theta\psi'$ if $i - j$ is even and $(X^{(i-j)}\Theta - 2\imath X^{(i-j-1)}\Sigma)\psi'$ if $i - j$ is odd. Using the fact $\Sigma_{|\mathbf{E}^{ij}} : \mathbf{E}^{ij} \to \mathbf{E}^{i+1,j-1} \oplus \mathbf{E}^{i+1,j} \oplus \mathbf{E}^{i+1,j+1}$, proved in the previous item, the statement follows by similar lines of reasoning as in the first item. $\square$

3 Metaplectic structures and symplectic curvature tensors

After we have finished the algebraic part of the paper, let us describe the geometric structure we shall be investigating. We begin with a recollection of results of Vaisman in [20] and of Gelfand et al. in [6]. Let (M, ω) be a symplectic manifold and ∇ be a symplectic torsion-free affine connection. By symplectic and torsion-free, we mean $\nabla\omega = 0$ and $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$ for all $X, Y \in \mathfrak{X}(M)$, respectively. Such connections are usually called Fedosov connections. In what follows, we shall call the triples (M, ω, ∇) Fedosov manifolds.

To fix our notation, let us recall the classical definition of the curvature tensor R^∇ of the connection ∇, which we shall use in this text. Let

$$R^\nabla(X, Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Let us choose a local symplectic frame $\{e_i\}_{i=1}^{2l}$ over an open subset $U \subseteq M$. We shall often write expressions in which indices i, j, k, l e.t.c. occur. We will implicitly mean i, j, k, l are running from 1 to $2l$ without mentioning it explicitly. We set

$$R_{ijkl} := \omega\left(R\left(e_k, e_l\right) e_j, e_i\right).$$

Let us mention that we are using the convention of Vaisman [20] which is different from that one used in Habarmann and Habermann [9].

From the symplectic curvature tensor field R^{∇}, we can build the symplectic Ricci curvature tensor field σ^{∇} defined by the classical formula

$$\sigma^{\nabla}(X, Y) := \mathrm{Tr}\left(V \mapsto R^{\nabla}(V, X)\, Y\right)$$

for each $X, Y \in \mathfrak{X}(M)$ (the variable V denotes a vector field on M). For the chosen frame and $i, j = 1, \ldots, 2l$, we set

$$\sigma_{ij} := \sigma^{\nabla}\left(e_i, e_j\right).$$

Further, let us define

$$\begin{aligned} 2(l+1)\widetilde{\sigma}^{\nabla}_{ijkl} &:= \omega_{il}\sigma_{jk} - \omega_{ik}\sigma_{jl} + \omega_{jl}\sigma_{ik} - \omega_{jk}\sigma_{il} + 2\sigma_{ij}\omega_{kl}, \quad (6)\\ \widetilde{\sigma}^{\nabla}(X, Y, Z, V) &:= \widetilde{\sigma}^{\nabla}_{ijkl} X^i Y^j Z^k V^l \quad \text{and}\\ W^{\nabla} &:= R^{\nabla} - \widetilde{\sigma}^{\nabla} \quad (7)\end{aligned}$$

for local vector fields $X = X^i e_i$, $Y = Y^j e_j$, $Z = Z^k e_k$ and $V = V^l e_l$. We will call the tensor field $\tilde{\sigma}^{\nabla}$ the extended symplectic Ricci curvature tensor field and W^{∇} the symplectic Weyl curvature tensor field. These tensor fields were already introduced in Vaisman [20]. We shall often drop the index ∇ in the previous expressions. Thus, we shall often write R, W, σ and $\widetilde{\sigma}$ instead of R^{∇}, W^{∇}, σ^{∇} and $\widetilde{\sigma}^{\nabla}$, respectively.

In Lemma 6, the symmetry of σ is stated.

Lemma 6 *For a Fedosov manifold (M, ω, ∇), the symplectic Ricci curvature tensor field σ is symmetric.*

Proof See Vaisman [20]. □

Let us describe the geometric structure with help of which the actions of the symplectic twistor operators are defined. This structure, called metaplectic, is a precise symplectic analogue of the notion of a spin structure in the Riemannian geometry. For a symplectic manifold (M^{2l}, ω) of dimension $2l$, let us denote the bundle of symplectic reperes in TM by $\mathcal{P}$ and the foot-point projection of $\mathcal{P}$ onto M by p. Thus $(p : \mathcal{P} \to M, G)$, where $G \simeq Sp(2l, \mathbb{R})$, is a principal G-bundle over M. As in the Sect. 2, let $\lambda : \tilde{G} \to G$ be a member of the isomorphism class of the non-trivial two-fold coverings of the symplectic group G. In particular, $\tilde{G} \simeq Mp(2l, \mathbb{R})$. Further, let us consider a principal $\tilde{G}$-bundle $(q : \mathcal{Q} \to M, \tilde{G})$ over the symplectic manifold

(M, ω). We call a pair $(\mathcal{Q}, \Lambda)$ metaplectic structure if $\Lambda : \mathcal{Q} \to \mathcal{P}$ is a surjective bundle homomorphism over the identity on M and if the following diagram,

$$\begin{tikzcd}
\mathcal{Q} \times \tilde{G} \arrow[r] \arrow[dd, "\Lambda\times\lambda"] & \mathcal{Q} \arrow[dd, "\Lambda"] \arrow[dr, "q"] & \\
& & M \\
\mathcal{P} \times G \arrow[r] & \mathcal{P} \arrow[ur, "p"'] &
\end{tikzcd}$$

with the horizontal arrows being respective actions of the displayed groups, commutes. See, e.g., Habermann and Habermann [9] and Kostant [12] for details on the metaplectic structures. Let us only remark, that typical examples of symplectic manifolds admitting a metaplectic structure are cotangent bundles of orientable manifolds (phase spaces), Calabi-Yau manifolds and complex projective spaces $\mathbb{CP}^{2k+1}$, $k \in \mathbb{N}_0$ (all considered with their standard symplectic forms).

Let us denote the vector bundle associated to the introduced principal $\tilde{G}$-bundle $(q : \mathcal{Q} \to M, \tilde{G})$ via the representation $meta$ on $\mathbf{S}$ by $\mathcal{S}$. We shall call this associated vector bundle symplectic spinor bundle. Thus, we have $\mathcal{S} = \mathcal{Q} \times_{meta} \mathbf{S}$. Sections $\phi \in \Gamma(M, \mathcal{S})$ will be called symplectic spinor fields. Let us denote the space of symplectic valued exterior differential forms $\Gamma(M, \mathcal{Q} \times_\rho (\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}))$ by $\Omega^\bullet(M, \mathcal{S})$ and call it the space of symplectic spinor valued forms simply. Further for $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, we define the associated vector bundles $\mathcal{E}^{ij}$ by the prescription $\mathcal{E}^{ij} := \mathcal{Q} \times_\rho \mathbf{E}^{ij}$.

Because the operators X, Y are $\tilde{G}$-equivariant (Lemma 3 item 1), they lift to operators acting on sections of the corresponding associated vector bundles. We shall use the same symbols as for the defined operators as for their "lifts" to the associated vector bundle structure. Because for each $i = 0, \ldots, 2l$, the decomposition $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S} \simeq \bigoplus_{j,(i,j)\in\Xi} \mathbf{E}^{ij}$ is multiplicity free (see the Remark below the Theorem 2), there exist uniquely defined projections $p^{ij} : \Omega^i(M, \mathcal{S}) \to \Gamma(M, \mathcal{E}^{ij})$, $(i, j) \in \mathbb{Z}\times\mathbb{Z}$.

Now, let us suppose that (M, ω) is not only equipped with a Fedosov connection ∇ but also admit a metaplectic structure $((q : \mathcal{Q} \to M, \tilde{G}), \Lambda)$. The connection ∇ determines the associated principal bundle connection Z on the principal bundle $(p : \mathcal{P} \to M, G)$. This principle bundle connection lifts to a principal bundle connection on the principal bundle $(q : \mathcal{Q} \to M, \tilde{G})$ and defines the associated covariant derivative on the symplectic bundle $\mathcal{S}$, which we shall denote by ∇^S and call it the symplectic spinor covariant derivative. See Habermann and Habermann [9] for details. The symplectic spinor covariant derivative induces the exterior symplectic spinor derivative d^{∇^S} acting on $\Omega^\bullet(M, \mathcal{S})$. The curvature tensor field $R^{\Omega^\bullet(M,\mathcal{S})}$ acting on the symplectic spinor valued forms is given by the classical formula

$$R^{\Omega^\bullet(M,\mathcal{S})} := d^{\nabla^S} d^{\nabla^S}.$$

In Theorem 7, a superset of the image of d^{∇^S} restricted to $\Gamma(M, \mathcal{E}^{ij})$, $(i, j) \in \Xi$, is described.

Theorem 7 *Let* (M, ω, ∇) *be a Fedosov manifold admitting a metaplectic structure. Then for the exterior symplectic spinor derivative* d^{∇^S} *and* $(i, j) \in \Xi$, *we have*

$$d^{\nabla^S}_{|\Gamma(M,\mathcal{E}^{ij})} : \Gamma\left(M, \mathcal{E}^{ij}\right) \to \Gamma\left(M, \mathcal{E}^{i+1,j-1} \oplus \mathcal{E}^{i+1,j} \oplus \mathcal{E}^{i+1,j+1}\right).$$

Proof See Krýsl [15]. □

Remark From the proof of the theorem, it is easy to see that it can be extended to the case (M, ω) is presymplectic and the symplectic connection ∇ has a non-zero torsion. For $l = 3$ and any $(i, j) \in \Xi_-$, the mapping d^{∇^S} restricted to $\Gamma(M, \mathcal{E}^{ij})$ is displayed as an arrow at the Figure 1 above. (The exterior covariant derivative d^{∇^S} maps $\Gamma(M, \mathcal{E}^{ij})$ into three "neighbor" subspaces.)

3.1 Curvature tensor on symplectic spinor valued forms and the complex of symplectic twistor operators

Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure $(\mathcal{Q}, \Lambda)$. In Lemma 8, the action of $R^S := d^{\nabla^S}\nabla^S$ on the space of symplectic spinor fields is described using just the symplectic curvature tensor field R of ∇.

Lemma 8 *Let* (M, ω, ∇) *be a Fedosov manifold admitting a metaplectic structure. Then for a symplectic spinor field* $\phi \in \Gamma(M, \mathcal{S})$, *we have*

$$R^S\phi = \frac{\imath}{2} R^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi.$$

Proof See Habermann, Habermann [9, pp. 42]. □

For our convenience, let us set $m_i := i$ for $i = 0, \ldots, l$ and $m_i := 2l - i$ for $i = l + 1, \ldots, 2l$. Now, we can define the symplectic twistor operators, with help of which we introduce the mentioned complex. For $i = 0, \ldots, 2l - 1$, we set

$$T_i : \Gamma(M, \mathcal{E}^{im_i}) \to \Gamma\left(M, \mathcal{E}^{i+1,m_{i+1}}\right), \quad T_i := p^{i+1,m_{i+1}} d^{\nabla^S}_{|\Gamma(M,\mathcal{E}^{im_i})}$$

and call these operators *symplectic twistor operators*. Informally, one can say that the operators are going on the two bottom edges of the triangle at the Fig. 1. Let us notice that up to a constant complex multiple $X\mathfrak{D} = \nabla^S - T_0$ where $\mathfrak{D}$ is the so called symplectic Dirac operator introduced by K. Habermann in [8]. In Riemannian spin geometry, the twistor operators fulfill a parallel relation.

Theorem 9 *Let (M^{2l}, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. If $l \geq 2$ and the symplectic Weyl tensor field $W^{\nabla} = 0$, then*

$$0 \longrightarrow \Gamma(M, \mathcal{E}^{00}) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^{11}) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-1}} \Gamma(M, \mathcal{E}^{ll}) \longrightarrow 0 \quad \textit{and}$$
$$0 \longrightarrow \Gamma(M, \mathcal{E}^{ll}) \xrightarrow{T_l} \Gamma(M, \mathcal{E}^{l+1,l+1}) \xrightarrow{T_{l+1}} \cdots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l,2l}) \longrightarrow 0$$

are complexes.

Proof (1) In this item, we prove that for an element $\psi \in \Omega^{\bullet}(M, \mathcal{S})$,

$$R^{\Omega^{\bullet}(M,\mathcal{S})}\psi = \frac{\imath}{l+1}\left(\imath X^2 \Theta^{\sigma} + X\Sigma^{\sigma}\right)\psi.$$

For $\psi = \alpha \otimes \phi \in \Omega^{\bullet}(M, \mathcal{S})$, we can write

$$\begin{aligned}
R^{\Omega^{\bullet}(M,\mathcal{S})}(\alpha \otimes \phi) &= d^{\nabla^S} d^{\nabla^S}(\alpha \otimes \phi) \\
&= d^{\nabla^S}(d\alpha \otimes \phi + (-1)^{deg(\alpha)}\alpha \wedge \nabla^S \phi) \\
&= d^2\alpha \otimes \phi + (-1)^{deg(\alpha)+1} d\alpha \wedge \nabla^S \phi + (-1)^{deg(\alpha)} d\alpha \wedge \nabla^S \phi \\
&\quad + (-1)^{deg(\alpha)}(-1)^{deg(\alpha)} \alpha \wedge d^{\nabla^S} \nabla^S \phi \\
&= \alpha \wedge \frac{\imath}{2} R^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij}.\phi \\
&= \frac{\imath}{2} R^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij}.\phi,
\end{aligned}$$

where we have used the Lemma 8. Using this computation, the definition of the symplectic Weyl curvature tensor field W^{∇} (Eq. (7)), the definition of the extended symplectic Ricci curvature tensor field $\widetilde{\sigma}^{\nabla}$ (Eq. (6)) and the assumption $W^{\nabla} = 0$, we get

$$\begin{aligned}
-4(l+1)\imath R^{\Omega^{\bullet}(M,\mathcal{S})}(\alpha \otimes \phi) &= 2(l+1) R^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij}.\phi \\
&= 2(l+1)(W^{ij}{}_{kl} + \widetilde{\sigma}^{ij}{}_{kl}) \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij}.\phi \\
&= 2(l+1)\widetilde{\sigma}^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij}.\phi \\
&= (\omega^i{}_l \sigma^j{}_k - \omega^i{}_k \sigma^j{}_l + \omega^j{}_l \sigma^i{}_k - \omega^j{}_k \sigma^i{}_l \\
&\quad + 2\sigma^{ij}\omega_{kl}) \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij}.\phi \\
&= 4\omega^i{}_l \sigma^j{}_k \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij}.\phi \\
&\quad + 2\sigma^{ij}\omega_{kl} \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij}.\phi \\
&= 4\imath X^2(\alpha \otimes \sigma^{ij} e_{ij}.\phi) + 4X(\sigma^j{}_k \epsilon^k \wedge \alpha \otimes e_j.\phi) \\
&= (4\imath X^2 \Theta^{\sigma} + 4X\Sigma^{\sigma})\psi,
\end{aligned}$$

where we have used the relation (1) in the second last step. Extending the result by linearity, we get the statement of this item for an arbitrary $\psi \in \Omega^{\bullet}(M, \mathcal{S})$.

(2) Using the formula for $R^{\Omega^\bullet(M,\mathcal{S})}$ derived in the previous item, the Proposition 5, the $\tilde{G}$-equivariance of X (Lemma 3 item 1) and the decomposition structure of $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ (see the Remark below the Theorem 2), we see that for $(i, j) \in \Xi$ and an element $\psi \in \Gamma(M, \mathcal{E}^{ij})$, the section $R^{\Omega^\bullet(M,\mathcal{S})}\psi \in \Gamma(M, \mathcal{E}^{i+2,j-1} \oplus \mathcal{E}^{i+2,j} \oplus \mathcal{E}^{i+2,j+1})$. Thus especially, $p^{i+2,m_{i+2}} R^{\Omega^\bullet(M,\mathcal{S})}\psi = 0$ for $i = 0, \dots, l-2, l, \dots, 2l-2$ and $\psi \in \Gamma(M, \mathcal{E}^{im_i})$. For $i = 0, \dots, l-2$, we get

$$\begin{aligned} 0 &= p^{i+2,i+2} R^{\Omega^\bullet(M,\mathcal{S})} = p^{i+2,i+2} d^{\nabla^S} d^{\nabla^S} \\ &= p^{i+2,i+2} d^{\nabla^S} (p^{i+1,0} + \cdots + p^{i+1,i+1}) d^{\nabla^S} \\ &= p^{i+2,i+2} d^{\nabla^S} p^{i+1,0} d^{\nabla^S} + \cdots + p^{i+2,i+2} d^{\nabla^S} p^{i+1,i+1} d^{\nabla^S} \\ &= T_{i+1} T_i, \end{aligned}$$

where we have used the Theorem 7 in the last step. Similarly, one proceeds in the case $i = l, \dots, 2l-2$. ◻

Corollary 10 *Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. If $l \geq 2$ and the symplectic Weyl tensor field $W^\nabla = 0$, then*

$$\begin{aligned} 0 \longrightarrow \Gamma\left(M, \mathcal{E}^{00}\right) \xrightarrow{T_0} \cdots \xrightarrow{T_{l-2}} \Gamma\left(M, \mathcal{E}^{l-1,l-1}\right) \xrightarrow{T_l T_{l-1}} \\ \xrightarrow{T_l T_{l-1}} \Gamma\left(M, \mathcal{E}^{l+1,l+1}\right) \xrightarrow{T_{l+1}} \cdots \xrightarrow{T_{2l-1}} \Gamma\left(M, \mathcal{E}^{2l,2l}\right) \longrightarrow 0 \end{aligned}$$

is a complex.

Proof Follows easily from the Theorem 9. ◻

The question of the existence of a symplectic connection with vanishing symplectic Weyl curvature tensor field was treated, e.g., in Cahen et al. [2]. These connections are called connections of Ricci type. For instance it is known that if a compact simply connected symplectic manifold (M, ω) admits a connection of Ricci type, then (M, ω) is affinely symplectomorphic to a $\mathbb{P}^n\mathbb{C}$ equipped with the symplectic form, given by the standard complex structure and the Fubini-Study metric, and the Levi-Civita connection of this metric. Let us refer an interested reader to the paper of Cahen et al. [3], where also a relation of symplectic connections to contact projective geometries is treated.

Further research could be devoted to the investigation and the interpretation of the cohomology of the introduced complex and to the investigation of analytic properties of the symplectic twistor operators.

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ELLIPTICITY OF THE SYMPLECTIC TWISTOR COMPLEX

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Abstract. For a Fedosov manifold (symplectic manifold equipped with a symplectic torsion-free affine connection) admitting a metaplectic structure, we shall investigate two sequences of first order differential operators acting on sections of certain infinite rank vector bundles defined over this manifold. The differential operators are symplectic analogues of the twistor operators known from Riemannian or Lorentzian spin geometry. It is known that the mentioned sequences form complexes if the symplectic connection is of Ricci type. In this paper, we prove that certain parts of these complexes are elliptic.

1. Introduction

In this article, we prove the ellipticity of certain parts of the so called symplectic twistor complexes. The symplectic twistor complexes are two sequences of first order differential operators defined over Ricci type Fedosov manifolds admitting a metaplectic structure. The mentioned parts of these complexes will be called truncated symplectic twistor complexes and will be defined later in this text.

Now, let us say a few words about the Fedosov manifolds. Formally speaking, a Fedosov manifold is a triple (M^{2l}, ω, ∇) where (M^{2l}, ω) is a (for definiteness $2l$ dimensional) symplectic manifold and ∇ is a symplectic torsion-free affine connection. Connections satisfying these two properties are usually called *Fedosov connections* in honor of Boris Fedosov who used them to obtain a deformation quantization for symplectic manifolds. (See Fedosov [5].) Let us also mention that in contrary to torsion-free Levi-Civita connections, the Fedosov ones are not unique. We refer an interested reader to Tondeur [18] and Gelfand, Retakh, Shubin [6] for more information.

To formulate the result on the ellipticity of the truncated symplectic twistor complexes, one should know some basic facts on the structure of the curvature tensor field of a Fedosov connection. In Vaisman [19], one can find a proof of a theorem which says that such curvature tensor field splits into two parts if $l \geq 2$, namely into the symplectic Ricci and symplectic Weyl curvature tensor fields. If

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$l = 1$, only the symplectic Ricci curvature tensor field occurs. Fedosov manifolds with zero symplectic Weyl curvature are usually called of Ricci type. (See also Cahen, Schwachhöfer [3] for another but related context.)

After introducing the underlying geometric structure, let us start describing the fields on which the differential operators from the symplectic twistor complexes act. These fields are certain exterior differential forms with values in the so called symplectic spinor bundle which is an associated vector bundle to the *metaplectic bundle*. We shall introduce the metaplectic bundle briefly now. Because the first homotopy group of the symplectic group $Sp(2l,\mathbb{R})$ is isomorphic to $\mathbb{Z}$, there exists a connected two-fold covering of this group. The covering space is called the metaplectic group, and it is usually denoted by $Mp(2l,\mathbb{R})$. Let us fix an element of the isomorphism class of all connected $2:1$ coverings of $Sp(2l,\mathbb{R})$ and denote it by λ. In particular, the mapping $\lambda\colon Mp(2l,\mathbb{R}) \to Sp(2l,\mathbb{R})$ is a Lie group homomorphism, and in this case it is also a Lie group representation. A metaplectic structure on a symplectic manifold (M^{2l},ω) is a notion parallel to that of a spin structure known from Riemannian geometry. In particular, one of its part is a principal $Mp(2l,\mathbb{R})$-bundle $\mathcal{Q}$ covering twice the bundle of symplectic repères $\mathcal{P}$ on (M,ω). This principal $Mp(2l,\mathbb{R})$-bundle is the mentioned metaplectic bundle and will be denoted by $\mathcal{Q}$ in this paper.

As we have already written, the fields we are interested in are certain exterior differential forms on M^{2l} with values in the *symplectic spinor bundle* which is a vector bundle over M associated to the chosen principal $Mp(2l,\mathbb{R})$-bundle $\mathcal{Q}$ via an 'analytic derivate' of the Segal-Sahle-Weil representation. The Segal-Shale-Weil representation is a faithful unitary representation of the metaplectic group $Mp(2l,\mathbb{R})$ on the vector space $L^2(\mathbb{L})$ of complex valued square Lebesgue integrable functions defined on a Lagrangian subspace $\mathbb{L}$ of the canonical symplectic vector space $(\mathbb{R}^{2l},\omega_0)$. For technical reasons, we shall use the so called Casselman-Wallach globalization of the underlying Harish-Chandra $(\mathfrak{g},\tilde{K})$-module of the Segal-Shale-Weil representation. Here, $\mathfrak{g}$ is the Lie algebra of the metaplectic group $\tilde{G}$ and $\tilde{K}$ is a maximal compact subgroup of the group $\tilde{G}$. The vector space carrying this globalization is the Schwartz space $\mathbf{S} := \mathcal{S}(\mathbb{L})$ of smooth functions on $\mathbb{L}$ rapidly decreasing in infinity with its usual Fréchet topology. This Schwartz space is the 'analytic derivate' mentioned above. We shall denote the resulting representation of $Mp(2n,\mathbb{R})$ on $\mathbf{S}$ by L and call it the *metaplectic representation*, i.e., we have $L\colon Mp(2l,\mathbb{R}) \to \mathrm{Aut}(\mathbf{S})$. Let us mention that $\mathbf{S}$ decomposes into two irreducible $Mp(2l,\mathbb{R})$-submodules $\mathbf{S}_+$ and $\mathbf{S}_-$, i.e., $\mathbf{S} = \mathbf{S}_+ \oplus \mathbf{S}_-$. The elements of $\mathbf{S}$ are usually called *symplectic spinors*. See Kostant [11] who used them in the context of geometric quantization.

The underlying algebraic structure of the symplectic spinor valued exterior differential forms is the vector space $\mathbf{E} := \bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S} = \bigoplus_{r=0}^{2l} \bigwedge^r(\mathbb{R}^{2l})^* \otimes \mathbf{S}$. Obviously, this vector space is equipped with the following tensor product representation ρ of the metaplectic group $Mp(2l,\mathbb{R})$. For $r = 0,\ldots,2l$, $g \in Mp(2l,\mathbb{R})$ and $\alpha \otimes s \in \bigwedge^r(\mathbb{R}^{2l})^* \otimes \mathbf{S}$, we set $\rho(g)(\alpha \otimes s) := \lambda(g)^{*\wedge r}\alpha \otimes L(g)s$ and extend this prescription by linearity. With this notation in mind, the symplectic spinor valued exterior differential forms are sections of the vector bundle $\mathcal{E}$ associated to the chosen principal $Mp(2l,\mathbb{R})$-bundle $\mathcal{Q}$ via ρ, i.e., $\mathcal{E} := \mathcal{Q} \times_\rho \mathbf{E}$. Now, we shall restrict our attention to the mentioned *specific* symplectic spinor valued exterior differential forms. For each $r = 0, \dots, 2l$, there exists a distinguished irreducible submodule of $\bigwedge^r (\mathbb{R}^{2l})^* \otimes \mathbf{S}_\pm$ which we denote by $\mathbf{E}^r_\pm$. Actually, the submodules $\mathbf{E}^r_\pm$ are the Cartan components of $\bigwedge^r (\mathbb{R}^{2l})^* \otimes \mathbf{S}_\pm$, i.e., the highest weight of each of them is the largest one of the highest weights of all irreducible constituents of $\bigwedge^r (\mathbb{R}^{2l})^* \otimes \mathbf{S}_\pm$ wrt. the standard choices. For $r = 0, \dots, 2l$, we set $\mathbf{E}^r := \mathbf{E}^r_+ \oplus \mathbf{E}^r_-$ and $\mathcal{E}^r := \mathcal{Q} \times_\rho \mathbf{E}^r$. Further, let us denote the corresponding $Mp(2l,\mathbb{R})$-equivariant projection from $\bigwedge^r (\mathbb{R}^{2l})^* \otimes \mathbf{S}$ onto $\mathbf{E}^r$ by p^r. We denote the lift of the projection p^r to the associated (or 'geometric') structures by the same symbol, i.e., $p^r \colon \Gamma(M, \mathcal{Q} \times_\rho (\bigwedge^r (\mathbb{R}^{2l})^* \otimes \mathbf{S})) \to \Gamma(M, \mathcal{E}^r)$.

Now, we are in a position to define the main subject of our investigation, namely the symplectic twistor complexes. Let us consider a Fedosov manifold (M, ω, ∇) and suppose that (M, ω) admits a metaplectic structure. Let d^{∇^S} be the exterior covariant derivative associated to ∇. For each $r = 0, \dots, 2l$, let us restrict the associated exterior covariant derivative d^{∇^S} to $\Gamma(M, \mathcal{E}^r)$ and compose the restriction with the projection p^{r+1}. The resulting operator, denoted by T_r, will be called *symplectic twistor operator*. In this way, we obtain two sequences of differential operators, namely $0 \longrightarrow \Gamma(M, \mathcal{E}^0) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^1) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-1}} \Gamma(M, \mathcal{E}^l) \longrightarrow 0$ and $0 \longrightarrow \Gamma(M, \mathcal{E}^l) \xrightarrow{T_l} \Gamma(M, \mathcal{E}^{l+1}) \xrightarrow{T_{l+1}} \cdots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l}) \longrightarrow 0$. It is known, see Krýsl [14], that these sequences form complexes provided the Fedosov manifold (M^{2l}, ω, ∇) is of Ricci type. These two complexes are the mentioned *symplectic twistor complexes*. Let us notice, that we did not choose the full sequence of all symplectic spinor valued exterior differential forms together with the exterior covariant derivative acting between them because for a general or even Ricci type Fedosov manifold, this sequence would not form a complex in general.

As we have mentioned, we shall prove that some parts of these two complexes are elliptic. To obtain these parts, one should remove the last (i.e., the zero) term and the second last term from the first complex and the first term (the zero space again) from the second complex. The complexes obtained in this way will be called truncated symplectic twistor complexes. Let us mention that by an elliptic complex, we mean a complex of differential operators such that its associated symbol sequence is an exact sequence of the sheaves in question. (See, e.g., Wells [21] for details.)

Let us make some remarks on the methods we have used to prove the ellipticity of the truncated symplectic twistor complexes. We decided to use the so called Schur-Weyl-Howe correspondence, which is referred to as the Howe correspondence for simplicity in this text. The Howe correspondence in our case, i.e., for the metaplectic group $Mp(2l,\mathbb{R})$ acting on the space $\mathbf{E}$ of symplectic spinor valued exterior forms, leads to the ortho-symplectic super Lie algebra $\mathfrak{osp}(1|2)$ and a certain representation of this algebra on $\mathbf{E}$. We decided to use the Howe type correspondence mainly because the spaces $\mathbf{E}^r$ (defined above) can be characterized via the mentioned representation of $\mathfrak{osp}(1|2)$ easily and in a way described in this paper. See R. Howe [10] for more information on the Howe type correspondence in general. Let us also mention that besides this duality, the Cartan lemma on exterior differential forms was used. For other examples of elliptic complexes, we refer an interested reader, e.g., to Stein and Weiss [17], Schmid [15], Hotta [9], and Branson [2].

For an application of symplectic spinors in mathematical physics, see, e.g., Shale [16] and Green, Hull [7] and the already mentioned article of Kostant [11]. In the first reference, one can find an application of these spinors in quantizing of Klein-Gordon fields and in the second one in the 10 dimensional super-string theory. The purpose for taking symplectic spinor valued forms might be justified by the intention to describe higher spin boson fields.

In the second section, we recall some known facts on symplectic spinors and the space of symplectic spinor valued exterior forms and its decomposition into irreducible submodules (Theorem 1). In the third chapter, basic information on Fedosov manifolds and their curvature are mentioned and the symplectic twistor complexes are introduced. In the fourth section, the symbol sequence of the symplectic twistor complexes is computed and the ellipticity of the truncated symplectic twistor complexes is proved (Theorem 7).

2. Symplectic spinor valued forms

In this paper the Einstein summation convention is used for finite sums, not mentioning it explicitly unless otherwise is stated. (We will not use this convention in the proof of the Lemma 6 and in the item 3 of the proof of the Theorem 7 only.) The category of representations of Lie groups we shall consider is that one the object of which are finite length admissible representations of a fixed reductive group G on Fréchet vector spaces and the morphisms are continuous G-equivariant maps between the objects. All manifolds, vector bundles and their sections in this text are supposed to be smooth. The only manifolds which are allowed to be of infinite dimension are the total spaces of vector bundles. If this is the case, the bundles are supposed to be Fréchet. The base manifolds are always finite dimensional. The sheaves we will consider are sheaves of smooth sections of vector bundles. If $E \to M$ is a Fréchet vector bundle, we denote the sheaf of sections by Γ, i.e., $\Gamma(U) := \Gamma(U, E)$ for each open set U in M. For $m \in M$, we denote the stalk of Γ at m by Γ_m.

2.1. *Symplectic linear algebra and basic notation.* In order to set the notation, let us start recalling some simple results from symplectic linear algebra. Let $(\mathbb{V}, \omega_0)$ be a real symplectic vector space of dimension $2l$, $l \geq 1$. Let us choose two Lagrangian subspaces $\mathbb{L}$ and $\mathbb{L}'$, such that $\mathbb{V} \simeq \mathbb{L} \oplus \mathbb{L}'$ [1]. It is easy to see that $\dim \mathbb{L} = \dim \mathbb{L}' = l$. Further, let us choose an adapted symplectic basis $\{e_i\}_{i=1}^{2l}$ of $(\mathbb{V} \simeq \mathbb{L} \oplus \mathbb{L}', \omega_0)$, i.e., $\{e_i\}_{i=1}^{2l}$ is a symplectic basis of $(\mathbb{V}, \omega_0)$ and $\{e_i\}_{i=1}^{l} \subseteq \mathbb{L}$ and $\{e_i\}_{i=l+1}^{2l} \subseteq \mathbb{L}'$. The basis dual to the basis $\{e_i\}_{i=1}^{2l}$ will be denoted by $\{\epsilon^i\}_{i=1}^{2l}$, i.e., for i, $j = 1, \ldots, 2l$ we have $\epsilon^j(e_i) = \iota_{e_i}\epsilon^j = \delta_i^j$, where $\iota_v \alpha$ for an element $v \in \mathbb{V}$ and an exterior form $\alpha \in \bigwedge^\bullet \mathbb{V}^*$, denotes the contraction of the form α by the vector v. Further for $i, j = 1, \ldots, 2l$, we set $\omega_{ij} := \omega_0(e_i, e_j)$ and define ω^{ij}, i, $j = 1, \ldots, 2l$, by

[1] Let us recall that by Lagrangian, we mean maximal isotropic wrt. ω_0.

the equation $\omega_{ij}\omega^{kj} = \delta_i^k$ for all $i,\ k = 1,\dots,2l$. Let us remark that not only $\omega_{ij} = -\omega_{ji}$, but also $\omega^{ij} = -\omega^{ji}$ for $i,\ j = 1,\dots,2l$.

As in the Riemannian case, we would like to rise and lower indices of tensor coordinates. In the symplectic case, one should be more careful because of the anti-symmetry of ω_0. For coordinates $K_{ab\dots c\dots d}{}^{rs\dots t\dots u}$ of a tensor K over $\mathbb{V}$. we denote the expression $\omega^{ic}K_{ab\dots c\dots d}{}^{rs\dots t}$ by $K_{ab\dots}{}^{i}{}_{\dots d}{}^{rs\dots t}$ and $K_{ab\dots c}{}^{rs\dots t\dots u}\omega_{ti}$ by $K_{ab\dots c}{}^{rs\dots}{}_{i}{}^{\dots u}$ and similarly for other types of tensors and also in the geometric setting when we will be considering tensor fields over a symplectic manifold (M^{2l},ω). Let us remark that $\omega_i{}^j = -\omega^j{}_i = \delta_i^j$, $i,\ j = 1,\dots,2l$. Further, one can also define an isomorphism $\sharp\colon \mathbb{V}^* \to \mathbb{V}, \mathbb{V}^* \ni \alpha \mapsto \alpha^\sharp \in \mathbb{V}$, by the formula

$$\alpha(w) = \omega_0(\alpha^\sharp, w) \quad \text{for each} \quad \alpha \in \mathbb{V}^* \quad \text{and} \quad w \in \mathbb{V}\,.$$

For $\alpha = \alpha_i\epsilon^i$ and $j = 1,\dots,2l$, we get $\alpha_j = \alpha(e_j) = \omega_0((\alpha^\sharp)^i e_i, e_j) = \omega_{ij}(\alpha^\sharp)^i = (\alpha^\sharp)_j$ which implies $\alpha^\sharp = (\alpha^\sharp)^i e_i = \alpha^i e_i$. Thus, we see that the rising of indices via the form ω_0 is realized by the isomorphism $\sharp$.

Finally, let us introduce the groups we will be using. Let us denote the symplectic group of $(\mathbb{V},\omega_0)$ by G, i.e., $G := Sp(\mathbb{V},\omega_0) \simeq Sp(2l,\mathbb{R})$. Because the fundamental group of $G = Sp(\mathbb{V},\omega_0)$ is $\mathbb{Z}$, there exists a connected $2\colon 1$, necessarily non-universal, covering of G by the so called metaplectic group $Mp(\mathbb{V},\omega_0)$ denoted by $\tilde{G}$ in this text. Let us denote the mentioned two-fold covering map by λ, in particular $\lambda\colon \tilde{G} \to G$. (See, e.g., Habermann, Habermann [8].)

2.2. *Segal-Shale-Weil representation and symplectic spinor valued forms.* The Segal-Shale-Weil representation is a distinguished representation of the metaplectic group $\tilde{G} = Mp(\mathbb{V},\omega_0)$.[2] This representation is unitary, faithful and does not descend to a representation of the symplectic group. Its underlying vector space is the vector space of complex valued square Lebesgue integrable functions $L^2(\mathbb{L})$ defined on the chosen Lagrangian subspace $\mathbb{L}$. Let us set $\mathbf{S} := V^\infty(HC(L^2(\mathbb{L})))$, where V^∞ is the Casselman-Wallach globalization functor and HC denotes the forgetful Harish-Chandra functor from the category of $\tilde{G}$-modules defined above into the category of Harish-Chandra $(\mathfrak{g},\tilde{K})$-modules[3]. We shall denote the resulting representation by L and call it the *metaplectic representation.* Thus, we have

$$L\colon\ Mp(\mathbb{V},\omega_0) \to \mathrm{Aut}(\mathbf{S})\,.$$

The elements of $\mathbf{S}$ will be called *symplectic spinors.* It is well known that $\mathbf{S}$ splits into two irreducible $Mp(\mathbb{V},\omega_0)$-submodules $\mathbf{S}_+$ and $\mathbf{S}_-$. Thus, we have $\mathbf{S} = \mathbf{S}_+ \oplus \mathbf{S}_-$. See the foundational paper of A. Weil [20] for more detailed information on the Segal-Shale-Weil representation and Casselman [4] on this type of globalization. Let us mention that choosing this particular globalization seems to be rather technical from the point of view of the aim of our article.

In the proof of the ellipticity of the truncated symplectic twistor complexes, we shall need some facts on the underlying vector space of the metaplectic representation. Let us mention that it is known that $\mathbf{S}$ is isomorphic to the Schwartz

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[2]The names oscillator and metaplectic are also used in the literature. See, e.g., Howe [10].

[3]Here, $\mathfrak{g}$ is the Lie algebra of $\tilde{G}$ and $\tilde{K}$ is the maximal compact Lie subgroup of $\tilde{G}$.

space $\mathcal{S}(\mathbb{L})$ of smooth functions rapidly decreasing in the infinity equipped with the standard (locally convex) Fréchet topology generated by the supremum semi-norms. (See, e.g., Habermann, Habermann [8] or Borel, Wallach [1].) For the convenience of the reader, let us briefly recall the definition of the involved semi-norms. For each a, $b \in \mathbb{N}_0^l$, the semi-norm $q_{a,b}$ is defined by the formula $q_{a,b}(f) := \sup_{x\in\mathbb{L}}|(x^a\partial^b f)(x)|$, $f \in \mathcal{S}(\mathbb{L})$. Let us order the set $(q_{a,b})_{a,b}$ in the standard 'lexicographical' way and denote the resulting sequence of semi-norms by $(q^k)_{k\in\mathbb{N}_0}$. These semi-norms generate a complete metric topology on $\mathcal{S}(\mathbb{L})$. Taking $a = b = 0$, one sees that the convergence with respect to the semi-norms implies the uniform convergence immediately. Further, it is well known that the Schwartz space $\mathcal{S}(\mathbb{L})$ possesses a Schauder basis. For a complex metric (e.g., Fréchet) space F, an ordered countable set $(f_i)_{i\in\mathbb{N}} \subseteq F$ is called a Schauder basis of F if each element $f \in F$ can be uniquely expressed as $f = \sum_{i=1}^\infty a_i f_i$ for some $a_i \in \mathbb{C}$. Notice that from the uniqueness of the coefficients a_i immediately follows that $0 = \sum_{i=1}^\infty a_i f_i$ implies $a_i = 0$ for all $i \in \mathbb{N}$. From the basic mathematical analysis courses, one knows that in the case of the Schwartz space $\mathcal{S}(\mathbb{L})$, one can take, e.g., the lexicographically ordered sequence of Hermite functions in l variables as the Schauder basis. We denote this basis by $(h_i)_{i\in\mathbb{N}}$.

Now, we may define the so called *symplectic Clifford multiplication* $\cdot\colon \mathbb{V}\times\mathbf{S} \to \mathbf{S}$. For $s \in \mathbf{S}$, $x = x^j e_j \in \mathbb{L}$, $x^j \in \mathbb{R}$ and $i, j = 1, \dots, l$, let us set

$$e_i \cdot s(x) := \imath x^i s(x) \quad \text{and} \quad e_{i+l} \cdot s(x) := \frac{\partial s}{\partial x^i}(x)\,.$$

In physics, this mapping (up to a constant multiple) is usually called the canonical quantization. Let us remark that the definition is correct due to the preceding paragraph. For each v, $w \in \mathbb{V}$ and $s \in \mathbf{S}$, one can easily derive the following commutation relation

$$v\cdot w\cdot s - w\cdot v\cdot s = -\imath\omega_0(v,w)s\,. \tag{1}$$

(See, e.g., Habermann, Habermann [8].) We shall use this relation repeatedly and without mentioning its use. Now, we prove that the symplectic Clifford multiplication by a fixed non-zero vector $v \in \mathbb{V}$ is injective as a mapping from $\mathbf{S}$ into $\mathbf{S}$. We shall use the $\tilde{G}$-equivariance of the symplectic Clifford multiplication, i.e., the fact $L(g)(v\cdot s) = [\lambda(g)v]\cdot L(g)s$ which holds for each $g \in \tilde{G}$, $v \in \mathbb{V}$ and $s \in \mathbf{S}$ (see Habermann, Habermann [8]). Thus, let us suppose that a fixed $s \in \mathbf{S}$ and a fixed $0 \neq v \in \mathbb{V}$ are given such that $v \cdot s = 0$. Because the action of the symplectic group G on $\mathbb{V} - \{0\}$ is transitive and λ is a covering, there exists an element $g \in \tilde{G}$ such that $\lambda(g)v = e_1$. Applying $L(g)$ on the equation $v\cdot s = 0$, we get $L(g)(v\cdot s) = 0$. Using the above mentioned equivariance of the symplectic Clifford multiplication, we get $0 = L(g)(v\cdot s) = [\lambda(g)v]\cdot(L(g)s) = e_1\cdot(L(g)s)$. Denoting $L(g)s =: \psi$ and using the definition of the symplectic Clifford multiplication, we obtain $\imath x^1\psi = 0$, which implies $\psi(x) = 0$ for each $x = (x^1, \dots, x^l) \in \mathbb{L}$ such that $x^1 \neq 0$. By continuity of $\psi \in \mathbf{S}$, we get $\psi = 0$. Because L is a group representation, we get $s = 0$ from $0 = \psi = L(g)s$, i.e., the injectivity of the symplectic Clifford multiplication.

Having defined the metaplectic representation and the symplectic Clifford multiplication, we shall introduce the underlying algebraic structure of the basic geometric object we are interested in, namely the space $\mathbf{E} := \bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ of symplectic spinor valued exterior forms. The vector space $\mathbf{E}$ is considered with its canonical (Fréchet) direct sum topology induced by the metric topology on the (finite dimensional) space of exterior forms and the Fréchet topology on $\mathbf{S}$. The metaplectic group $\tilde{G}$ acts on $\mathbf{E}$ by the representation

$$\rho\colon \tilde{G} \to \mathrm{Aut}(\mathbf{E}) \quad \text{defined by the formula}$$

$$\rho(g)(\alpha \otimes s) := (\lambda(g)^*)^{\wedge r} \alpha \otimes L(g)s\,,$$

where $\alpha \in \bigwedge^r \mathbb{V}^*$, $s \in \mathbf{S}$, $r = 0, \dots, 2l$, and it is extended by linearity also for non-homogeneous elements.

For $\psi = \alpha \otimes s \in \mathbf{E}$, $v \in \mathbb{V}$ and $\beta \in \bigwedge^{\bullet} \mathbb{V}^*$, we set $\iota_v \psi := \iota_v \alpha \otimes s$, $\beta \wedge \psi := \beta \wedge \alpha \otimes s$ and $v \cdot \psi := \alpha \otimes v \cdot s$ and extend these definitions by linearity to non-homogeneous elements. Obviously, the contraction, the exterior multiplication and the Clifford multiplication by a fixed vector or co-vector are continuous on $\mathbf{E}$.

Now, we shall describe the decomposition of the space $\mathbf{E}$ into irreducible $\tilde{G}$-submodules. For $i = 0, \dots, l$, let us set $m_i := i$, and for $i = l+1, \dots 2l$, $m_i := 2l - i$, and define the set Ξ of pairs of non-negative integers

$$\Xi := \big\{(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid i = 0, \dots, 2l,\ j = 0, \dots, m_i\big\}\,.$$

One can say the set Ξ has a shape of a triangle if visualized in a 2-plane. (See the Figure 1. below.) We use the elements of Ξ for parameterizing the irreducible submodules of $\mathbf{E}$.

In Krýsl [12] for each $(i,j) \in \Xi$, two irreducible $\tilde{G}$-modules $\mathbf{E}^{ij}_{\pm}$ were uniquely defined via the highest weights of their underlying Harish-Chandra modules and by the fact that they are irreducible submodules of $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}_{\pm}$. For convenience for each $(i,j) \in \mathbb{Z} \times \mathbb{Z} \setminus \Xi$, we set $\mathbf{E}^{ij}_{\pm} := 0$, and for each $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, we define $\mathbf{E}^{ij} := \mathbf{E}^{ij}_{+} \oplus \mathbf{E}^{ij}_{-}$.

In the following theorem, the decomposition of $\mathbf{E}$ into irreducible $\tilde{G}$-submodules is described.

Theorem 1. *For $r = 0, \dots, 2l$, the following decomposition into irreducible $\tilde{G}$-modules*

$$\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}_{\pm} \simeq \bigoplus_{\substack{j \\ (r,j)\in\Xi}} \mathbf{E}^{rj}_{\pm} \quad \textit{holds}\,.$$

Proof. See Krýsl [12]. □

The following remark on the multiplicity structure of the module $\mathbf{E}$ is crucial. It follows from the prescriptions for the highest weights of the underlying Harish-Chandra modules of $\mathbf{E}^{ij}_{\pm}$ (see Krýsl [13]).

Remark. 1. For any $(r,j), (r,k) \in \Xi$ such that $j \neq k$, we have

$$\mathbf{E}^{rj}_{\pm} \not\simeq \mathbf{E}^{rk}_{\pm}$$

$$\begin{array}{ccccccc}
\mathbf{E}^{00}_{\pm} & \mathbf{E}^{10}_{\pm} & \mathbf{E}^{20}_{\pm} & \mathbf{E}^{30}_{\pm} & \mathbf{E}^{40}_{\pm} & \mathbf{E}^{50}_{\pm} & \mathbf{E}^{60}_{\pm} \\
 & \mathbf{E}^{11}_{\pm} & \mathbf{E}^{21}_{\pm} & \mathbf{E}^{31}_{\pm} & \mathbf{E}^{41}_{\pm} & \mathbf{E}^{51}_{\pm} & \\
 & & \mathbf{E}^{22}_{\pm} & \mathbf{E}^{32}_{\pm} & \mathbf{E}^{42}_{\pm} & & \\
 & & & \mathbf{E}^{33}_{\pm} & & &
\end{array}$$

FIG. 1: Decomposition of $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}_{\pm}$ for $2l = 6$.

(any combination of $\pm$ at both sides of the preceding relation is allowed). Thus in particular, $\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$ is multiplicity-free for each $r = 0, \ldots, 2l$.

2. Moreover, it is known that $\mathbf{E}^{rj}_{\pm} \simeq \mathbf{E}^{sj}_{\mp}$ for each $(r,j), (s,j) \in \Xi$. One cannot change the order of $+$ and $-$ at precisely one side of the preceding isomorphism without changing its trueness.
3. From the preceding two items, one gets immediately that there are no submodules of $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}$ isomorphic to $\mathbf{E}^{i+1,i+1}_{\pm}$ for each $i = 0, \ldots, l-1$.

In the Figure 1, one can see the decomposition structure of $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}_{\pm}$ in the case of $l = 3$. For $i = 0, \ldots, 6$, the i^{th} column constitutes of the irreducible modules in which the $\mathbf{S}_{\pm}$-valued exterior forms of form-degree i decompose.

In the next theorem, the decomposition of $\mathbb{V}^* \otimes \mathbf{E}^{ij}$, $(i,j) \in \Xi$, into irreducible $\tilde{G}$-submodules is described. Let us remind the reader that due to our convention $\mathbf{E}^{ij} = 0$ for $(i,j) \in \mathbb{Z} \times \mathbb{Z} \setminus \Xi$. We will use this theorem in the proofs of Lemma 6 and Theorem 7 on the ellipticity of the truncated symplectic twistor complexes.

Theorem 2. *For $(i,j) \in \Xi$, we have*

$$(\mathbb{V}^* \otimes \mathbf{E}^{ij}) \cap (\bigwedge^{i+1} \mathbb{V}^* \otimes \mathbf{S}) \simeq \mathbf{E}^{i+1,j-1} \oplus \mathbf{E}^{i+1,j} \oplus \mathbf{E}^{i+1,j+1} \, .$$

Proof. See Krýsl [13]. $\square$

Remark. Roughly speaking, the theorem says that the wedge multiplication sends each irreducible module $\mathbf{E}^{ij}$ into at most three "neighbor" modules in the $(i+1)^{st}$ column. (See the Figure 1.)

2.3. *Operators related to a Howe type correspondence***.** In this section, we will introduce five continuous linear operators acting on the space $\mathbf{E}$ of symplectic spinor valued exterior forms. Let us mention that these operators are related to the so called Howe type correspondence for the metaplectic group $Mp(\mathbb{V}, \omega_0)$ acting on

$\mathbf{E}$ via the representation ρ. For $r = 0, \dots, 2l$ and $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, we set

$$F^+ \colon \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S}\,, \qquad F^+(\alpha \otimes s) := \frac{\imath}{2} \sum_{i=1}^{2l} \epsilon^i \wedge \alpha \otimes e_i \cdot s$$

and

$$F^- \colon \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r-1} \mathbb{V}^* \otimes \mathbf{S}\,, \qquad F^-(\alpha \otimes s) := \frac{1}{2} \sum_{i=1}^{2l} \omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot s$$

and extend them linearly. Further, we shall introduce the operators H, E^+ and E^- acting also continuously on the space $\mathbf{E} = \bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$. We define

$$H := 2\{F^+, F^-\} \quad \text{and} \quad E^\pm := \pm 2\{F^\pm, F^\pm\}\,,$$

where $\{\,,\}$ denotes the anti-commutator in the associative algebra $\mathrm{End}(\mathbf{E})$. By a direct computation, we get

$$E^-(\alpha \otimes s) = \frac{\imath}{2} \omega^{ij} \iota_{e_i} \iota_{e_j} \alpha \otimes s \tag{2}$$

for any $\alpha \otimes s \in \bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$. Thus, we see that the operator E^- acts on the form-part of a symplectic spinor valued exterior form only. Because of that we will write $E^- \alpha \otimes s$ instead of $E^-(\alpha \otimes s)$ simply.

In the next lemma, we sum-up some known facts and derive some new information on the operators $F^\pm$, $E^\pm$ and H which we shall need in the proof of the ellipticity of the truncated symplectic twistor complexes.

Lemma 3. 1. *The operators $F^\pm$, $E^\pm$ and H are $\tilde{G}$-equivariant.*

2. *For $i = 0, \dots, l$, the operator $F^-_{|\mathbf{E}^{im_i}} = 0$ and for $i = l, \dots, 2l$, the operator $F^+_{|\mathbf{E}^{im_i}} = 0$.*
3. *The associative algebra $End_{\tilde{G}}(\mathbf{E}) := \{A \colon \mathbf{E} \to \mathbf{E}$ continuous $\mid A\rho(g) = \rho(g)A$ for all $g \in \tilde{G}\}$ is, as an associative algebra, finitely generated by F^+ and F^- and the $\tilde{G}$-equivariant projections $p_\pm \colon \mathbf{S} \to \mathbf{S}_\pm$.*
4. *For $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, the following relations hold on $\mathbf{E}$*

$$[E^+, E^-] = H\,, \qquad [E^-, F^+] = -F^-\,, \tag{3}$$

$$H(\alpha \otimes s) = \frac{1}{2}(r - l)\alpha \otimes s\,, \tag{4}$$

$$\{F^+, \iota_v\}(\alpha \otimes s) = \frac{\imath}{2} \alpha \otimes v \cdot s \quad \textit{and} \quad [F^-, v\cdot](\alpha \otimes s) = \frac{\imath}{2} \iota_v \alpha \otimes s\,. \tag{5}$$

Proof. See Krýsl [13] for the proof of the items 1 and 2, and Krýsl [12] for a proof of the item 3 and of the relations in the rows (3) and (4). Now, suppose we are given an element $v = v^i e_i \in \mathbb{V}$, $v^i \in \mathbb{R}$, $i = 1, \dots, 2l$, and a homogeneous element $\alpha \otimes s \in \bigwedge^j \mathbb{V}^* \otimes \mathbf{S}$, $j = 0, \dots, 2l$. First, let us prove the first relation in the row (5). Using the definition of F^+, we may write $\{F^+, \iota_v\}(\alpha \otimes s) = F^+(\iota_v \alpha \otimes s) + \frac{\imath}{2} \iota_v(\epsilon^i \wedge \alpha \otimes e_i \cdot s) = \frac{\imath}{2}[\epsilon^i \wedge \iota_v \alpha \otimes e_i \cdot s + v^i \alpha \otimes e_i \cdot s - \epsilon^i \wedge \iota_v \alpha \otimes e_i \cdot s] = \frac{\imath}{2} \alpha \otimes v \cdot s$. Thus, the first relation of (5) follows now by linearity. Now, let us prove the second relation at the row (5). Using the definition of F^- and the commutation relation (1), we get $F^-(\alpha\otimes v\cdot s) = \frac{1}{2}(\omega^{ij}\iota_{e_i}\alpha\otimes e_j\cdot v\cdot s) = \frac{1}{2}\omega^{ij}\iota_{e_i}\alpha\otimes(v\cdot e_j\cdot s - \imath\omega_0(e_j,v)s) = v\cdot F^-(\alpha\otimes s) + \frac{\imath}{2}\omega^{ij}\iota_{e_i}\alpha\otimes v_j s = v\cdot F^-(\alpha\otimes s) + \frac{\imath}{2}\iota_v\alpha\otimes s$. Thus, the second relation at the row (5) is proved. $\square$

Remark. The operators $F^{\pm}$, $E^{\pm}$ and H satisfy the commutation and anti-commutation relations identical to that ones which are satisfied by the usual generators of the ortho-symplectic super Lie algebra $\mathfrak{osp}(1|2)$.

3. Symplectic twistor complexes and their elliptic parts

In this section, we define the notion of a Fedosov manifold, recall some information on its curvature, introduce a symplectic analogue of the spin structure (the metaplectic structure) and define the symplectic twistor complexes.

Let (M,ω) be a symplectic manifold. Let us consider an affine torsion-free symplectic connection ∇ on (M,ω) and denote the induced connection on $\Gamma(M,\bigwedge^2 T^*M)$ by ∇ as well. Let us recall that by torsion-free and symplectic, we mean $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0$ for all $X,Y\in\mathfrak{X}(M)$ and $\nabla\omega = 0$. Such connections are usually called Fedosov connections, and the triple (M,ω,∇) a Fedosov manifold. See the Introduction and the references therein for more information on these connections. The curvature tensor R^∇ of a Fedosov connection is defined in the classical way, i.e., formally by the same formula as in the Riemannian geometry. It is known, see Vaisman [19], that R^∇ splits into two parts, namely into the extended symplectic Ricci and Weyl curvature tensor fields, here denoted by $\widetilde{\sigma}^\nabla$ and W^∇ respectively. Let us display the definitions of these two curvature parts although we shall not use them explicitly. For a symplectic frame $(U,\{e_i\}_{i=1}^{2l})$, $U\subseteq M$, we have the following local formulas

$$\begin{aligned}\sigma_{ij} &:= R^k{}_{ikj}\,,\\ 2(l+1)\widetilde{\sigma}^\nabla_{ijkn} &:= \omega_{in}\sigma_{jk} - \omega_{ik}\sigma_{jn} + \omega_{jn}\sigma_{ik} - \omega_{jk}\sigma_{in} + 2\sigma_{ij}\omega_{kn} \quad \text{and}\\ W^\nabla &:= R^\nabla - \widetilde{\sigma}^\nabla\,,\end{aligned}$$

where $i,j,k,n = 1,\dots,2l$. Let us call a Fedosov manifold (M,ω,∇) of *Ricci type* if $W^\nabla = 0$.

Remark. Because the Ricci curvature tensor field σ_{ij} is symmetric (see Vaisman [19]), a possible candidate for the scalar curvature, namely $\sigma^{ij}\omega_{ij}$, is zero.

Example. It is easy to see that each Riemann surface equipped with its volume form as the symplectic form and with the Riemann connection is a Fedosov manifold of Ricci type. Further for any $l\geq 1$, the Fedosov manifold $(\mathbb{CP}^l,\omega_{FS},\nabla)$ is also a Fedosov manifold of Ricci type. Here, ω_{FS} is the Kähler form associated to the Fubini-Study metric and to the complex structure on the complex projective space $\mathbb{CP}^l$, and ∇ is the Riemannian connection associated to the Fubini-Study metric.

Now, let us introduce the metaplectic structure the definition of which we have sketched briefly in the Introduction. For a symplectic manifold (M^{2l},ω) of dimension $2l$, let us denote the bundle of symplectic repères in TM by $\mathcal{P}$ and the foot-point projection from $\mathcal{P}$ onto M by p. Thus $(p\colon \mathcal{P} \to M, G)$, where $G \simeq Sp(2l, \mathbb{R})$, is a principal G-bundle over M. As in the subsection 2.1, let $\lambda\colon \tilde{G} \to G$ be a member of the isomorphism class of the non-trivial two-fold coverings of the symplectic group G. In particular, $\tilde{G} \simeq Mp(2l, \mathbb{R})$. Now, let us consider a principal $\tilde{G}$-bundle $(q\colon \mathcal{Q} \to M, \tilde{G})$ over the chosen symplectic manifold (M, ω). We call the pair $(\mathcal{Q}, \Lambda)$ metaplectic structure if $\Lambda\colon \mathcal{Q} \to \mathcal{P}$ is a surjective bundle morphism *compatible* with the actions of G on $\mathcal{P}$ and that of $\tilde{G}$ on $\mathcal{Q}$ and with the covering λ in the same way as in the Riemannian spin geometry. (For a more elaborate definition see, e.g., Habermann, Habermann [8].) Let us remark, that typical examples of symplectic manifolds admitting a metaplectic structure are cotangent bundles of orientable manifolds (phase spaces), Calabi-Yau manifolds and the complex projective spaces $\mathbb{CP}^{2k+1}$, $k \in \mathbb{N}_0$.

Now, let us denote the Fréchet vector bundle associated to the introduced principal $\tilde{G}$-bundle $(q : \mathcal{Q} \to M, \tilde{G})$ via the metaplectic representation L on $\mathbf{S}$ by $\mathcal{S}$. Thus, we have $\mathcal{S} = \mathcal{Q} \times_L \mathbf{S}$. We shall call this associated vector bundle $\mathcal{S} \to M$ the *symplectic spinor bundle*. The sections $\phi \in \Gamma(M, \mathcal{S})$ will be called *symplectic spinor fields*. Let us put $\mathcal{E} := \mathcal{Q} \times_\rho \mathbf{E}$. For $r = 0, \ldots, 2l$, we define $\mathcal{E}^r := \mathcal{Q} \times_\rho \mathbf{E}^r$, where $\mathbf{E}^r$ abbreviates $\mathbf{E}^{rm_r}$. The smooth sections $\Gamma(M, \mathcal{E})$ will be called symplectic spinor valued exterior differential forms. Because the operators $E^{\pm}$, $F^{\pm}$ and H are $\tilde{G}$-equivariant (see the Lemma 3 item 1), they lift to operators acting on sections of the corresponding associated vector bundles. The same is true about the projections p^{ij}, $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. We shall use the same symbols as for the mentioned operators as for their "lifts" to the associated vector bundle structure.

Now, we shall make a use of the Fedosov connection. The Fedosov connection ∇ determines the induced principal G-bundle connection on the principal bundle $(p\colon \mathcal{P} \to M, G)$. This connection lifts to a principal $\tilde{G}$-bundle connection on the principal bundle $(q\colon \mathcal{Q} \to M, \tilde{G})$ and defines the associated covariant derivative on the symplectic bundle $\mathcal{S}$, which we shall denote by ∇^S, and call it the symplectic spinor covariant derivative. See, e.g., Habermann, Habermann [8] for this classical construction. The symplectic spinor covariant derivative ∇^S induces the exterior covariant derivative d^{∇^S} acting on $\Gamma(M, \mathcal{E})$. For $r = 0, \ldots, 2l$, we have $d^{\nabla^S} : \Gamma(M, \mathcal{Q} \times_\rho (\bigwedge^r \mathbb{V}^* \otimes \mathbf{S})) \to \Gamma(M, \mathcal{Q} \times_\rho (\bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S}))$. Now, we are able to define the symplectic twistor operators. For $r = 0, \ldots, 2l$, we set

$$T_r\colon \Gamma(M, \mathcal{E}^r) \to \Gamma(M, \mathcal{E}^{r+1}), \qquad T_r := p^{r+1, m_{r+1}} d^{\nabla^S}_{|\Gamma(M, \mathcal{E}^r)}$$

and call these operators *symplectic twistor operators.* Informally, one can say that the operators are going on the lower edges of the triangle at the Figure 1. Let us notice that $F^-(\nabla^S - T_0)$ is, up to a non-zero scalar multiple, the so called symplectic Dirac operator introduced by K. Habermann. See, e.g., Habermann, Habermann [8].

In the next theorem, we state that the sequences consisting of the symplectic twistor operators form complexes. These sequences will be called symplectic twistor sequences or complexes.

Theorem 4. *Let $l \geq 2$ and (M^{2l}, ω, ∇) be a Fedosov manifold of Ricci type admitting a metaplectic structure. Then*

$$0 \longrightarrow \Gamma(M, \mathcal{E}^{00}) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^{11}) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-1}} \Gamma(M, \mathcal{E}^{ll}) \longrightarrow 0$$

and

$$0 \longrightarrow \Gamma(M, \mathcal{E}^{ll}) \xrightarrow{T_l} \Gamma(M, \mathcal{E}^{l+1,l+1}) \xrightarrow{T_{l+1}} \cdots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l,2l}) \longrightarrow 0$$

are complexes.

Proof. See Krýsl [14]. □

4. Ellipticity of the symplectic twistor complex

After the preceding summarizing parts, we now tend to the proof the ellipticity of the truncated symplectic twistor complexes. Let us recall that by an elliptic complex of differential operators we mean a complex of differential operators acting on the sections of Fréchet bundles such that the associated complex of symbols of the considered differential operators forms an exact sequence of sheaves. Let us recall that a sequence $(\Gamma(\mathcal{F}^\bullet), \pi^\bullet)$ in the category of complexes of sheaves of sections of Fréchet bundles $\mathcal{F}^\bullet$ is called exact if the stalks $[\mathrm{Ker}(\pi^i)]_m$, $[\mathrm{Im}(\pi^{i-1})]_m$ satisfy the equality $[\mathrm{Ker}(\pi^i)]_m = [\mathrm{Im}(\pi^{i-1})]_m$ for each $i \in \mathbb{Z}$ and each $m \in M$, where always when arriving at a preshaef and not at a sheaf, we consider its sheafification not distinguishing it at the notation level. Let us notice that in the case of symbols, we may speak about fibers and not necessarily about stalks because the symbols are bundle and not only sheaf morphisms. See the classical text-book of Wells [21] for more on ellipticity of complexes of differential operators.

After this introductory paragraph, we start with a simple lemma in which the symbol of the exterior covariant symplectic spinor derivative associated to a Fedosov manifold admitting a metaplectic structure is computed.

Lemma 5. *Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure, $\mathcal{S} \to M$ be the corresponding symplectic spinor bundle and d^{∇^S} denotes the exterior covariant derivative. Then for each $\xi \in \Gamma(M, T^*M)$ and $\alpha \otimes \phi \in \Gamma(M, \mathcal{E})$, the symbol σ^ξ of d^{∇^S} is given by*

$$\sigma^\xi(\alpha \otimes \phi) = \xi \wedge \alpha \otimes \phi\,.$$

Proof. For $f \in \mathcal{C}^\infty(M)$, $\xi \in \Gamma(M, T^*M)$ and $\alpha \otimes s \in \Gamma(M, \mathcal{E})$, let us compute $d^{\nabla^S}(f\alpha \otimes s) - f d^{\nabla^S}(\alpha \otimes s) = df \wedge \alpha \otimes s + f d^{\nabla^S}(\alpha \otimes s) - f d^{\nabla^S}(\alpha \otimes s) = df \wedge \alpha \otimes s$. Using this computation, we get the statement of the lemma. □

From now on, we shall denote the projections p^{im_i} onto $\mathbf{E}^i$ by p^i simply, $i = 0, \ldots, 2l$. (In order not to cause a possible confusion, we will make no use of the projections from $\mathbf{E}$ onto $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}$ or of their lifts to the associated geometric structures.) Due to the previous lemma and the definition of the symplectic twistor operators, we get easily that for each $i = 0, \ldots, 2l$ and $\xi \in \Gamma(M, T^*M)$, the symbol σ_i^ξ of the symplectic twistor operator T_i is given by the formula

$$\sigma_i^\xi(\alpha \otimes s) := p^{i+1}(\xi \wedge \alpha \otimes s)$$

for each $\alpha \otimes s \in \Gamma(M, \mathcal{E}^i)$.

In order to prove the ellipticity of the appropriate parts of the symplectic twistor complexes, we need to compare the kernels and the images of the symbols maps σ_i^ξ for any $\xi \in \Gamma(M, T^*M) \setminus \{0\}$. Therefore, we prove the following statement in which the projections p^i are more specified.

Lemma 6. *For $i = 0, \dots, l-1$, $\xi \in \mathbb{V}^*$ and $\alpha \otimes s \in \mathbf{E}^i$, we have*

$$p^{i+1}(\xi \wedge \alpha \otimes s) = \xi \wedge \alpha \otimes s + \beta F^+(\alpha \otimes \xi^\sharp \cdot s) + \gamma(E^+ \iota_{\xi^\sharp} \alpha \otimes s) \tag{6}$$

where $\beta = \frac{2}{i-l}$ and $\gamma = \frac{\imath}{i-l}$.

For $i = l+1, \dots, 2l$ and $\psi \in \mathbf{E}^{i-1, m_{i-1}} \oplus \mathbf{E}^{i-1, m_{i-1}-1} \oplus \mathbf{E}^{i-1, m_{i-1}-2}$, we have

$$p^{i-1}\psi = \psi + \frac{4}{l-i} F^- F^+ \psi + \frac{1}{l-i} E^- E^+ \psi\,. \tag{7}$$

Proof. We prove the first relation only. The second formula can be derived following the same lines of reasoning used for proving the first one. We split the proof of (6) into four parts.

1. In this item, we prove that for a fixed $i \in \{0, \dots, l\}$ and any $k = 0, \dots, i$, there exists $\alpha_k^i \in \mathbb{C}$ such that

$$p^i = \sum_{k=0}^{i} \alpha_k^i (F^+)^k (F^-)^k$$

with $\alpha_0^i = 1$ for each $i = 0, \dots, l$. Because for each $i = 0, \dots, l$, the projections p^i are $\tilde{G}$-equivariant, they can be expressed as (finite) linear combinations of the elements of the finite dimensional vector space $\mathrm{End}_{\tilde{G}}(\mathbf{E})$. Due to the Lemma 3 item 3 (cf. also Krýsl [12]), we know that the complex associative algebra $\mathrm{End}_{\tilde{G}}(\mathbf{E})$ is generated by F^+ and F^- and by the projections $p_\pm$. It is easy to see that the projections $p_\pm$ can be omitted from any expression for p^i and thus, each projection p^i can be expressed just using F^+ and F^-. Due to the defining relation $H = 2\{F^+, F^-\}$ and the relation (4) on the values of H on homogeneous elements, one can order the operators F^+ and F^- in an expression for p^i in the way that the operators F^+ appear on the left-hand and the operators F^- on the right-hand side. In this way, we express p^i as a linear combination of the expressions of type $(F^+)^a(F^-)^b$ for some $a, b \in \mathbb{N}_0$. Since the projection p^i does not change the form degree of a symplectic spinor valued exterior form and F^- and F^+ decreases and increases the form degree by one, respectively, the relation $a = b$ follows. Because the operator F^- decreases the form degree by one, the summands $(F^+)^k(F^-)^k$ for $k > i$ actually do not occur in the expression for the projection p^i written above. Thus,

$$p^i = \sum_{k=0}^{i} \alpha_k^i (F^+)^k (F^-)^k \tag{8}$$

for some $\alpha_k^i \in \mathbb{C}$, $k = 0, \dots, i$.

Now, we shall prove the equation $\alpha_0^i = 1$, $i = 0, \dots, l$. By evaluating the left-hand side of (8) on an element $\phi \in \mathbf{E}^i$ we get ϕ, whereas at the right-hand side the only summand which remains is the one indexed by zero. (The other summands vanish because F^- is $\tilde{G}$-equivariant, decreases the form degree by one and there is no summand in $\bigwedge^{i-1} \mathbb{V}^* \otimes \mathbf{S}$ isomorphic to $\mathbf{E}^i_+$ or to $\mathbf{E}^i_-$. See the Remark item 3 below the Theorem 1.)

2. Now, suppose $\xi \in \mathbb{V}^*$ and $\alpha \otimes s \in \mathbf{E}^i$, $i = 0, \dots, l-1$. Due to the Theorem 2, we know that $\phi := \xi \wedge \alpha \otimes s \in \mathbf{E}^{i+1,i-1} \oplus \mathbf{E}^{i+1,i} \oplus \mathbf{E}^{i+1,i+1}$. Applying p^{i+1} to the element ϕ, only the zeroth, first, and second summand in the expression $p^{i+1}\phi = \sum_{k=0}^{i+1} \alpha_k^{i+1} (F^+)^k (F^-)^k \phi$ remains. (For $k > 2$, the k^{th} summand vanishes in the expression for $p^{i+1}\phi$ because F^- is $\tilde{G}$-equivariant, decreases the form degree by one and there is no summand in $\bigwedge^{i-2} \mathbb{V}^* \otimes \mathbf{S}$ isomorphic to $\mathbf{E}_{\pm}^{i+1,i-1}$ or $\mathbf{E}_{\pm}^{i+1,i}$ or $\mathbf{E}_{\pm}^{i+1,i+1}$. See the item 3 of the Remark below the Theorem 1.)
3. Due to the previous item, we already know that for the element $\phi = \xi \wedge \alpha \otimes s$ chosen above, we get
$$p^{i+1}\phi = \sum_{k=0}^{2} \alpha_k^{i+1} (F^+)^k (F^-)^k \phi\,.$$
Using the relations (4) and (2), we may write
$$\begin{aligned}
p^{i+1}(\xi \wedge \alpha \otimes s) &= \xi \wedge \alpha \otimes s + \alpha_1^{i+1} F^+ F^- (\xi \wedge \alpha \otimes s) \\
&\quad + \alpha_2^{i+1} (F^+)^2 (F^-)^2 (\xi \wedge \alpha \otimes s) \\
&= \xi \wedge \alpha \otimes s + \alpha_1^{i+1} \frac{1}{2} F^+ \omega^{ij} [(\iota_{e_i}\xi)\alpha \otimes e_j \cdot s - \xi \wedge \iota_{e_i}\alpha \otimes e_j \cdot s] \\
&\quad - \alpha_2^{i+1} E^+ \frac{\imath}{32} \omega^{ij} \iota_{e_i}\iota_{e_j} (\xi \wedge \alpha \otimes s) \\
&= \xi \wedge \alpha \otimes s - \alpha_1^{i+1} \frac{1}{2} F^+ [\alpha \otimes \xi^\sharp \cdot s + 2\xi \wedge F^-(\alpha \otimes s)] \\
&\quad - \alpha_2^{i+1} E^+ \frac{\imath}{32} \omega^{ij} \iota_{e_i} (\xi_j \alpha \otimes s - \xi \wedge \iota_{e_j}\alpha \otimes s)\,.
\end{aligned}$$
Because $\alpha \otimes s \in \mathbf{E}^i$, we get $F^-(\alpha \otimes s) = 0$ by Lemma 3 item 2. Using the last written equation, we may write
$$\begin{aligned}
p^{i+1}(\xi \wedge \alpha \otimes s) &= \xi \wedge \alpha \otimes s - \frac{\alpha_1^{i+1}}{2} F^+ (\alpha \otimes \xi^\sharp \cdot s) \\
&\quad - \frac{\imath \alpha_2^{i+1}}{32} E^+ (2\xi^i \iota_{e_i}\alpha \otimes s + \frac{2\alpha_2^{i+1}}{\imath} \xi \wedge E^- \alpha \otimes s)\,.
\end{aligned}$$
The last summand in this expression vanishes due to the Lemma 3 item 2 because first $E^- = -4F^-F^-$ (Eqn. (2)) and second $\alpha \otimes s \in \mathbf{E}^i$. Summing-up, we have
$$p^{i+1}\phi = \xi \wedge \alpha \otimes s - \alpha_1^{i+1} \frac{1}{2} F^+ (\alpha \otimes \xi^\sharp \cdot s) - \alpha_2^{i+1} \frac{\imath}{16} E^+ \iota_{\xi^\sharp} \alpha \otimes s\,,$$
which is a formula of the form written in the statement of the lemma.

4. In this item, we shall determine the numbers $\beta, \gamma \in \mathbb{C}$. Using the fact that p^{i+1} is an idempotent ($(p^{i+1})^2 = p^{i+1}$), we get $\alpha_1^{i+1} = 4/(l-i)$ and $\alpha_2^{i+1} = 16/(l-i)$ after a tedious but straightforward calculation.

   Thus, comparing the last written formula of the preceding item and the Eqn. (6), we get $\beta = 2/(i-l)$ and $\gamma = \imath/(i-l)$.

$\square$

Remark. For $i = l, \ldots, 2l$, $\xi \in \mathbb{V}^*$ and $\alpha \otimes s \in \mathbf{E}^i$, the formula for p^{i+1} reads simply

$$p^{i+1}(\xi \wedge \alpha \otimes s) = \xi \wedge \alpha \otimes s$$

because of the Theorem 2 and the items 1 and 2 of the Remark below the Theorem 1. (Notice that one may also use the relation (7).)

Now, we are prepared to prove the ellipticity of the truncated symplectic twistor complexes.

Theorem 7. *Let (M^{2l}, ω, ∇) be a Fedosov manifold of Ricci type admitting a metaplectic structure, $l \geq 2$. Then the truncated symplectic twistor complexes*

$$0 \longrightarrow \Gamma(M, \mathcal{E}^0) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^1) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-2}} \Gamma(M, \mathcal{E}^{l-1})$$

and

$$\Gamma(M, \mathcal{E}^l) \xrightarrow{T_l} \Gamma(M, \mathcal{E}^{l+1}) \xrightarrow{T_{l+1}} \cdots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l}) \longrightarrow 0$$

are elliptic.

Proof. We should prove the equations $\mathrm{Ker}(\sigma_i^\xi)_m = \mathrm{Im}(\pi_{i-1}^\xi)_m$ for the appropriate indices i and for each point $m \in M$. Here the constituents of the previous equation are fibers of the corresponding shaeves.

1. First, we prove that the sequences mentioned in the formulation of the theorem are complexes. For $i = 0, \ldots, l-2, l, \ldots, 2l-1$, $\psi \in \Gamma(M, \mathcal{E}^i)$ and a differential 1-form $\xi \in \Gamma(M, T^*M)$, we may write $0 = p^{i+2}(0) = p^{i+2}((\xi \wedge \xi) \wedge \psi) = p^{i+2}(\xi \wedge \mathrm{Id}(\xi \wedge \psi)) = p^{i+2}(\xi \wedge \sum_{j=0}^{m_{i+1}} p^{i+1,j}(\xi \wedge \psi))$. Due to the Theorem 2, we know that the last written expression equals $p^{i+2}(\xi \wedge p^{i+1}(\xi \wedge \psi)) = \sigma_{i+1}^\xi \sigma_i^\xi \psi$ and thus $\sigma_{i+1}^\xi \sigma_i^\xi = 0$.
2. Second, we prove the relation $\mathrm{Ker}(\sigma_i^\xi)_m \subseteq \mathrm{Im}(\sigma_{i-1}^\xi)_m$ for each $0 \neq \xi \in T_m^*M$ and $i = 0, \ldots, l-2$. Here $\sigma_{-1}^\xi = 0$ is to be understood. Suppose a homogeneous element $\alpha \otimes s \in \mathcal{E}_m^i$ is given such that $\sigma_i^\xi(\alpha \otimes s) = 0$. (In the next item, we will treat the general non-homogeneous case.) Due to the paragraph below the Lemma 5, we know that $0 = \sigma_i^\xi(\alpha \otimes s) = p^{i+1}(\xi \wedge \alpha \otimes s)$. We shall find an element $\psi \in \mathcal{E}_m^{i-1}$ such that $p^i(\xi \wedge \psi) = \alpha \otimes s$.

   Using formula (6) for the projection (Lemma 6), we may rewrite the equation $p^{i+1}(\xi \wedge \alpha \otimes s) = 0$ into

$$\xi \wedge \alpha \otimes s + \beta F^+(\alpha \otimes \xi^\sharp \cdot s) + \gamma E^+ \iota_{\xi^\sharp} \alpha \otimes s = 0\,. \tag{9}$$

Applying the operator E^- (formula (2)) on the both sides of the previous equation and using the first commutation relation in the row (3) from Lemma 3, we get

$$\frac{\imath}{2}\omega^{ij}\iota_{e_i}\iota_{e_j}(\xi\wedge\alpha)\otimes s+\beta E^-F^+(\alpha\otimes\xi^\sharp\cdot s)+\gamma(E^+E^--2H)\iota_{\xi^\sharp}\alpha\otimes s=0\,.$$

Using the graded Leibniz property of $\iota_{\xi^\sharp}$, the relation (4) for the values of H on form-homogeneous elements and the second relation in the row (3) from Lemma 3, we obtain

$$\begin{aligned}\frac{\imath}{2}(-2\iota_{\xi^\sharp}-2\imath\xi\wedge E^-)(\alpha\otimes s)+\beta F^+E^-(\alpha\otimes\xi^\sharp\cdot s)-\beta F^-(\alpha\otimes\xi^\sharp\cdot s)\\ +\gamma E^+E^-\iota_{\xi^\sharp}\alpha\otimes s+\gamma(l-i+1)\iota_{\xi^\sharp}\alpha\otimes s=0\,.\end{aligned}$$

The operator E^- commutes with the operator of the symplectic Clifford multiplication (by the vector field $\xi^\sharp$) and also with the contraction $\iota_{\xi^\sharp}$ because $E^-=\frac{\imath}{2}\omega^{ij}\iota_{e_i}\iota_{e_j}$ (formula (2)). Using these two facts, we get

$$\begin{aligned}\frac{\imath}{2}(-2\iota_{\xi^\sharp}-2\imath\xi\wedge E^-)(\alpha\otimes s)+\beta F^+\xi^\sharp\cdot E^-(\alpha\otimes s)-\beta F^-(\alpha\otimes\xi^\sharp.s)\\ +\gamma E^+\iota_{\xi^\sharp}E^-\alpha\otimes s+\gamma(l-i+1)\iota_{\xi^\sharp}\alpha\otimes s=0\,.\end{aligned}$$

Because $F^-(\alpha\otimes s)=0$ (Lemma 3 item 2), we have $E^-\alpha\otimes s=4F^-F^-(\alpha\otimes s)=0$. Thus, we obtain the identity

$$-\imath\iota_{\xi^\sharp}\alpha\otimes s-\beta F^-(\alpha\otimes\xi^\sharp\cdot s)+\gamma(l-i+1)\iota_{\xi^\sharp}\alpha\otimes s=0\,.$$

Substituting the second relation in the row (5) into the previous equation and using the fact $F^-(\alpha\otimes s)=0$ again, we get

$$\begin{aligned}-\imath\iota_{\xi^\sharp}\alpha\otimes s-\beta\xi^\sharp\cdot F^-(\alpha\otimes s)-\beta\frac{\imath}{2}\iota_{\xi^\sharp}\alpha\otimes s\\ +\gamma(l-i+1)\iota_{\xi^\sharp}\alpha\otimes s=0\,.\end{aligned}$$

Using the prescription for the numbers β and γ (Lemma 6) and the already twice used relation $F^-(\alpha\otimes s)=0$, we get $(-\imath+\gamma(l-i+1)-\beta\frac{\imath}{2})\iota_{\xi^\sharp}\alpha\otimes s=-2\imath\iota_{\xi^\sharp}\alpha\otimes s=0$ from which the equation

$$\iota_{\xi^\sharp}\alpha\otimes s=0 \tag{10}$$

follows.

Substituting this relation into the prescription for the projection p^i (Eqn. (9)), we get for $i=0,\dots,l-2$ the equation

$$0=p^{i+1}(\xi\wedge\alpha\otimes s)=\xi\wedge\alpha\otimes s+\beta F^+(\alpha\otimes\xi^\sharp\cdot s)\,. \tag{11}$$

Applying the contraction operator $\iota_{\xi^\sharp}$ to the previous equation and using the first formula in the row (5) from Lemma 3, we obtain

$$0=-\xi\wedge\iota_{\xi^\sharp}\alpha\otimes s-\beta F^+\iota_{\xi^\sharp}(\alpha\otimes\xi^\sharp\cdot s)+\beta\frac{\imath}{2}\alpha\otimes\xi^\sharp.(\xi^\sharp\cdot s)\,.$$

Using the fact that the contraction and symplectic Clifford multiplication commute, we have

$$0=-\xi\wedge\iota_{\xi^\sharp}\alpha\otimes s-\beta F^+\xi^\sharp\cdot(\iota_{\xi^\sharp}\alpha\otimes s)+\beta\frac{\imath}{2}\alpha\otimes\xi^\sharp\cdot(\xi^\sharp\cdot s)\,.$$

Substituting the Eqn. (10) into the previous equation, we obtain

$$\alpha \otimes \xi^\sharp \cdot (\xi^\sharp \cdot s) = 0 \, .$$

Substituting the definition of F^+ into the equation (11) multiplying it by $\xi^\sharp$ and using the equation $\iota_{\xi^\sharp}\alpha \otimes s = 0$ (Eqn. (10)) again, we get

$$0 = \xi \wedge \alpha \otimes \xi^\sharp \cdot s + \beta \frac{\imath}{2} \epsilon^i \wedge \alpha \otimes \xi^\sharp \cdot e_i \cdot \xi^\sharp \cdot s \, ,$$

$$0 = \xi \wedge \alpha \otimes \xi^\sharp \cdot s + \beta \frac{\imath}{2} \epsilon^i \wedge \alpha \otimes (e_i \cdot \xi^\sharp \cdot \xi^\sharp \cdot - \imath \omega_0(\xi^\sharp, e_i)\xi^\sharp \cdot)s \, .$$

Substituting the identity $\alpha \otimes \xi^\sharp \cdot \xi^\sharp \cdot s = 0$ into the previous equation, we obtain

$$0 = (1 + \frac{1}{2}\beta)\xi \wedge \alpha \otimes \xi^\sharp \cdot s \, .$$

If $i = 0, \ldots, l-2$, the coefficient $1 + \beta/2 \neq 0$, and thus by dividing, we get $\xi \wedge \alpha \otimes \xi^\sharp \cdot s = 0$. Because the symplectic Clifford multiplication by a non-zero vector is injective (see the subsection 2.2), we have

$$0 = \xi \wedge \alpha \otimes s \, . \tag{12}$$

3. In this item, we will still suppose $i = 0, \ldots, l-2$. Let us consider a general element $\phi \in \mathrm{Ker}(\sigma_i^\xi)_m \subseteq \mathcal{E}^i_m$ and denote the basis of $\bigwedge^i T^*_m M$ by $(\alpha_{ik})_{k=1}^{n_i}$, $n_i \in \mathbb{N}$. Due to the finite dimensionality of $\bigwedge^i T^*_m M$, there exist complex numbers a_{jk}, $j \in \mathbb{N}$, $k = 1, \ldots, n_i$, such that $\phi = \sum_{k=1}^{n_i} \sum_{j=1}^{\infty} a_{jk} \alpha^{ik} \otimes h_j$ where $(h_j)_{j \in \mathbb{N}}$ is the Schauder basis of $\mathcal{S}_m$ corresponding to the Schauder basis of $\mathcal{S}(\mathbb{L}) \simeq \mathcal{S}_m$. Because the operators $F^\pm$, H, $E^\pm$, ι_ξ and $\xi \wedge$ are continuous on $\mathcal{E}_m$, we get $0 = \sum_{k=1}^{n_i} \sum_{j=1}^{\infty} a_{jk} \xi \wedge \alpha^{ik} \otimes h_j$ precisely in the same way as we obtained the formula (12) in the homogeneous situation (item 2 of this proof). Using the definition of the Schauder basis again, we have for each $j \in \mathbb{N}$ the equation $\sum_{k=1}^{n_i} a_{jk} \xi \wedge \alpha^{ik} = 0$. Using the Cartan lemma on exterior differential systems, we get the existence of a family $(\beta_j)_{j \in \mathbb{N}}$ of $(i-1)$ forms such that $\xi \wedge \beta_j = \sum_{k=1}^{n_i} a_{jk} \alpha^{ik}$. It is possible to see (e.g. by taking the standard Hodge-type metric on the space of forms) that one can choose the family $(\beta_j)_{j \in \mathbb{N}}$ in such a way that $\psi := \sum_{j=1}^{\infty} \beta_j \otimes h_j$ converges. Thus, we may write $\sigma^\xi_{i-1}(\sum_{j=1}^{\infty} \beta_j \otimes h_j) = p^i(\sum_{j=1}^{\infty} \xi \wedge \beta_j \otimes h_j) = p^i(\sum_{j=1}^{\infty} \sum_{k=1}^{n_i} a_{jk} \alpha^{ik} \otimes h_j) = p^i(\phi) = \phi$. Summing-up, we have that $\psi = \sum_{j=1}^{\infty} \beta_j \otimes h_j$ is the desired preimage. Thus, $\phi \in \mathrm{Im}(\sigma^\xi_{i-1})_m$.
4. Now, we prove that $\mathrm{Ker}(\sigma_i^\xi)_m \subseteq \mathrm{Im}(\sigma^\xi_{i-1})_m$ for $i = l+1, \ldots, 2l$, $0 \neq \xi \in \Gamma(M, T^*M)$. If $\phi = \alpha \otimes s \in \mathrm{Ker}(\sigma_i^\xi)_m$, then $0 = p^{i+1}(\xi \wedge \phi) = \xi \wedge \alpha \otimes s$. Due to the Cartan lemma, we know that there is a form $\beta \in \bigwedge^{i-1} T^*_m M$ such that $\xi \wedge \beta \otimes s = \alpha \otimes s$. Define $\psi := p^{i-1}(\beta \otimes s)$. Using the formula (7), the equation $\xi \wedge \beta = \alpha$ and the assumption $F^+(\alpha \otimes s) = 0$ (implied by $\alpha \otimes s \in \mathbf{E}^{im_i}$), one can prove that $\xi \wedge \psi = \alpha \otimes s$ in an analogous way as we proceeded the item 2 of this proof. The dehomogenization goes in the steps similar to that ones written in the preceding item.

$\square$

In the future, we would like to interpret the appropriate (reduced) cohomology groups of the truncated symplectic twistor complexes. Eventually, one can search for an application of the symplectic twistor complexes in representation theory. One can also try to prove that the full (i.e., not truncated) symplectic twistor complexes are not elliptic by finding an example of a suitable Ricci type Fedosov manifold admitting a metaplectic structure.

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Symplectic Killing spinors

Svatopluk Krýsl

Abstract. Let (M,ω) be a symplectic manifold admitting a metaplectic structure (a symplectic analogue of the Riemannian spin structure) and a torsion-free symplectic connection ∇. Symplectic Killing spinor fields for this structure are sections of the symplectic spinor bundle satisfying a certain first order partial differential equation and they are the main object of this paper. We derive a necessary condition which has to be satisfied by a symplectic Killing spinor field. Using this condition one may easily compute the symplectic Killing spinor fields for the standard symplectic vector spaces and the round sphere S^2 equipped with the volume form of the round metric.

Keywords: Fedosov manifolds, symplectic spinors, symplectic Killing spinors, symplectic Dirac operators, Segal-Shale-Weil representation

Classification: 58J60, 53C07

1. Introduction

In this article we shall study the so called symplectic Killing spinor fields on Fedosov manifolds admitting a metaplectic structure. A Fedosov manifold is a structure consisting of a symplectic manifold (M^{2l},ω) and the so called Fedosov connection on (M,ω). A Fedosov connection ∇ is an affine connection on (M,ω) such that it is symplectic, i.e., $\nabla\omega = 0$, and torsion-free. Let us notice that in contrary to the Riemannian geometry, a Fedosov connection is not unique. Thus, it seems natural to add the Fedosov connection into the studied structure and obtain the notion of a Fedosov manifold. See, e.g., Tondeur [13] for symplectic connections for presymplectic structures and Gelfand, Retakh, Shubin [3] for Fedosov connections.

It is known that if $l > 1$, the curvature tensor of a Fedosov connection decomposes into two invariant parts, namely into the so called symplectic Ricci curvature and symplectic Weyl curvature tensor fields. If $l = 1$, only the symplectic Ricci curvature occurs. See Vaisman [14] for details.

In order to define a symplectic Killing spinor field, we shall briefly describe the so called metaplectic structures with help of which these fields are defined. Any symplectic group $\mathrm{Sp}(2l,\mathbb{R})$ admits a non-trivial, i.e., connected, two-fold covering,

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the so called metaplectic group, denoted by $\mathrm{Mp}(2l,\mathbb{R})$ in this paper. A metaplectic structure over a symplectic manifold is a symplectic analogue of the Riemannian spin structure. In particular, one of its parts is a principal $\mathrm{Mp}(2l,\mathbb{R})$-bundle which covers twice the bundle of symplectic frame of (M^{2l},ω). Let us denote this principal $\mathrm{Mp}(2l,\mathbb{R})$-bundle by $q:\ \mathcal{Q}\to M$.

Now, let us say a few words about the symplectic spinor fields. These fields are sections of the so called symplectic spinor bundle $\mathcal{S}\to M$. This vector bundle is the bundle associated to the principal $\mathrm{Mp}(2l,\mathbb{R})$-bundle $q:\mathcal{Q}\to M$ via the so called *Segal-Shale-Weil representation*. The Segal-Shale-Weil representation is a distinguished representation of the metaplectic group and plays a similar role in the quantization of boson particles as the spinor representations of spin groups play in the quantization of fermions. See, e.g., Shale [12]. The Segal-Shale-Weil representation is unitary and does not descend to a representation of the symplectic group. The vector space of the underlying Harish-Chandra $(\mathfrak{g},K)$-module of the Segal-Shale-Weil representation is isomorphic to $S^\bullet(\mathbb{R}^l)$, the *symmetric* power of a Lagrangian subspace $\mathbb{R}^l$ of the symplectic vector space $\mathbb{R}^{2l}$. Thus, the situation is parallel to the complex orthogonal case, where the spinor representation can be realized on the *exterior* power of a maximal isotropic subspace. The Segal-Shale-Weil representation and some of its analytic versions are sometimes called oscillatory representation, metaplectic representation or symplectic spinor representation. For a detailed explanation of the last name, see, e.g., Kostant [8].

The symplectic Killing spinor field is a non-zero section of the symplectic spinor bundle $\mathcal{S}\to M$ satisfying certain linear first order partial differential equation formulated by the connection $\nabla^S:\Gamma(M,\mathcal{S})\times\Gamma(M,TM)\to\Gamma(M,\mathcal{S})$, the associated connection to the Fedosov connection ∇. This partial differential equation is a symplectic analogue of the classical symplectic Killing spinor equation from at least two aspects. One of them is rather formal. Namely, the defining equation for a symplectic Killing spinor is of the "same shape" as that one for a Killing spinor field on a Riemannian spin manifold. The second similarity can be expressed by comparing this equation with the so called symplectic Dirac equation and the symplectic twistor equation and will be discussed below in this paper. Let us mention that any symplectic Killing spinor field determines a unique complex number, the so called symplectic Killing spinor number. Let us notice that the symplectic Killing spinor fields were considered already in a connection with the existence of a linear embedding of the spectrum of the so called symplectic Dirac operator into the spectrum of the so called symplectic Rarita-Schwinger operator. The symplectic Killing spinor fields represent an obstruction for the mentioned embedding. See Krýsl [10] for this aspect.

In many particular cases, the equation for symplectic Killing spinor fields seems to be rather complicated. On the other hand, in many cases it is known that its solutions are rare. Therefore it is reasonable to look for a necessary condition satisfied by a symplectic Killing spinor field which is simpler than the defining equation itself. Let us notice that similar necessary conditions are known and parallel methods were used in Riemannian or Lorentzian spin geometry. See, e.g., Friedrich [2].

In this paper, we shall prove that any symplectic Killing spinor field necessarily satisfies certain zeroth order differential equation. More precisely, we prove that any symplectic Killing spinor is necessarily a section of the kernel of a symplectic spinor bundle morphism. We derive this equation by prolongating the symplectic Killing spinor equation. We make such a prolongation that enables us to compare the result with an appropriate part of the curvature tensor of the associated connection ∇^S acting on symplectic spinors. An explicit formula for this part of the curvature action was already derived in Krýsl [11]. Especially, it is known that the symplectic Weyl curvature of ∇ does not show up in this part and thus, the mentioned morphism depends on the symplectic Ricci part of the curvature of the Fedosov connection ∇ only. This will make us able to prove that the only symplectic Killing number of a Fedosov manifold of Weyl type is zero. This will in turn imply that any symplectic Killing spinor on the standard symplectic vector space of an arbitrary finite dimension and equipped with the standard flat connection is constant. This result can be obtained easily when one knows the prolongated equation, whereas computing the symplectic Killing spinors without this knowledge is rather complicated. This fact will be illustrated when we will compute the symplectic Killing spinors on the standard symplectic 2-plane using just the defining equation for symplectic Killing spinor field.

The cases when the prolongated equation does not help so easily as in the case of the Weyl type Fedosov manifolds are the Ricci type ones. Nevertheless, we prove that there are no symplectic Killing spinors on the 2-sphere, equipped with the volume form of the round metric as the symplectic form and the Riemannian connection as the Fedosov connection. Let us remark that in this case, the prolongated equation has a shape of a stationary Schrödinger equation. More precisely, it has the shape of the equation for the eigenvalues of certain oscillator-like quantum Hamiltonian determined completely by the symplectic Ricci curvature tensor of the Fedosov connection.

Let us notice that there are some applications of symplectic spinors in physics besides those in the mentioned article of Shale [12]. For an application in string theory physics, see, e.g., Green, Hull [4].

In the second section, some necessary notions from symplectic linear algebra and representation theory of reductive Lie groups are explained and the Segal-Shale-Weil representation and the symplectic Clifford multiplication are introduced. In the third section, the Fedosov connections are introduced and some properties of their curvature tensors acting on symplectic spinor fields are summarized. In the fourth section, the symplectic Killing spinors are defined and symplectic Killing spinors on the standard symplectic 2-plane are computed. In this section, a connection of the symplectic Killing spinor fields to the eigenfunctions of symplectic Dirac and symplectic twistor operators is formulated and proved. Further, the mentioned prolongation of the symplectic Killing spinor equation is derived and the symplectic Killing spinor fields on the standard symplectic vector spaces are computed. At the end, the case of the round sphere S^2 is treated.

2. Symplectic spinors and symplectic spinor valued forms

Let us start recalling some notions from symplectic linear algebra. Let us mention that we shall often use the Einstein summation convention without mentioning it explicitly. Let $(\mathbb{V}, \omega_0)$ be a symplectic vector space of dimension $2l$, i.e., ω_0 is a non-degenerate anti-symmetric bilinear form on the vector space $\mathbb{V}$ of dimension $2l$. Let $\mathbb{L}$ and $\mathbb{L}'$ be two Lagrangian subspaces[1] of $(\mathbb{V}, \omega_0)$ such that $\mathbb{L} \oplus \mathbb{L}' = \mathbb{V}$. Let $\{e_i\}_{i=1}^{2l}$ be an adapted symplectic basis of $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega_0)$, i.e., $\{e_i\}_{i=1}^{2l}$ is a symplectic basis and $\{e_i\}_{i=1}^{l} \subseteq \mathbb{L}$ and $\{e_i\}_{i=l+1}^{2l} \subseteq \mathbb{L}'$. Because the definition of a symplectic basis is not unique, we shall fix one which we shall use in this text. A basis $\{e_i\}_{i=1}^{2l}$ of $(\mathbb{V}, \omega_0)$ is called symplectic, if $\omega_0(e_i, e_j) = 1$ iff $1 \leq i \leq l$ and $j = l + i$; $\omega_0(e_i, e_j) = -1$ iff $l + 1 \leq i \leq 2l$ and $j = i - l$ and $\omega_0(e_i, e_j) = 0$ in the remaining cases. Whenever a symplectic basis will be chosen, we will denote the basis of $\mathbb{V}^*$ dual to $\{e_i\}_{i=1}^{2l}$ by $\{\epsilon^i\}_{i=1}^{2l}$. Further for $i, j = 1, \ldots, 2l$, we set $\omega_{ij} := \omega_0(e_i, e_j)$ and similarly for other type of tensors. For $i, j = 1, \ldots, 2l$, we define ω^{ij} by the equation $\sum_{k=1}^{2l} \omega_{ik}\omega^{jk} = \delta^i_j$.

As in the orthogonal case, we would like to rise and lower indices. Because the symplectic form ω_0 is antisymmetric, we should be more careful in this case. For coordinates $K_{ab\ldots c\ldots d}{}^{rs\ldots t\ldots u}$ of a tensor K over $\mathbb{V}$, we denote the expression $\omega^{ic}K_{ab\ldots c\ldots d}{}^{rs\ldots t}$ by $K_{ab\ldots}{}^{i}{}_{\ldots d}{}^{rs\ldots t}$ and $K_{ab\ldots c}{}^{rs\ldots t\ldots u}\omega_{ti}$ by $K_{ab\ldots c}{}^{rs\ldots}{}_{i}{}^{\ldots u}$ and similarly for other types of tensors and also in a geometric setting when we will be considering tensor fields over a symplectic manifold (M, ω).

Let us denote the symplectic group $\mathrm{Sp}(\mathbb{V}, \omega_0)$ of $(\mathbb{V}, \omega_0)$ by G. Because the maximal compact subgroup of G is isomorphic to the unitary group $U(l)$ which is of homotopy type $\mathbb{Z}$, we have $\pi_1(G) \simeq \mathbb{Z}$. From the theory of covering spaces, we know that there exists up to an isomorphism a unique connected double cover of G. This double cover is the so called metaplectic group $\mathrm{Mp}(\mathbb{V}, \omega_0)$ and will be denoted by $\tilde{G}$ in this text. We shall denote the covering homomorphism by λ, i.e., $\lambda : \tilde{G} \to G$ is a fixed member of the isomorphism class of all connected 2:1 coverings.

Now, let us recall some notions from representation theory of reductive Lie groups which we shall need in this paper. Let us mention that these notions are rather of technical character for the purpose of our article. For a reductive Lie group G in the sense of Vogan [15], let $\mathcal{R}(G)$ be the category the object of which are complete, locally convex, Hausdorff vector spaces with a continuous action of G which is admissible and of finite length; the morphisms are continuous linear G-equivariant maps between the objects. Let us notice that, e.g., finite covers of the classical groups are reductive. It is known that any irreducible unitary representation of a reductive group is in $\mathcal{R}(G)$. Let $\mathfrak{g}$ be the Lie algebra of G

[1] i.e., maximal isotropic with respect to ω_0, in particular $\dim \mathbb{L} = \dim \mathbb{L}' = l$

and K be a maximal compact subgroup of G. It is well known that there exists the so called L^2-globalization functor, denoted by L^2 here, from the category of admissible Harish-Chandra modules to the category $\mathcal{R}(G)$. See Vogan [15] for details. Let us notice that this functor behaves compatibly with respect to Hilbert tensor products. See, e.g., Vogan [15] again. For an object $\mathbf{E}$ in $\mathcal{R}(G)$, let us denote its underlying Harish-Chandra $(\mathfrak{g}, K)$-module by E and when we will be considering only its $\mathfrak{g}^{\mathbb{C}}$-module structure, we shall denote it by $\mathbb{E}$. If $\mathfrak{g}^{\mathbb{C}}$ happens to be a simple complex Lie algebra of rank l, let us denote its Cartan subalgebra by $\mathfrak{h}^{\mathbb{C}}$. The set Φ of roots for $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ is then uniquely determined. Further let us choose a set $\Phi^+ \subseteq \Phi$ of positive roots and denote the corresponding set of fundamental weights by $\{\varpi_i\}_{i=1}^{l}$. For $\lambda \in \mathfrak{h}^{\mathbb{C}}$, let us denote the irreducible highest weight module with the highest weight λ by $L(\lambda)$.

Let us denote by $\mathcal{U}(\mathbf{W})$ the group of unitary operators on a Hilbert space $\mathbf{W}$ and let $L : \mathrm{Mp}(\mathbb{V}, \omega_0) \to \mathcal{U}(\mathbf{L}^2(\mathbb{L}))$ be the Segal-Shale-Weil representation of the metaplectic group. It is an infinite dimensional unitary representation of the metaplectic group on the complex valued square Lebesgue integrable functions defined on the Lagrangian subspace $\mathbb{L}$. This representation does not descend to a representation of the symplectic group $\mathrm{Sp}(\mathbb{V}, \omega_0)$. See, e.g., Weil [16] and Kashiwara, Vergne [7]. For convenience, let us set $\mathbf{S} := \mathbf{L}^2(\mathbb{L})$ and call it the symplectic spinor module and its elements symplectic spinors. It is well known that as a $\tilde{G}$-module, $\mathbf{S}$ decomposes into the direct sum $\mathbf{S} = \mathbf{S}_+ \oplus \mathbf{S}_-$ of two irreducible submodules. The submodule $\mathbf{S}_+$ ($\mathbf{S}_-$) consists of even (odd) functions in $\mathbf{L}^2(\mathbb{L})$. Further, let us notice that in Krýsl [9], a slightly different analytic version (based on the so called minimal globalizations) of this representation was used.

As in the orthogonal case, we may multiply spinors by vectors. The multiplication $. : \mathbb{V} \times \mathbf{S} \to \mathbf{S}$ will be called symplectic Clifford multiplication and it is defined as follows. For $f \in \mathbf{S}$ and $i = 1, \ldots, l$, we set

$$(e_i.f)(x) := \imath x^i f(x),$$
$$(e_{l+i}.f)(x) := \frac{\partial f}{\partial x^i}(x), \ x \in \mathbb{L}$$

and extend it linearly to get the symplectic Clifford multiplication. The symplectic Clifford multiplication (by a fixed vector) has to be understood as an unbounded operator on $\mathbf{L}^2(\mathbb{L})$. See Habermann, Habermann [6] for details. Let us also notice that the symplectic Clifford multiplication corresponds to the so called Heisenberg canonical quantization known from quantum mechanics. (For brevity, we shall write $v.w.s$, instead of $v.(w.s)$, $v, w \in \mathbb{V}$ and $s \in \mathbf{S}$.)

It is easy to check that the symplectic Clifford multiplication satisfies the relation described in the following

Lemma 1. *For $v, w \in \mathbb{V}$ and $s \in \mathbf{S}$, we have*

$$v.(w.s) - w.(v.s) = -\imath\omega_0(v, w)s.$$

PROOF: See Habermann, Habermann [6]. □

Let us consider the representation

$$\rho : \tilde{G} \to \mathrm{Aut}(\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S})$$

of the metaplectic group $\tilde{G}$ on $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ given by

$$\rho(g)(\alpha \otimes s) := \lambda^{*\wedge r}(g)\alpha \otimes L(g)s,$$

where $r = 0, \ldots, 2l$, $\alpha \in \bigwedge^r \mathbb{V}^*$, $s \in \mathbf{S}$ and $\lambda^{*\wedge r}$ denotes the rth wedge power of the representation λ^* dual to λ, and extended linearly. For definiteness, let us consider the vector space $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ equipped with the topology of the Hilbert tensor product. Because the L^2-globalization functor behaves compatibly with respect to the Hilbert tensor products, one can easily see that the representation ρ belongs to the class $\mathcal{R}(\tilde{G})$.

In the next theorem, the space o symplectic valued exterior two-forms is decomposed into irreducible summands.

Theorem 2. *For $\frac{1}{2}\dim(\mathbb{V}) = l > 2$, the following isomorphism*

$$\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}_\pm \simeq \mathbf{E}^{20}_\pm \oplus \mathbf{E}^{21}_\pm \oplus \mathbf{E}^{22}_\pm$$

holds. For $j_2 = 0, 1, 2$, the $\mathbf{E}^{2j_2}$ are uniquely determined by the conditions that first, they are submodules of the corresponding tensor products and second,

$$\mathbb{E}^{20}_- \simeq \mathbb{S}_- \simeq L(\varpi_{l-1} - \frac{3}{2}\varpi_l), \quad \mathbb{E}^{20}_+ \simeq \mathbb{S}_+ \simeq L(-\frac{1}{2}\varpi_l),$$
$$\mathbb{E}^{21}_- \simeq L(\varpi_1 - \frac{1}{2}\varpi_l), \quad \mathbb{E}^{21}_+ \simeq L(\varpi_1 + \varpi_{l-1} - \frac{3}{2}\varpi_l),$$
$$\mathbb{E}^{22}_+ \simeq L(\varpi_2 - \frac{1}{2}\varpi_l) \ \ \textit{and} \ \ \mathbb{E}^{22}_- \simeq L(\varpi_2 + \varpi_{l-1} - \frac{3}{2}\varpi_l).$$

PROOF: This theorem is proved in Krýsl [10] or Krýsl [9] for the so called minimal globalizations. Because the L^2-globalization behaves compatibly with respect to the considered Hilbert tensor product topology, the statement remains true. □

Remark. Let us notice that for $l = 2$, the number of irreducible summands in symplectic spinor valued two-forms is the same as that one for $l > 2$. In this case ($l = 2$), one only has to change the prescription for the highest weights described in the preceding theorem. For $l = 1$, the number of the irreducible summands is different from that one for $l \geq 2$. Nevertheless, in this case the decomposition is also multiplicity-free. See Krýsl [9] for details.

In order to make some proofs in the section on symplectic Killing spinor fields simpler and more clear, let us introduce the operators

$$
\begin{aligned}
F^{+} &: \bigwedge^{\bullet}\mathbb{V}^{*}\otimes\mathbf{S}\to\bigwedge^{\bullet+1}\mathbb{V}^{*}\otimes\mathbf{S},\quad F^{+}(\alpha\otimes s):=\sum_{i=1}^{2l}\epsilon^{i}\wedge\alpha\otimes e_{i}.s,\\
F^{-} &: \bigwedge^{\bullet}\mathbb{V}^{*}\otimes\mathbf{S}\to\bigwedge^{\bullet-1}\mathbb{V}^{*}\otimes\mathbf{S},\quad F^{-}(\alpha\otimes s):=-\sum_{i,j=1}^{2l}\omega^{ij}\iota_{e_{i}}\alpha\otimes e_{j}.s,\\
H &: \bigwedge^{\bullet}\mathbb{V}^{*}\otimes\mathbf{S}\to\bigwedge^{\bullet}\mathbb{V}^{*}\otimes\mathbf{S},\quad H:=\{F^{+},F^{-}\}.
\end{aligned}
$$

Remark. (1) One easily finds out that the operators are independent of the choice of an adapted symplectic basis $\{e_i\}_{i=1}^{2l}$.

(2) Let us remark that the operators F^+, F^- and H defined here differ from the operators F^+, F^-, H defined in Krýsl [9]. Though, by a constant real multiple only.

(3) The operators F^+ and F^- are used to prove the Howe correspondence for $\mathrm{Mp}(\mathbb{V}, \omega_0)$ acting on $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ via the representation ρ. More or less, the ortho-symplectic super Lie algebra $\mathfrak{osp}(1|2)$ plays the role of a (super Lie) algebra, a representation of which is the appropriate commutant. See Krýsl [9] for details.

In the next lemma the $\tilde{G}$-equivariance of the operators F^+, F^- and H is stated, some properties of $F^\pm$ are mentioned and the value of H on degree-homogeneous elements is computed. We shall use this lemma when we will be treating the symplectic Killing spinor fields in the fourth section.

Lemma 3. *Let* $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega_0)$ *be a* $2l$ *dimensional symplectic vector space. Then*

(1) *the operators* F^+, F^+ *and* H *are* $\tilde{G}$*-equivariant,*

(2) (a) $F^-_{|\mathbf{E}^{11}} = 0$,

  (b) $F^+_{|\mathbf{E}^{00}}$ *is an isomorphism onto* $\mathbf{E}^{10}$,

  (c) $(F^+)^2_{|\mathbf{S}} = -\frac{\imath}{2}\omega \otimes \mathrm{Id}_{|\mathbf{S}}$ *and it is an isomorphism onto* $\mathbf{E}^{20}$.

(3) *For* $r = 0, \ldots, 2l$, *we have*

$$H_{|\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}} = \imath(r - l)\,\mathrm{Id}_{|\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}}\,.$$

Proof: See Krýsl [9]. □

Let us remark that the items 1 and 3 of the preceding lemma follow by a direct computation, and the second item follows from the first item, decomposition theorem (Theorem 2), a version of the Schur lemma and a direct computation.

3. Curvature of Fedosov manifolds and its actions on symplectic spinors

After we have finished the "algebraic part" of this paper, let us recall some basic facts on Fedosov manifolds, their curvature tensors, metaplectic structures and the action of the curvature tensor on symplectic spinor fields.

Let us start recalling some notions and results related to the so called Fedosov manifolds. Let (M^{2l}, ω) be a symplectic manifold of dimension $2l$. Any torsion-free affine connection ∇ on M preserving ω, i.e., $\nabla\omega = 0$, is called *Fedosov connection*. The triple (M, ω, ∇), where ∇ is a Fedosov connection, will be called *Fedosov manifold*. As we have already mentioned in the Introduction, a Fedosov connection for a given symplectic manifold (M, ω) is not unique. Let us remark that Fedosov manifolds are used for a construction of geometric quantization of symplectic manifolds due to Fedosov. See, e.g., Fedosov [1].

To fix our notation, let us recall the classical definition of the curvature tensor R^∇ of the connection ∇, we shall be using here. We set

$$R^\nabla(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Let us choose a local adapted symplectic frame $\{e_i\}_{i=1}^{2l}$ on a fixed open subset $U \subseteq M$. By a local adapted symplectic frame $\{e_i\}_{i=1}^{2l}$ over U, we mean such a local frame that for each $m \in U$ the basis $\{(e_i)_m\}_{i=1}^{2l}$ is an adapted symplectic basis of $(T_m M, \omega_m)$. Whenever a symplectic frame is chosen, we denote its dual frame by $\{\epsilon^i\}_{i=1}^{2l}$. Although some of the formulas below hold only locally, containing a local adapted symplectic frame, we will not mention this restriction explicitly.

From the symplectic curvature tensor field R^∇, we can build the symplectic Ricci curvature tensor field σ^∇ defined by the classical formula

$$\sigma^\nabla(X, Y) := \mathrm{Tr}(V \mapsto R^\nabla(V, X)Y)$$

for each $X, Y \in \mathfrak{X}(M)$ (the variable V denotes a vector field on M). For the chosen frame and $i, j = 1, \ldots, 2l$, we define

$$\sigma_{ij} := \sigma^\nabla(e_i, e_j).$$

Let us define the extended Ricci tensor field by the equation

$$\widetilde{\sigma}(X, Y, Z, U) := \widetilde{\sigma}_{ijkn} X^i Y^j Z^k U^n, \ X, Y, Z, U \in \mathfrak{X}(M),$$

where for $i, j, k, n = 1, \ldots, 2l$,

$$2(l+1)\widetilde{\sigma}_{ijkn} := \omega_{in}\sigma_{jk} - \omega_{ik}\sigma_{jn} + \omega_{jn}\sigma_{ik} - \omega_{jk}\sigma_{in} + 2\sigma_{ij}\omega_{kn}.$$

A Fedosov manifold (M, ω, ∇) is called of Weyl type, if $\sigma = 0$. Let us notice, that it is called of Ricci type, if $R = \widetilde{\sigma}$. In Vaisman [14], one can find more information on the $\mathrm{Sp}(2l, \mathbb{R})$-invariant decomposition of the curvature tensors of Fedosov connections.

Now, let us describe the geometric structure with help of which the symplectic Killing spinor fields are defined. This structure, called *metaplectic*, is a symplectic analogue of the notion of a spin structure in the Riemannian geometry. For a symplectic manifold (M^{2l}, ω) of dimension $2l$, let us denote the bundle of symplectic frame in TM by $\mathcal{P}$ and the foot-point projection of $\mathcal{P}$ onto M by p. Thus $(p : \mathcal{P} \to M, G)$, where $G \simeq \mathrm{Sp}(2l, \mathbb{R})$, is a principal G-bundle over M. As in Subsection 2, let $\lambda : \tilde{G} \to G$ be a member of the isomorphism class of the non-trivial two-fold coverings of the symplectic group G. In particular, $\tilde{G} \simeq \mathrm{Mp}(2l, \mathbb{R})$. Further, let us consider a principal $\tilde{G}$-bundle $(q : \mathcal{Q} \to M, \tilde{G})$ over the symplectic manifold (M, ω). We call a pair $(\mathcal{Q}, \Lambda)$ metaplectic structure if $\Lambda : \mathcal{Q} \to \mathcal{P}$ is a surjective bundle homomorphism over the identity on M and if the following diagram,

$$\begin{tikzcd}
\mathcal{Q} \times \tilde{G} \arrow[r] \arrow[dd, "\Lambda \times \lambda"] & \mathcal{Q} \arrow[dr, "q"] \arrow[dd, "\Lambda"] & \\
 & & M \\
\mathcal{P} \times G \arrow[r] & \mathcal{P} \arrow[ur, "p"'] &
\end{tikzcd}$$

with the horizontal arrows being respective actions of the displayed groups, commutes. See, e.g., Habermann, Habermann [6] and Kostant [8] for details on metaplectic structures. Let us only remark that typical examples of symplectic manifolds admitting a metaplectic structure are cotangent bundles of orientable manifolds (phase spaces), Calabi-Yau manifolds and complex projective spaces $\mathbb{CP}^{2k+1}$, $k \in \mathbb{N}_0$.

Let us denote the vector bundle associated to the introduced principal $\tilde{G}$-bundle $(q : \mathcal{Q} \to M, \tilde{G})$ via the representation ρ acting on $\mathbf{S}$ by $\mathcal{S}$, and call this associated vector bundle *symplectic spinor bundle*. Thus, we have $\mathcal{S} = \mathcal{Q} \times_\rho \mathbf{S}$. The sections $\phi \in \Gamma(M, \mathcal{S})$ will be called *symplectic spinor fields*. Further for $j_2 = 0, 1, 2$, we define the associated vector bundles $\mathcal{E}^{2j_2}$ by the prescription $\mathcal{E}^{2j_2} := \mathcal{Q} \times_\rho \mathbf{E}^{2j_2}$. Further, we define $\mathcal{E}^r := \Gamma(M, \mathcal{Q} \times_\rho \bigwedge^r \mathbb{V}^* \otimes \mathbf{S})$, i.e., the space o symplectic spinor valued differential r-forms, $r = 0, \ldots, 2l$. Because the symplectic Clifford multiplication is $\tilde{G}$-equivariant (see Habermann, Habermann [6]), we can lift it to the associated vector bundle structure, i.e., to let it act on the elements from $\Gamma(M, \mathcal{S})$. For $j_2 = 0, 1, 2$, let us denote the vector bundle projections $\Gamma(M, \mathcal{E}^2) \to \Gamma(M, \mathcal{E}^{2j_2})$ by p_{2j_2}, i.e., $p_{2j_2} : \Gamma(M, \mathcal{E}^2) \to \Gamma(M, \mathcal{E}^{2j_2})$ for all appropriate j_2. This definition makes sense because due to the decomposition result (Theorem 2) and Remark below Theorem 2, the $\tilde{G}$-module of symplectic spinor valued exterior 2-forms is multiplicity-free.

Let Z be the principal bundle connection on the principal G-bundle $(p : \mathcal{P} \to M, G)$ associated to the chosen Fedosov connection ∇ and $\tilde{Z}$ be a lift of Z to the principal $\tilde{G}$-bundle $(q : \mathcal{Q} \to M, \tilde{G})$. Let us denote by ∇^S the covariant derivative associated to $\tilde{Z}$. Thus, in particular, ∇^S acts on the symplectic spinor fields.

Any section ϕ of the associated vector bundle $\mathcal{S} = \mathcal{Q} \times_\rho \mathbf{S}$ can be equivalently considered as a $\tilde{G}$-equivariant $\mathbf{S}$-valued function on $\mathcal{Q}$. Let us denote this function by $\hat{\phi}$, i.e., $\hat{\phi} : \mathcal{Q} \to \mathbf{S}$. For a local adapted symplectic frame $s : U \to \mathcal{P}$, let us denote by $\overline{s} : U \to \mathcal{Q}$ one of the lifts of s to $\mathcal{Q}$. Finally, let us set $\phi_s := \hat{\phi} \circ \overline{s}$. Further for $q \in \mathcal{Q}$ and $\psi \in \mathbf{S}$, let us denote by $[q, \psi]$ the equivalence class in $\mathcal{S}$ containing (q, ψ). (As it is well known, the total space $\mathcal{S}$ of the symplectic spinor bundle is the product $\mathcal{Q} \times \mathbf{S}$ modulo an equivalence relation.)

Lemma 4. *Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Then for each $X \in \mathfrak{X}(M)$, $\phi \in \Gamma(M, \mathcal{S})$ and a local adapted symplectic frame $s : U \to \mathcal{P}$, we have*

$$\nabla^S_X \phi = [\overline{s}, X(\phi_s)] - \frac{\imath}{2} \sum_{i=1}^{l} [e_{i+l}.(\nabla_X e_i). - e_i.(\nabla_X e_{i+l}).]\phi \text{ and}$$

$$\nabla^S_X (Y.\phi) = (\nabla^S_X Y).\phi + X.\nabla^S_Y \phi.$$

Proof: See Habermann, Habermann [6]. □

The curvature tensor on symplectic spinor fields is defined by the formula

$$R^S(X, Y)\phi = \nabla^S_X \nabla^S_Y \phi - \nabla^S_Y \nabla^S_X \phi - \nabla^S_{[X,Y]} \phi,$$

where $\phi \in \Gamma(M, \mathcal{S})$ and $X, Y \in \mathfrak{X}(M)$.

In the next lemma, a part of the action of R^S on the space of symplectic spinors is described using just the symplectic Ricci curvature tensor field σ.

Lemma 5. *Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Then for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, we have*

$$p^{20} R^S \phi = \frac{\imath}{2l} \sigma^{ij} \omega_{kl} \epsilon^k \wedge \epsilon^l \otimes e_i.e_j.\phi.$$

Proof: See Krýsl [11]. □

4. Symplectic Killing spinor fields

In this section, we shall focus our attention to the symplectic Killing spinor fields. More specifically, we compute the symplectic Killing spinor fields on some Fedosov manifolds admitting a metaplectic structure and derive a necessary condition satisfied by a symplectic Killing spinor field.

Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. We call a non-zero section $\phi \in \Gamma(M, \mathcal{S})$ *symplectic Killing spinor field* if

$$\nabla^S_X \phi = \lambda X.\phi$$

for a complex number $\lambda \in \mathbb{C}$ and each vector field $X \in \mathfrak{X}(M)$. The complex number λ will sometimes be called symplectic Killing spinor number. (Allowing the zero section to be a symplectic Killing spinor would make the notion of a symplectic Killing spinor number meaningless.)

Let us note that one can rewrite equivalently the preceding defining equation for a symplectic Killing spinor into

$$\nabla^S \phi = \lambda F^+ \phi.$$

Indeed, if this equation is satisfied, we get by inserting the local vector field $X = X^i e_i$ the equation $\nabla^S_X \phi = \iota_X(\lambda \epsilon^i \otimes e_i.\phi) = \lambda \epsilon^i(X) e_i.\phi = \lambda X^i e_i.\phi = \lambda X.\phi$, i.e., the defining equation. Conversely, one can prove that $\nabla^S_X \phi = \lambda X.\phi$ is equivalent to $\iota_X \nabla^S \phi = \iota_X(\lambda F^+ \phi)$. Because this equation holds for each vector field X, we get $\nabla^S \phi = \lambda F^+ \phi$. We shall call both the defining equation and the equivalent equation for a symplectic Killing spinor field the symplectic Killing spinor equation.

In the next example, we compute the symplectic Killing spinors on the standard symplectic 2-plane.

Example 1. Let us solve the symplectic Killing spinor equation for the standard symplectic vector space $(\mathbb{R}^2[s,t], \omega_0)$ equipped with the standard flat Euclidean connection ∇. In this case, $(\mathbb{R}^2, \omega_0, \nabla)$ is also a Fedosov manifold. The bundle of symplectic frame in $T\mathbb{R}^2$ defines a principal $\mathrm{Sp}(2,\mathbb{R})$-bundle. Because $H^1(\mathbb{R}^2, \mathbb{R}) = 0$, we know that there exists, up to a bundle isomorphism, only one metaplectic bundle over $\mathbb{R}^2$, namely the trivial principal $\mathrm{Mp}(2,\mathbb{R})$-bundle $\mathbb{R}^2 \times \mathrm{Mp}(2,\mathbb{R}) \to \mathbb{R}^2$ and thus also a unique metaplectic structure $\Lambda : \mathrm{Mp}(2,\mathbb{R}) \times \mathbb{R}^2 \to \mathrm{Sp}(2,\mathbb{R}) \times \mathbb{R}^2$ given by $\Lambda(g,(s,t)) := (\lambda(g),(s,t))$ for $g \in \mathrm{Mp}(2,\mathbb{R})$ and $(s,t) \in \mathbb{R}^2$. Let $\mathcal{S} \to \mathbb{R}^2$ be the symplectic spinor bundle. In this case $\mathcal{S} \to \mathbb{R}^2$ is isomorphic to the trivial vector bundle $\mathbf{S} \times \mathbb{R}^2 = L^2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$. Thus, we may think of a symplectic spinor field ϕ as of a mapping $\phi : \mathbb{R}^2 \to \mathbf{S} = L^2(\mathbb{R})$. Let us define $\psi : \mathbb{R}^3 \to \mathbb{C}$ by $\psi(s,t,x) := \phi(s,t)(x)$ for each $(s,t,x) \in \mathbb{R}^3$. One easily shows that ϕ is a symplectic Killing spinor if and only if the function ψ satisfies the system

$$\begin{aligned} \frac{\partial \psi}{\partial s} &= \lambda \imath x \psi \quad \text{and} \\ \frac{\partial \psi}{\partial t} &= \lambda \frac{\partial \psi}{\partial x}. \end{aligned}$$

If $\lambda = 0$, the solution of this system of partial differential equations is necessarily $\psi(s,t,x) = \overline{\psi}(x)$, $(s,t,x) \in \mathbb{R}^3$, for any $\overline{\psi} \in L^2(\mathbb{R})$.

If $\lambda \neq 0$, let us consider the independent variable and corresponding dependent variable transformation $s = s, y = t + \lambda^{-1}x$, $z = t - \lambda^{-1}x$ and $\psi(s,t,x) = \widetilde{\psi}(s, t + \lambda^{-1}x, t - \lambda^{-1}x) = \widetilde{\psi}(s,y,z)$. The Jacobian of this transformation is $-2/\lambda \neq 0$ and the transformation is obviously a diffeomorphism. Substituting this transformation in the studied system, one gets the following equivalent transformed system

$$\begin{aligned}\frac{\partial\widetilde{\psi}}{\partial s} &= \frac{\imath}{2}\lambda^2(y-z)\widetilde{\psi}\\ \frac{\partial\widetilde{\psi}}{\partial y}+\frac{\partial\widetilde{\psi}}{\partial z} &= \lambda(\frac{\partial\widetilde{\psi}}{\partial y}\lambda^{-1}+\frac{\partial\widetilde{\psi}}{\partial z}(-\lambda^{-1})).\end{aligned}$$

(Let us notice that the substitution we have used is similar to that one which is usually used to obtain the d'Alemebert's solution of the wave equation in two dimensions.) The first equation implies $\frac{\partial\widetilde{\psi}}{\partial z}=0$, and thus $\widetilde{\psi}(s,y,z)=\overline{\overline{\psi}}(s,y)$ for a function $\overline{\overline{\psi}}$. Substituting this relation into the second equation of the transformed system, we get

$$\frac{\partial\overline{\overline{\psi}}}{\partial s}=\frac{\imath}{2}(y-z)\lambda^2\overline{\overline{\psi}}.$$

The solution of this equation is $\overline{\overline{\psi}}(s,y)=e^{\frac{\imath}{2}\lambda^2(y-z)s}\widetilde{\overline{\psi}}(y)$ for a suitable function $\widetilde{\overline{\psi}}$. Because of the dependence of the right hand side of the last written equation on z, we see that $\overline{\overline{\psi}}$ does not exist unless $\lambda=0$ or $\widetilde{\overline{\psi}}=0$ (More formally, one gets these restrictions by substituting the last written formula for $\overline{\overline{\psi}}$ into the first equation of the transformed system.) Thus, necessarily $\psi=0$ or $\lambda=0$. The case $\lambda=0$ is excluded by the assumption at the beginning of this calculation.

Summing up, we have proved that any symplectic Killing spinor field ϕ on $(\mathbb{R}^2,\omega_0,\nabla)$ is constant, i.e., for each $(s,t)\in\mathbb{R}^2$, we have $\phi(s,t)=\overline{\psi}$ for a function $\overline{\psi}\in L^2(\mathbb{R})$. The only symplectic Killing spinor number is zero in this case.

Remark. More generally, one can treat the case of a standard symplectic vector space $(\mathbb{R}^{2l}[s^1,\dots,s^l,t^1,\dots,t^l],\omega_0)$ equipped with the standard flat Euclidean connection ∇. One gets by similar lines of reasoning that any symplectic Killing spinor for this Fedosov manifold is also constant, i.e.,

$$\psi(s^1,\dots,s^l,t^1,\dots,t^l)=\overline{\psi},$$

for $(s^1,\dots,s^l),(t^1,\dots,t^l)\in\mathbb{R}^l$ and $\overline{\psi}\in L^2(\mathbb{R}^l)$. But we shall see this result more easily below when we will be studying the prolongated equation mentioned in the Introduction.

Now, in order to make a connection of the symplectic Killing spinor equation to some slightly more known equations, let us introduce the following operators.

The operator

$$\mathfrak{D}:\Gamma(M,\mathcal{S})\to\Gamma(M,\mathcal{S}),\ \mathfrak{D}:=-F^{-}\nabla^{S}$$

is called symplectic Dirac operator and its eigenfunctions are called symplectic Dirac spinors. Let us notice that the symplectic Dirac operator was introduced by Katharina Habermann in 1992. See, e.g., Habermann [5].

The operator

$$\mathfrak{T} : \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{E}^{11}), \ \mathfrak{T} := \nabla^S - p^{10}\nabla^S$$

is called (the first) symplectic twistor operator.

In the next theorem, the symplectic Killing spinor fields are related to the symplectic Dirac spinors and to the kernel of the symplectic twistor operator.

Theorem 6. *Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. A symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$ is a symplectic Killing spinor field if and only if ϕ is a symplectic Dirac spinor lying in the kernel of the symplectic twistor operator.*

Proof: We prove this equivalence in two steps.

(1) Suppose $\phi \in \Gamma(M, \mathcal{S})$ is a symplectic Killing spinor to a symplectic Killing number $\lambda \in \mathbb{C}$. Thus it satisfies the equation $\nabla^S\phi = \lambda F^+\phi$. Applying the operator $-F^-$ to the both sides of the preceding equation and using the definition of the symplectic Dirac operator, we get $\mathfrak{D}\phi = -\lambda F^- F^+\phi = \lambda(-H + F^+F^-)\phi = -\lambda H\phi = -\lambda(-\imath l\phi) = \imath\lambda l\phi$ due to the definition of H and Lemma 3(2)(a) and (3). Thus ϕ is a symplectic Dirac spinor.

Now, we compute $\mathfrak{T}\phi$. Using the definition of $\mathfrak{T}$, we get $\mathfrak{T}\phi = (\nabla^S - p^{10}\nabla^S)\phi = \lambda(F^+\phi - p^{10}F^+\phi) = \lambda p^{11}F^+\phi = 0$, because $F^+\phi \in \Gamma(M, \mathcal{E}^{10})$ due to Lemma 3(2)(a).

(2) Conversely, let $\phi \in \Gamma(M, \mathcal{E}^{00})$ be in the kernel of the symplectic twistor operator and also a symplectic Dirac spinor. Thus, we have $\nabla^S\phi - p^{10}\nabla^S\phi = 0$ and $\mathfrak{D}\phi = -F^-\nabla^S\phi = \mu\phi$ for a complex number $\mu \in \mathbb{C}$. From the first equation, we deduce that $\psi := \nabla^S\phi \in \Gamma(M, \mathcal{E}^{10})$. Because $F^+_{|\Gamma(M,\mathcal{E}^{00})}$ is surjective onto $\Gamma(M, \mathcal{E}^{10})$ (see Lemma 3(2)(b)), there exists a $\psi' \in \Gamma(M, \mathcal{E}^{00})$ such that $\psi = F^+\psi'$. Let us compute $F^+F^-\psi = F^+F^-F^+\psi' = F^+(H - F^+F^-)\psi' = F^+(-\imath l\psi') = -\imath l\psi$, where we have used the defining equation for H and Lemma 3(2)(a) and (3). Thus we get

$$-F^+F^-\psi = \imath l\psi. \tag{1}$$

From the symplectic Dirac equation, we get $\mu\phi = -F^-\psi$. Thus $-F^+F^-\psi = \mu F^+\phi$. Using the equation (1), we obtain $\imath l\psi = \mu F^+\phi$, i.e., $\nabla^S\phi = -\imath\frac{\mu}{l}F^+\phi$. Thus, ϕ is a symplectic Killing spinor to the symplectic Killing spinor number $-\imath\mu/l$. □

In the next theorem, we derive the mentioned prolongation of the symplectic Killing spinor equation. It is a zeroth order equation. More precisely, it is an equation for the sections of the kernel of an endomorphism of the symplectic spinor bundle $\mathcal{S} \to M$. A similar computation is well known from the Riemannian spin geometry. See, e.g., Friedrich [2].

Theorem 7. *Let (M^{2l}, ω, ∇) be a Fedosov manifold admitting a metaplectic structure and a symplectic Killing spinor field $\phi \in \Gamma(M, \mathcal{S})$ to the symplectic Killing spinor number λ. Then*

$$\sigma^{ij} e_i.e_j.\phi = 2l\lambda^2\phi.$$

Proof: Let $\phi \in \Gamma(M^{2l}, \mathcal{S})$ be a symplectic spinor Killing field, i.e., $\nabla^S_X \phi = \lambda X.\phi$ for a complex number λ and any vector field $X \in \mathfrak{X}(M)$. For vector fields $X, Y \in \mathfrak{X}(M)$, we may write

$$\begin{aligned}
R^S(X,Y)\phi &= (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\phi \\
&= \lambda\nabla_X(Y.\phi) - \lambda\nabla_Y(X.\phi) - \lambda[X,Y].\phi \\
&= \lambda(\nabla_X Y).\phi + \lambda Y.(\nabla_X \phi) - \lambda(\nabla_Y X).\phi - \lambda X.\nabla_Y.\phi - \lambda[X,Y].\phi \\
&= \lambda T(X,Y).\phi + \lambda^2(Y.X. - Y.X.)\phi \\
&= \lambda T(X,Y).\phi + \imath\lambda^2\omega(X,Y)\phi = \imath\lambda^2\omega(X,Y)\phi,
\end{aligned}$$

where we have used the symplectic Killing spinor equation and the compatibility of the symplectic spinor covariant derivative and the symplectic Clifford multiplication (Lemma 4).

Thus $R^S\phi = \imath\lambda^2\omega \otimes \phi$. Because of Lemma 3(2)(c), we know that the right hand side is in $\Gamma(M, \mathcal{E}^{20})$. Thus also $R^S\phi = p^{20}R^S\phi$. Using Lemma 5, we get $\frac{\imath}{2l}\omega \otimes \sigma^{ij} e_i.e_j.\phi = \imath\lambda^2\omega \otimes \phi$. Thus $\sigma^{ij} e_i.e_j.\phi = 2l\lambda^2\phi$ and the theorem follows. □

Remark. Let us recall that in the Riemannian spin geometry (positive definite case), the existence of a non-zero Killing spinor implies that the manifold is Einstein. Further, let us notice that if the symplectic Ricci curvature tensor σ is (globally) diagonalizable by a symplectomorphism, the prolongated equation has the shape of the equation for eigenvalues of the Hamiltonian of an elliptic l dimensional harmonic oscillator with possibly varying axes lengths. An example of a diagonalizable symplectic Ricci curvature will be treated in Example 3. Although, in this case the axis will be constant and the harmonic oscillator will be spherical.

Now, we derive a simple consequence of the preceding theorem in the case of Fedosov manifolds of Weyl type, i.e., $\sigma = 0$.

Corollary 8. *Let (M, ω, ∇) be a Fedosov manifold of Weyl type. Let (M, ω) admit a metaplectic structure and a symplectic Killing spinor ϕ field to the symplectic Killing spinor number λ. Then the symplectic Killing spinor number $\lambda = 0$ and ϕ is locally covariantly constant.*

Proof: Follows immediately from the preceding theorem and the symplectic Killing spinor equation. □

Example 2. Let us go back to the case of $(\mathbb{R}^{2l}, \omega_0, \nabla)$ from Remark below Example 1. Corollary 8 implies that any symplectic Killing spinor field for this structure is covariantly constant, i.e., in fact constant in this case, and any symplectic Killing number is zero. In this case, we see that the prolongated equation from Theorem 7 makes it possible to compute the symplectic Killing spinor fields without any big effort, compared to the calculations in Example 1 where the 2-plane was treated.

In the next example, we compute the symplectic Killing spinor fields on S^2 equipped with the standard symplectic structure and the Riemannian connection of the round metric. This is an example of a Fedosov manifold (specified more carefully below) for which one cannot use Corollary 8, because it is not of Weyl type. But still, one can use Theorem 7.

Example 3. Consider the round sphere $(S^2, r^2(d\theta^2 + \sin^2\theta d\phi^2))$ of radius $r > 0$, θ being the longitude a ϕ the latitude. Then $\omega := r^2 \sin\theta d\theta \wedge d\phi$ is the volume form of the round sphere. Because ω is also a symplectic form, (S^2, ω) is a symplectic manifold. Let us consider the Riemannian connection ∇ of the round sphere. Then ∇ preserves the symplectic volume form ω being a metric connection of the round sphere. Because ∇ is torsion-free, we see that (S^2, ω, ∇) is a Fedosov manifold. Now, we will work in a coordinate patch without mentioning it explicitly. Let us set $e_1 := \frac{1}{r}\frac{\partial}{\partial\theta}$ and $e_2 := \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$. Clearly, $\{e_1, e_2\}$ is a local adapted symplectic frame and it is a local orthogonal frame as well. With respect to this basis, the Ricci form σ of ∇ takes the form

$$[\sigma^{ij}]_{i,j=1,2} = \begin{pmatrix} 1/r & 0 \\ 0 & 1/r \end{pmatrix}.$$

Let us consider S^2 as the complex projective space $\mathbb{CP}^1$. It is easy to see that the (unique) complex structure on $\mathbb{CP}^1$ is compatible with the volume form. The first Chern class of the tangent bundle to $\mathbb{CP}^1$ is known to be even. Thus, the symplectic manifold (S^2, ω) admits a metaplectic structure and we may consider a symplectic Killing spinor field $\phi \in \Gamma(S^2, \mathcal{S})$ corresponding to a symplectic Killing spinor number λ. Because the first homology group of the sphere S^2 is zero, the metaplectic structure is unique and thus the trivial one. Because of the triviality of the associated symplectic spinor bundle $\mathcal{S} \to S^2$, we may write $\phi(m) = (m, f(m))$ where $f(m) \in L^2(\mathbb{R})$ for each $m \in S^2$. Using Theorem 7 and the prescription for the Ricci form, we get that $\sigma^{ij} e_i.e_j.[f(m)] = \frac{1}{r}H[f(m)] = 2\lambda^2 f(m)$, where $H = \frac{\partial^2}{\partial x^2} - x^2$ is the quantum Hamiltonian of the one dimensional harmonic oscillator. The solutions of the Sturm-Liouville type equation $H[f(m)] = 2r\lambda^2 f(m)$, $m \in S^2$, are well known. The eigenfunctions of H are the Hermite functions $f_l(m)(x) = h_l(x) := e^{x^2/2}\frac{d^l}{dx^l}(e^{-x^2})$ for $m \in S^2$ and $x \in \mathbb{R}$ and the corresponding eigenvalues are $-(2l+1)$, $l \in \mathbb{N}_0$. Thus $2r\lambda^2 = -(2l+1)$ and consequently

$$\lambda = \pm\imath\sqrt{\frac{2l+1}{2r}}.$$

Using the fact that $\{e_1, e_2\}$ is a local orthonormal frame and ∇ is metric and torsion-free, we easily get

$$\begin{array}{ll} \nabla_{e_1} e_1 = 0 & \nabla_{e_1} e_2 = 0 \\ \nabla_{e_2} e_1 = \frac{\cot\theta}{r} e_2 & \nabla_{e_2} e_2 = -\frac{\cot\theta}{r} e_1. \end{array}$$

From the definition of differentiability of functions with values in a Hilbert space, we see easily as a consequence of the preceding computations that any symplectic Killing spinor field is necessarily of the form $\phi(m) = (m, c(m) f_l(m))$ for a smooth function $c \in \mathcal{C}^\infty(S^2, \mathbb{C})$. Substituting this Ansatz into the symplectic Killing spinor equation, we get for each vector field $X \in \mathfrak{X}(S^2)$ the equation

$$\nabla_X(c f_l) = (Xc) f_l + c\nabla_X f_l = \lambda c(X.f_l).$$

Due to Lemma 4, we have for a local adapted symplectic frame $s : U \subseteq S^2 \to \mathcal{P} = \mathrm{Sp}(2, \mathbb{R}) \times S^2$,

$$\nabla_X f_l = [\overline{s}, X(f_l)_s] - \frac{\imath}{2}[e_2.(\nabla_X e_1). - e_1.(\nabla_X e_2).] f_l.$$

(See the paragraph above Lemma 4 for an explanation of the notation used in this formula.)

Because $m \mapsto (m, f_l(m))$ is constant as a section of the trivial bundle $\mathcal{S} \to S^2$, the first summand of the preceding expression vanishes. Thus for $X = e_1$, we get

$$(e_1 c) f_l + \frac{\imath c}{2}[e_2.(\nabla_{e_1} e_1). - e_1.(\nabla_{e_1} e_2).] f_l = \lambda c(e_1.f_l).$$

Using the knowledge of the values of $\nabla_{e_1} e_j$, for $j = 1, 2$, computed above, the second summand at the left hand side of the last written equation vanishes and thus, we get

$$\frac{1}{r}\frac{\partial c}{\partial \theta} f_l = \lambda c \imath x f_l.$$

This equation implies $c(\theta, \phi) = \psi(x, \phi) e^{\imath r x \lambda \theta}$ for x such that $h_l(x) \neq 0$ and a suitable function ψ. (The set of such $x \in \mathbb{R}$, such that $h_l(x) \neq 0$ is the complement in $\mathbb{R}$ of a finite set.) Because $r > 0$ is given and λ is certainly non-zero (see the prescription for λ above), the only possibility for c to be independent of x is $\psi = 0$. Therefore $c = 0$ and consequently $\phi = 0$. On the other hand, $\phi = 0$ (the zero section) is clearly a solution, but according to the definition not a symplectic Killing spinor. Thus, there is no symplectic Killing spinor field on the round sphere.

Remark. In the future, one can study holonomy restrictions implied by the existence of a symplectic Killing spinor. One can also try to extend the results to general symplectic connections, i.e., to drop the condition on the torsion-freeness or study also the symplectic Killing fields on Ricci type Fedosov manifolds admitting a metaplectic structure in more detail.

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Howe Duality for the Metaplectic Group Acting on Symplectic Spinor Valued Forms

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Abstract. Let $\mathbb{S}$ denote the oscillatory module over the complex symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$. Consider the $\mathfrak{g}$-module $\mathbb{W} = \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ of forms with values in the oscillatory module. We prove that the associative commutant algebra $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$ is generated by the image of a certain representation of the ortho-symplectic Lie super algebra $\mathfrak{osp}(1|2)$ and two distinguished projection operators. The space $\mathbb{W}$ is then decomposed with respect to the joint action of $\mathfrak{g}$ and $\mathfrak{osp}(1|2)$. This establishes a Howe type duality for $\mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$ acting on $\mathbb{W}$.
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1. Introduction

Let $(\mathbb{V}, \omega)$ be a real finite dimensional symplectic vector space. We denote the symplectic group $Sp(\mathbb{V}, \omega)$ by G, and its connected double cover, i.e., the metaplectic group $Mp(\mathbb{V}, \omega)$, by $\widetilde{G}$. Further, let K denote the maximal compact subgroup of $\widetilde{G}$ and $\mathfrak{g}$ the complexification of the Lie algebra of G. The complexification of the Lie algebra of the metaplectic group $\widetilde{G}$ is isomorphic to $\mathfrak{g}$ and thus, we may denote it by $\mathfrak{g}$ as well.

There exists a distinguished faithful unitary representation of the metaplectic group $\widetilde{G}$ – the so called Segal-Shale-Weil or symplectic spinor representation. (Let us note that also the names oscillatory or metaplectic representation are used in the literature.) For a justification of the latter name, see Kostant [8]. Now, let us consider the underlying Harish-Chandra $(\mathfrak{g}, K)$-module of the Segal-Shale-Weil representation. When we think of this $(\mathfrak{g}, K)$-module as equipped with its $\mathfrak{g}$-module structure only, we denote it by $\mathbb{S}$ and call it the *oscillatory module.* It is known that $\mathbb{S}$ splits into two irreducible $\mathfrak{g}$-modules, $\mathbb{S} \simeq \mathbb{S}^+ \oplus \mathbb{S}^-$.

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Further, let us set $\mathbb{W} = \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ and denote the appropriate tensor product representation of $\mathfrak{g}$ on $\mathbb{W}$ by ρ. In this paper, we first decompose the module $\mathbb{W}$ into irreducible $\mathfrak{g}$-modules. Next, we shall find generators of the commutant algebra

$$\mathrm{End}_{\mathfrak{g}}(\mathbb{W}) = \{T \in \mathrm{End}(\mathbb{W}) \mid T\rho(X) = \rho(X)T \text{ for all } X \in \mathfrak{g}\}$$

of the symplectic Lie algebra $\mathfrak{g}$ acting on $\mathbb{W}$. Let $p^{\pm} : \mathbb{S} \to \mathbb{S}^{\pm}$ be the unique $\mathfrak{g}$-equivariant projections. These projections induce projection operators acting on the whole space $\mathbb{W}$ in an obvious way. We denote them by $p^{\pm}$ as well. Further, we shall introduce a representation $\sigma : \mathfrak{osp}(1|2) \to \mathrm{End}(\mathbb{W})$ of the complex ortho-symplectic super Lie algebra $\mathfrak{osp}(1|2)$ on the space $\mathbb{W}$ and prove that the image of σ together with p^+ and p^- generate the commutant $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$. At the end, we decompose the $(\mathfrak{g} \times \mathfrak{osp}(1|2))$-module $\mathbb{W}$ into a direct sum

$$\bigoplus_{j=0}^{l}[(\mathbb{E}^-_{jj} \oplus \mathbb{E}^+_{jj}) \otimes \mathbb{G}^j],$$

where $\mathbb{E}^+_{jj}$ and $\mathbb{E}^-_{jj}$ are certain irreducible infinite dimensional highest weight $\mathfrak{g}$-modules and $\mathbb{G}^j$ is a finite dimensional irreducible $\mathfrak{osp}(1|2)$-module. This establishes a Howe type duality for $\mathfrak{g}$ acting on $\mathbb{W}$. One may call this duality of type $2:1$ because each irreducible $\mathfrak{osp}(1|2)$-module $\mathbb{G}^j$ from the decomposition above is paired to two irreducible $\mathfrak{g}$-modules, namely to $\mathbb{E}^+_{jj}$ and $\mathbb{E}^-_{jj}$.

The basic tool used to obtain these results was the decomposition of the $\mathfrak{g}$-module $\mathbb{W}$ into irreducible summands. This decomposition was achieved using a theorem of Britten, Hooper, Lemire [1] on a decomposition of the tensor product of an irreducible finite dimensional $\mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$-module and the oscillatory module $\mathbb{S}$. Let us remark that the so called Howe dualities are generalizations of classical results of Schur and Weyl. Whereas Schur studied the case of $GL(\mathbb{V})$ acting on the k-fold product $\bigotimes^k \mathbb{V}$, Weyl (see, e.g., Weyl [15]) considered the $SO(\mathbb{V})$-module $\bigotimes^k \mathbb{V}$, $k \in \mathbb{N}$. See Howe [5] for a historical treatment on the cases studied by Schur and Weyl and for their generalizations. In Howe [5], one can find several applications of these dualities and also a classical version of our 2:1 or say, quantum duality. Let us remark that a similar result to the one presented here was obtained by Slupinski in [13]. In his paper, Slupinski considers the case of spinor valued forms as a module over the appropriate spin group. Roughly speaking, he proves that $\mathfrak{sl}(2, \mathbb{C})$ is the Howe dual partner to the spin group. One may rephrase this fact by saying that the situation studied in [13] is super symmetric to the one we are interested in.

The motivation for our study of the Howe duality for forms with values in the oscillatory module comes from differential geometry and mathematical physics. See, e.g., Habermann, Habermann [4] or Krýsl [10] for applications and examples in differential geometry. For applications of symplectic spinors in mathematical physics, we refer an interested reader to Shale [12], who used them to quantize Klein-Gordon fields, and to Kostant [8] for a use in geometric quantization of Hamiltonian mechanics.

In the second section of the paper, we introduce basic notation, summarize known facts on the oscillatory module and derive the decomposition of $\mathbb{W} = \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ into irreducible $\mathfrak{g}$-modules (Theorem 2.3). The generators of the commutant $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$ are given in the third section (Theorem 3.7). In the fourth section, the representation $\sigma : \mathfrak{osp}(1|2) \to \mathrm{End}(\mathbb{W})$ is introduced and the fact that it is a representation is proved (Theorem 4.1). In this section, the space $\mathbb{W}$ is also decomposed into submodules with respect to the joint action of $\mathfrak{g}$ and $\mathfrak{osp}(1|2)$, i.e., the Howe duality is proved (Theorem 4.5).

2. Decomposition of $\mathbb{W} = \bigwedge^\bullet(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$

Let us suppose that $\mathfrak{g}$ is a complex simple Lie algebra and let us choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a set of positive roots Φ^+. We denote the complex irreducible highest weight $\mathfrak{g}$-module with a highest weight $\mu \in \mathfrak{h}^*$ by $L(\mu)$. If μ happens to be dominant and integral with respect to the choice $(\mathfrak{h}, \Phi^+)$, we denote the module $L(\mu)$ by $F(\mu)$, emphasizing the fact that the module $L(\mu)$ is finite dimensional. For a dominant integral weight μ with respect to $(\mathfrak{h}, \Phi^+)$, we denote the set of weights of the irreducible representation $F(\mu)$ by $\Pi(\mu)$.

Now, let us restrict our attention to the studied symplectic case. Consider a $2l$ dimensional real symplectic vector space $(\mathbb{V}, \omega)$. Let $\mathbb{V} = \mathbb{L} \oplus \mathbb{L}'$ be a direct sum decomposition of the vector space $\mathbb{V}$ into two Lagrangian subspaces $\mathbb{L}$ and $\mathbb{L}'$. Further, let $\{e_i\}_{i=1}^{2l}$ be an adapted symplectic basis of $(\mathbb{V}, \omega)$, i.e., $\{e_i\}_{i=1}^{2l}$ is a symplectic basis of $(\mathbb{V}, \omega)$ and $\{e_i\}_{i=1}^{l} \subseteq \mathbb{L}$ and $\{e_i\}_{i=l+1}^{2l} \subseteq \mathbb{L}'$. Because the notion of a symplectic basis is not unique, let us fix it now. We call a basis $\{e_i\}_{i=1}^{2l}$ of $\mathbb{V}$ a symplectic basis of $(\mathbb{V}, \omega)$ if for $\omega_{ij} = \omega(e_i, e_j)$, we have

$\omega_{ij} = 1$ if an only if $i \leq l$ and $j = i + l$,

$\omega_{ij} = -1$ if and only if $i > l$ and $j = i - l$ and

$\omega_{ij} = 0$ in other cases.

The basis of $\mathbb{V}^*$ dual to the basis $\{e_i\}_{i=1}^{2l}$ will be denoted by $\{\epsilon^i\}_{i=1}^{2l}$.

Let us denote the symplectic group $Sp(\mathbb{V}, \omega)$ by G and the metaplectic group by $\widetilde{G}$. We shall denote the complex symplectic Lie algebra, i.e., the Lie algebra $\mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$, by $\mathfrak{g}$. The complexified symplectic form on $\mathbb{V}^{\mathbb{C}}$ will still be denoted by ω. Because the complexification of the Lie algebra of $\widetilde{G}$ is isomorphic to $\mathfrak{g}$, we will identify them and denote both of them by $\mathfrak{g}$. If a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and a set of positive roots Φ^+ are chosen, the set of fundamental weights $\{\varpi_i\}_{i=1}^{l}$ is uniquely determined. Now, we shall consider a basis $\{\epsilon_i\}_{i=1}^{l}$ of $\mathfrak{h}^*$ defined by the equations $\varpi_i = \sum_{j=1}^{i} \epsilon_j$, $i = 1, \ldots, l$. For $\mu = \sum_{i=1}^{l} \mu_i \epsilon_i$, we shall often denote $L(\mu)$ by $L(\mu_1, \ldots, \mu_l)$, or even by $L(\mu_1 \ldots \mu_l)$ only.

The Segal-Shale-Weil representation is a faithful unitary representation of the metaplectic group $\widetilde{G}$ on the complex vector space $L^2(\mathbb{L})$ of complex valued square Lebesgue integrable functions defined on $\mathbb{L}$. Because we would like to omit problems caused by dealing with unbounded operators, we shall consider the underlying Harish-Chandra $(\mathfrak{g}, K)$-module of the Segal-Shale-Weil representation. When we consider this $(\mathfrak{g}, K)$-module with its $\mathfrak{g}$-module structure only, we denote it by $\mathbb{S}$ and call it the oscillatory module. The appropriate representation will be denoted by L. In particular, we have the Lie algebra homomorphism

$$L : \mathfrak{g} \to \mathrm{End}(\mathbb{S}) \text{ at our disposal.}$$

It is known that $\mathbb{S}$ splits into two irreducible $\mathfrak{g}$-modules, $\mathbb{S} \simeq \mathbb{S}^+ \oplus \mathbb{S}^-$.

Further, one can define a representation of $\mathfrak{g}$ on the space $\mathbb{C}[z^1, \ldots, z^l]$ of polynomials such that $\mathbb{C}[z^1, \ldots, z^l] \simeq \mathbb{S}$ as $\mathfrak{g}$-modules. From now on, we shall consider $\mathbb{S}$ in this polynomial realization. Let us notice that in this realization, $\mathbb{S}^+$ is isomorphic to the space of even polynomials in $\mathbb{C}[z^1, \ldots, z^l]$ and $\mathbb{S}^-$ to the space of the odd ones. Moreover, one can prove that $\mathbb{S}^+ \simeq L(\lambda^0)$ and $\mathbb{S}^- \simeq L(\lambda^1)$, where $\lambda^0 = -\frac{1}{2}\varpi_l$ and $\lambda^1 = \varpi_{l-1} - \frac{3}{2}\varpi_l$. For more information on the Segal-Shale-Weil representation, see Weil [14] and Kashiwara, Vergne [7]. For information on the oscillatory module, see Britten, Hooper, Lemire [1].

In order to derive the studied type of Howe duality, we shall need the symplectic Clifford multiplication $\mathbb{V}^{\mathbb{C}} \times \mathbb{S} \to \mathbb{S}$ which enables us to multiply elements from the oscillatory module by elements from $\mathbb{V}^{\mathbb{C}}$. It is given by the following prescription

$$(e_i.s)(x) = \frac{\partial s}{\partial x^i}(x), \ \ (e_{i+l}.s)(x) = \imath x^i s(x), \ i = 1, \ldots, l, \tag{1}$$

where $x = \sum_{i=1}^{l} x^i e_i \in \mathbb{L}, s \in \mathbb{S}$, and it is extended linearly to the whole space $\mathbb{V}^{\mathbb{C}}$. The symplectic Clifford multiplication is basically the canonical quantization prescription.

Now, for $i = 0, 1$ and a dominant integral weight $\lambda = \sum_{j=1}^{l} \lambda_j \varpi_j \in \mathfrak{h}^*$, let us introduce a set $T_\lambda^i \subseteq \mathfrak{h}^*$. A weight $\mu \in \mathfrak{h}^*$ is an element of T_λ^i if and only if the numbers d_j, $j = 1, \ldots, l$, defined by $\lambda - \mu = \sum_{j=1}^{l} d_j \epsilon_j$ satisfy the following conditions

1) $d_j + \delta_{l,j}\delta_{1,i} \in \mathbb{N}_0$ for $j = 1, \ldots, l$,

2) $0 \le d_j \le \lambda_j$ for $j = 1, \ldots, l-1$, $0 \le d_l + \delta_{1,i} \le 2\lambda_l + 1$ and

3) $\sum_{j=1}^{l} d_j$ is even.

In what follows, we will need a result on the decomposition of the tensor product of a finite dimensional $\mathfrak{g}$-module with one of the modules $L(\lambda^i)$, $i = 0, 1$, into irreducible $\mathfrak{g}$-modules. This result was published in Britten, Hooper, Lemire [1].

Theorem 2.1. *For $i = 0, 1$ and a dominant integral weight μ, we have*

$$F(\mu) \otimes L(\lambda^i) \simeq \bigoplus_{\kappa \in T_\mu^i \cap \Pi(\mu)} L(\lambda^i + \kappa).$$

Proof. See Britten, Hooper, Lemire [1]. ■

Let us remark that there is a misprint in the original article of Britten, Hooper, Lemire [1].

For convenience, let us introduce a function $\mathrm{sgn} : \{+, -\} \to \{0, 1\}$ given by the prescription $\mathrm{sgn}(+) = 0$ and $\mathrm{sgn}(-) = 1$ and the $\mathfrak{g}$-modules

$$\mathbb{E}_{ij}^{\pm} = L(\underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{j}, \underbrace{-\tfrac{1}{2}, \cdots, -\tfrac{1}{2}}_{l-j-1}, -1 + \tfrac{1}{2}(-1)^{i+j+\mathrm{sgn}(\pm)}),$$

where $i = 0, \ldots, l-1$, $j = 0, \ldots, i$ and $i = l$, $j = 0, \ldots, l-1$. For $i = j = l$, we set $\mathbb{E}^+_{ll} = L(\frac{1}{2} \cdots \frac{1}{2})$ and $\mathbb{E}^-_{ll} = L(\frac{1}{2} \cdots \frac{1}{2} - \frac{5}{2})$. For $i = l+1, \ldots, 2l$ and $j = 0, \ldots, 2l-i$, we assume $\mathbb{E}^{\pm}_{ij} = \mathbb{E}^{\pm}_{(2l-i)j}$. In order to write the results as short as possible, for $i = 0, \ldots, l$, let us set $m_i = i$ and for $i = l+1, \ldots, 2l$, $m_i = 2l - i$. With these conventions, we define

$$\Xi = \{(i, j) \,|\, i = 0, \ldots, 2l, \ j = 0, \ldots, m_i\}$$

and consider $\mathbb{E}^{\pm}_{ij} = 0$ for $(i, j) \in \mathbb{Z}^2 \setminus \Xi$. Finally, we set $\mathbb{E}_{ij} = \mathbb{E}^+_{ij} \oplus \mathbb{E}^-_{ij}$. Now, let us derive the next

Lemma 2.2. *For $r = 1, \ldots, l$, we have*

$$\Pi(\varpi_r) \supseteq \{\sum_{s=1}^{r} \pm\epsilon^{i_s} \,|\, 1 \leq i_1 < \ldots < i_r \leq l\}.$$

Proof. It is not hard to see (see, e.g., Corollary 5.1.11. pp. 237 and Theorem 5.1.8. (3) pp. 236 in Goodman, Wallach [3]) that for $r = 1, \ldots, l$, the $\mathfrak{g}$-module $F(\varpi_r)$ is isomorphic to the $\mathbb{C}$-linear span of isotropic r-vectors in $\mathbb{V}^{\mathbb{C}}$ (i.e., of the multi-vectors $w = u_1 \wedge \ldots \wedge u_r$, where $\omega(u_i, u_j) = 0$ for $i, j = 1, \ldots, r$), on which $\mathfrak{g}$ acts via the linear extension of the dual to the defining representation of $\mathfrak{g} \subseteq \mathrm{End}(\mathbb{V}^{\mathbb{C}})$ on $\mathbb{V}^{\mathbb{C}}$. Second, it is easy to realize that one can choose the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the set of positive roots Φ^+ in a way that the following is true. For $i = 1, \ldots, l$, the basis vector $e_i \in \mathbb{V}^{\mathbb{C}}$ is a weight vector of weight ϵ_i and the vector e_{i+l} is a weight vector of weight $-\epsilon_i$, both for the defining representation of $\mathfrak{g}$ on $\mathbb{V}^{\mathbb{C}}$. Using this fact, the result follows. ■

Now, we define the module $\mathbb{W}$, which we have mentioned in the Introduction. As a vector space

$$\mathbb{W} = \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}.$$

The representation $\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{W})$ of $\mathfrak{g}$ on $\mathbb{W}$ is defined by the prescription

$$\rho(X)(\alpha \otimes s) = X\alpha \otimes L(X)s,$$

where $X \in \mathfrak{g}$, $\alpha \in \bigwedge^i(\mathbb{V}^*)^{\mathbb{C}}$, $s \in \mathbb{S}$ and $i = 0, \ldots, 2l$. In the prescription above, the symbol $X\alpha$ refers to the action of $X \in \mathfrak{g} \subseteq \mathrm{End}(\mathbb{V}^{\mathbb{C}})$ on $\bigwedge^i(\mathbb{V}^*)^{\mathbb{C}}$, i.e., to the representation dual to the defining one and extended to the exterior i-forms linearly.

Now, we can state the decomposition theorem. Its proof is based on a direct use of Theorem 2.1 and Lemma 2.2.

Theorem 2.3. *For $i = 0, \ldots, 2l$, the following decomposition into irreducible $\mathfrak{g}$-modules*

$$\bigwedge^{i}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} \simeq \bigoplus_{j=0}^{m_i} \mathbb{E}^{\pm}_{ij} \qquad \textit{holds.}$$

Proof. Using Theorem 5.1.8. pp. 236 and Corollary 5.1.9. pp. 237 in Goodman, Wallach [3], we get for $i = 2k,\ k \in \mathbb{N}_0$,

$$\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} = (F(\varpi_0) \oplus F(\varpi_2) \oplus \ldots \oplus F(\varpi_i)) \otimes \mathbb{S}^{\pm}, \qquad (2)$$

where $\varpi_0 = 0$ and $F(\varpi_0) \simeq \mathbb{C}$ denotes the trivial $\mathfrak{g}$-module.

Using the cited theorems in Goodman, Wallach [3] again, we obtain for $i = 2k+1,\ k \in \mathbb{N}_0$,

$$\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} = (F(\varpi_1) \oplus F(\varpi_3) \oplus \ldots \oplus F(\varpi_i)) \otimes \mathbb{S}^{\pm}. \qquad (3)$$

We shall consider the mentioned tensor products for $i = 0, \ldots, l$ only, because the result for $i = l+1, \ldots, 2l$, follows from the one for $i = 0, \ldots, l$ immediately due to the $\mathfrak{g}$-isomorphism $\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} \simeq \bigwedge^{2l-i} (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm}$ and the definition of $\mathbb{E}^{\pm}_{ij}$. Let us consider the tensor products by $\mathbb{S}^+$ and $\mathbb{S}^-$ separately.

1) First, let us consider the tensor product $\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^+$. Using Lemma 2.2 and Theorem 2.1, we easily compute that for $j = 1, \ldots, l$, $T^0_{\varpi_j} = \{\epsilon_1 + \ldots + \epsilon_j, \epsilon_1 + \ldots + \epsilon_{j-1} - \epsilon_l\} \subseteq \Pi(\varpi_j)$ and thus,

$$F(\varpi_j) \otimes \mathbb{S}^+ = L(\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{j}, \underbrace{-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}}_{l-j}) \oplus L(\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{j-1}, \underbrace{-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}}_{l-j}, -\tfrac{3}{2}),$$

where the relation $\varpi_j = \sum_{i=1}^{j} \epsilon_i$ was used. Adding up these terms according to (2) and (3), we obtain the statement of the theorem for both of the cases i is odd and i is even.

2) Now, let us consider the tensor product $\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^-$. Using Lemma 2.2, we easily compute that for $j = 1, \ldots, l-1$, we have $T^1_{\varpi_j} = \{\epsilon_1 + \ldots + \epsilon_j, \epsilon_1 + \ldots + \epsilon_{j-1} + \epsilon_l\} \subseteq \Pi(\varpi_j)$ and $T^1_{\varpi_l} = \{\epsilon_1 + \ldots + \epsilon_l, \epsilon_1 + \ldots + \epsilon_{l-1} - \epsilon_l\} \subseteq \Pi(\varpi_l)$. Therefore using Theorem 2.1, we get

$$F(\varpi_j) \otimes \mathbb{S}^- = L(\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{j-1}, \underbrace{-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}}_{l-j+1}) \oplus L(\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{j}, \underbrace{-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}}_{l-j-1}, -\tfrac{3}{2})$$

for $j = 1, \ldots, l-1$. For $j = l$, we obtain $F(\varpi_l) \otimes \mathbb{S}^- = L(\frac{1}{2} \ldots \frac{1}{2} - \frac{1}{2}) \oplus L(\frac{1}{2} \ldots \frac{1}{2} - \frac{5}{2})$ using Theorem 2.1 again. Adding up these terms according to (2) and (3), we obtain the statement of the theorem for both cases i is odd and i is even. ■

From now on, we shall consider $\mathbb{E}^{\pm}_{ij} \subseteq \bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm}$, $(i, j) \in \Xi$.

Remark 2.4. Due to Theorem 2.3 and the definitions of $\mathbb{E}^{\pm}_{ij}$, we know that for $i = 0, \ldots, 2l$, the $\mathfrak{g}$-module $\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ is multiplicity-free.

3. The commutant algebra $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$

We shall prove that the associative commutant algebra $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$ is generated by the below introduced elements – a "raising" operator, a "lowering" operator and two projections.

For $i = 0, \dots, 2l$ and $\alpha \otimes s = \alpha \otimes (s_+, s_-) \in \bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes (\mathbb{S}^+ \oplus \mathbb{S}^-)$, we set

$$F^+ : \bigwedge^{i} (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S} \to \bigwedge^{i+1} (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S},\ F^+(\alpha \otimes s) = \frac{\imath}{2} \sum_{j=1}^{2l} \epsilon^j \wedge \alpha \otimes e_j.s,$$

$$F^- : \bigwedge^{i} (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S} \to \bigwedge^{i-1} (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S},\ F^-(\alpha \otimes s) = \tfrac{1}{2} \sum_{j,k=1}^{2l} \omega_{jk} \iota_{e_j} \alpha \otimes e_k.s \text{ and}$$

$$p^{\pm} : \bigwedge^{i} (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S} \to \bigwedge^{i} (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S},\ p^{\pm}(\alpha \otimes s) = \alpha \otimes s_{\pm}$$

and extend them linearly to the whole space $\mathbb{W}$. Next, we consider the operator H defined by the formula

$$H = 2(F^+F^- + F^-F^+).$$

The values of the operator H are determined in the next

Lemma 3.1. *Let $(\mathbb{V}, \omega)$ be a symplectic vector space of dimension $2l$. Then for $i = 0, \dots, 2l$, we have*

$$H_{|\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}} = \tfrac{1}{2}(i - l) Id_{|\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}}.$$

Proof. The proof is straightforward, see Krýsl [9]. □ ■

Lemma 3.2. *The maps $F^{\pm}, p^{\pm}$ and H are $\mathfrak{g}$-equivariant with respect to the representation ρ of $\mathfrak{g}$ on $\mathbb{W}$.*

Proof. The operators $p^{\pm}$ are clearly $\mathfrak{g}$-equivariant. The $\mathfrak{g}$-equivariance of $F^{\pm}$ and H can be checked straightforwardly. See Krýsl [9] for a proof. ■

Definition 3.3. Let us denote the associative algebra generated by $F^{\pm}$ and $p^{\pm}$ by $\mathfrak{C}$.

Let us recall the definition of the commutant algebra

$$\mathrm{End}_{\mathfrak{g}}(\mathbb{W}) = \{T \in \mathrm{End}(\mathbb{W}) \,|\, T\rho(X) = \rho(X)T \text{ for all } X \in \mathfrak{g}\}.$$

Due to the previous lemma, we already know that $\mathfrak{C} \subseteq \mathrm{End}_{\mathfrak{g}}(\mathbb{W})$. Now, we shall prove that $\mathfrak{C}$ exhausts the whole commutant $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$. For convenience, let us set $\Xi_- = \Xi \setminus \{(i, 2l - i) \,|\, i = l, \dots, 2l\}$ and $\Xi_+ = \Xi \setminus \{(i, i) \,|\, i = 0, \dots, l\}$.

Lemma 3.4. *For each $(i, j) \in \Xi$, we have*

$$F^+_{|\mathbb{E}^{\pm}_{ij}} : \mathbb{E}^{\pm}_{ij} \xrightarrow{\sim} \mathbb{E}^{\mp}_{i+1,j} \quad \text{if } (i,j) \in \Xi_- \text{ and}$$

$$F^-_{|\mathbb{E}^{\pm}_{ij}} : \mathbb{E}^{\pm}_{ij} \xrightarrow{\sim} \mathbb{E}^{\mp}_{i-1,j} \quad \text{if } (i,j) \in \Xi_+.$$

Proof. First, for $(i,j) \in \Xi$, we prove that

$$F^- F^+_{|\mathbb{E}_{ij}} = \begin{cases} \frac{1}{4}\left(\frac{1+i-j}{2}\right) \mathrm{Id}_{|\mathbb{E}_{ij}} & \text{if } i+j \text{ is odd} \\ \frac{1}{4}\left(\frac{i+j}{2} - l\right) \mathrm{Id}_{|\mathbb{E}_{ij}} & \text{if } i+j \text{ is even.} \end{cases} \tag{4}$$

Let us fix an integer $j \in \{0,\ldots,l\}$ and proceed by the induction on the form degree i.

I. For $i = j$ and $\phi \in \mathbb{E}_{ii}$, let us compute $F^-F^+\phi = (\frac{1}{2}H - F^+F^-)\phi = \frac{1}{4}(i-l)\phi - F^+F^-\phi$ due to the definition of H and Lemma 3.1. We have $F^-_{|\mathbb{E}_{ii}} = 0$ because F^- is $\mathfrak{g}$-equivariant (Lemma 3.2), lowers the form degree by one and there is no summand isomorphic to $\mathbb{E}^+_{ii}$ or $\mathbb{E}^-_{ii}$ in $\bigwedge^{i-1}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ (see Theorem 2.3). Summing up, we have $F^-F^+\phi = \frac{1}{4}(i-l)\phi$ according to (4).

II. Now, let us suppose the statement is true for $(i,j) \in \Xi$, $i+j$ odd. For $(i+1,j) \in \Xi$ and $\phi \in \mathbb{E}_{i+1,j}$, let us compute $F^-F^+\phi = \frac{1}{2}H\phi - F^+F^-\phi = \frac{1}{4}(i+1-l)\phi - F^+F^-\phi$ due to the definition of H and Lemma 3.1. Using the induction hypothesis, we have $F^-F^+_{|\mathbb{E}_{ij}} = \frac{1}{4}(\frac{1+i-j}{2})\mathrm{Id}_{|\mathbb{E}_{ij}}$. Thus, $F^+_{|\mathbb{E}_{ij}}$ is injective. Because F^+ is $\mathfrak{g}$-equivariant, raises the form degree by one and there is no other summand in $\bigwedge^{i+1}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ isomorphic to $\mathbb{E}_{ij}$ than $\mathbb{E}_{i+1,j}$, we see that $F^+_{|\mathbb{E}_{ij}} : \mathbb{E}_{ij} \to \mathbb{E}_{i+1,j}$. Because of the proved injectivity, $F^+_{|\mathbb{E}_{ij}}$ is actually an isomorphism. Thus, there exists $\widetilde{\phi} \in \mathbb{E}_{ij}$ such that $\phi = F^+\widetilde{\phi}$. We may write $F^+F^-\phi = F^+F^-(F^+\widetilde{\phi}) = F^+(F^-F^+\widetilde{\phi}) = \frac{1}{4}(\frac{1+i-j}{2})F^+\widetilde{\phi} = \frac{1}{4}(\frac{1+i-j}{2})\phi$ by the induction hypothesis. Substituting this relation into the already derived $F^-F^+\phi = \frac{1}{4}(i+1-l)\phi - F^+F^-\phi$, we get $F^-F^+\phi = \frac{1}{4}(i+1-l)\phi - \frac{1}{4}(\frac{1+i-j}{2})\phi = \frac{1}{4}(\frac{i+1+j}{2} - l)\phi$ according to the formula (4).

Now, let us suppose the statement is true for $(i,j) \in \Xi$, $i+j$ even. For $(i+1,j) \in \Xi$ and $\phi \in \mathbb{E}_{i+1,j}$, we compute $F^-F^+\phi = \frac{1}{2}H\phi - F^+F^-\phi = \frac{1}{4}(i+1-l)\phi - F^+F^-\phi$ due to the definition of H and Lemma 3.1. Similarly to the case $i+j$ is odd, we get the existence of $\widetilde{\phi} \in \mathbb{E}_{ij}$ such that $\phi = F^+\widetilde{\phi}$. Using the induction hypothesis, we may write $F^+F^-\phi = F^+F^-(F^+\widetilde{\phi}) = F^+(F^-F^+\widetilde{\phi}) = \frac{1}{4}(\frac{i+j}{2} - l)F^+\widetilde{\phi} = \frac{1}{4}(\frac{i+j}{2} - l)\phi$. Substituting this expression into the computation above, we get $F^-F^+\phi = \frac{1}{4}(i+1-l)\phi - \frac{1}{4}(\frac{i+j}{2} - l)\phi = \frac{1}{4}(\frac{1+(i+1)-j}{2})\phi$. Thus, the formula follows.

Using the derived formula (4), we see that $F^-F^+_{|\mathbb{E}_{ij}}$ is injective if and only if $i+j \neq 2l$ and $j \neq i+1$, i.e., $(i,j) \in \Xi_-$, the second condition being empty. Especially, F^+ is injective for $(i,j) \in \Xi_-$. Thus, F^+ is an isomorphism on $\mathbb{E}_{ij}$, $(i,j) \in \Xi_-$. From this, we may further conclude that F^- is injective on the image of F^+, i.e., it is an isomorphism on $\mathbb{E}_{ij}$ for $(i,j) \in \Xi_+$. ■

Remark 3.5. It is easy to see that F^- is zero when restricted to $\mathbb{E}^{\pm}_{ii}$, $i = 0,\ldots,l$. Namely, we know that F^- lowers the form degree by one, it is $\mathfrak{g}$-equivariant and there is no submodule of the module $\bigwedge^{i-1}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ isomorphic to $\mathbb{E}^+_{ii}$ or to $\mathbb{E}^-_{ii}$ (see Theorem 2.3). A similar discussion can be made for F^+ restricted to $\mathbb{E}_{im_i}$, $i = l,\ldots,2l$.

For $(i,j) \in \Xi$, let us denote the unique $\mathfrak{g}$-equivariant projections from the space $\mathbb{W}$ onto the submodules $\mathbb{E}^{\pm}_{ij}$ by $S^{\pm}_{ij}$, i.e.,

$$S^{\pm}_{ij} : \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} \to \mathbb{E}^{\pm}_{ij} \subseteq \bigwedge^{i}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm}.$$

Lemma 3.6. *For each $(i,j) \in \Xi$, the projections $S^{\pm}_{ij} \in \mathfrak{C}$.*

Proof. For $i = 0, \dots, 2l$, let us define the projection operators

$$S^{\pm}_{i} : \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} \to \bigwedge^{i}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm}$$

by the formula

$$S^{\pm}_{i} = \left(\prod_{j=0, j\neq i}^{2l} \frac{2H - j + l}{i - j} \right) p^{\pm}.$$

Using Lemma 3.1, we see that the image of each $S^{\pm}_{i}$ is the prescribed space and the normalization is correct, i.e., that the formula defines a projection. Recall that due to its definition, H can be expressed using the operators F^{+} and F^{-} only and thus, for $i = 0, \dots, 2l$, $S^{\pm}_{i} \in \mathfrak{C}$. Further, let us fix an integer $i \in \{0, \dots, 2l\}$. We prove that for each j, such that $(i,j) \in \Xi$, the projection $S^{\pm}_{ij} \in \mathfrak{C}$. We proceed by induction on j.

I. For $j = 0$, we define $S''_{i0} = (F^{+})^{i}(F^{-})^{i}$. Using the fact that applying F^{-} (or F^{+}) lowers (or raises) the form degree by 1, we see that $S''_{i0} : \bigwedge^{i}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} \to \mathbb{E}^{\pm}_{i0}$. Using the Schur lemma for complex irreducible highest weight modules (see Dixmier [2]), we conclude that there exists a complex number $\lambda_{i0} \in \mathbb{C}$ such that $S''_{i0|\mathbb{E}^{\pm}_{i0}} = \lambda_{i0} \mathrm{Id}_{|\mathbb{E}^{\pm}_{i0}}$. Due to Lemma 3.4, we know that $\lambda_{i0} \neq 0$. Thus, $S^{\pm}_{i0} = \frac{1}{\lambda_{i0}} S''_{i0} \circ S^{\pm}_{i}$. Because the operators F^{+}, F^{-}, p^{+} and p^{-} were used only, we get $S^{\pm}_{i0} \in \mathfrak{C}$.

II. Let us suppose that for $k = 0, \dots, j$, the operators $S^{\pm}_{ik}$ can be written as linear combinations of compositions of the operators $F^{\pm}$ and $p^{\pm}$. Now, we shall use the operators $S^{\pm}_{i0}, \dots, S^{\pm}_{ij}$ in order to define the operator $S^{\pm}_{i,j+1}$. Let us take an element $\xi \in \bigwedge^{i}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm}$ and define $\zeta := S'_{i,j+1}\xi := \xi - \sum_{k=0}^{j} S^{\pm}_{ik}\xi \in \bigoplus_{k=j+1}^{m_i} \mathbb{E}^{\pm}_{ik}$. Now, form an element $\zeta' := S''_{i,j+1}\zeta := (F^{+})^{i-j-1}(F^{-})^{i-j-1}\zeta$. In the same way as in item I., we conclude that $\zeta' \in \mathbb{E}^{\pm}_{i,j+1}$. Let us define $S'''_{i,j+1} = S''_{i,j+1} \circ S'_{i,j+1}$. Using the Schur lemma for $S'''_{i,j+1|\mathbb{E}^{\pm}_{i,j+1}} : \mathbb{E}^{\pm}_{i,j+1} \to \mathbb{E}^{\pm}_{i,j+1}$, we conclude that there is a complex number $\lambda_{i,j+1} \in \mathbb{C}$ such that $S'''_{i,j+1|\mathbb{E}^{\pm}_{i,j+1}} = \lambda_{i,j+1} \mathrm{Id}_{|\mathbb{E}^{\pm}_{i,j+1}}$. Due to Lemma 3.4, we know that $\lambda_{i,j+1} \neq 0$. Thus, $S^{\pm}_{i,j+1} = \frac{1}{\lambda_{i,j+1}} S'''_{i,j+1} \circ S^{\pm}_{i}$. Going through the construction back, we see that for constructing the operator $S^{\pm}_{i,j+1}$, only the operators $F^{\pm}$ and $p^{\pm}$ were used. ∎

Now we prove that the algebra $\mathfrak{C}$ exhausts the whole commutant $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$.

Theorem 3.7. *We have*

$$End_{\mathfrak{g}}(\mathbb{W}) = \mathfrak{C}.$$

Proof. Due to Lemma 3.2, we know that $\mathfrak{C} \subseteq \mathrm{End}_{\mathfrak{g}}(\mathbb{W})$. We prove the opposite inclusion. For $T \in \mathrm{End}_{\mathfrak{g}}(\mathbb{W})$, we may write $T = \bigoplus_{(i,j),(r,s)\in\Xi}(S_{ij}^{+}+S_{ij}^{-})T(S_{rs}^{+}+S_{rs}^{-})$. For fixed (i,j) and (r,s), let us consider the operator $A = S_{ij}^{+}TS_{rs}^{-} : \mathbb{W} \to \mathbb{E}_{ij}^{+}$. Due to Theorem 2.3, the operator is non-zero ony if $j = s$ and there is a $k \in \mathbb{Z}$ such that $i - r = 2k + 1$. Suppose $k \geq 0$. Due to the Schur lemma, A does not change if we replace the operator T, occurring in the middle of the expression for A, by a complex multiple of $(F^{+})^{2k+1}$ (Lemma 3.4). Thus, we have $A = cS_{ij}^{+}(F^{+})^{2k+1}S_{rs}^{-}$ for a complex number $c \in \mathbb{C}$. Because $S_{ij}^{+}, S_{rs}^{-} \in \mathfrak{C}$ (Lemma 3.6), we see that $A \in \mathfrak{C}$. Similarly, one can proceed in the case $k < 0$ and also when treating the remaining operators $S_{ij}^{+}TS_{rs}^{+}$, $S_{ij}^{-}TS_{rs}^{-}$ and $S_{ij}^{-}TS_{rs}^{+}$. ∎

4. Howe duality for $\mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$ acting on $\mathbb{W}$

We start this section by introducing a representation of the complex ortho-symplectic super Lie algebra $\mathfrak{g}' = \mathfrak{osp}(1|2)$ on the vector space $\mathbb{W}$. The super Lie bracket of two $\mathbb{Z}_2$-homogeneous elements $u, v \in \mathfrak{g}' = \mathfrak{g}_0' \oplus \mathfrak{g}_1'$ will be denoted by $[u, v]$ if and only if at least one of them is an element of the even part $\mathfrak{g}_0'$. In the other cases, we will denote it by $\{u, v\}$. Further, there exists a basis $\{h, e^{+}, e^{-}, f^{+}, f^{-}\}$ of $\mathfrak{g}'$ such that the set $\{e^{+}, h, e^{-}\}$ spans the even part $\mathfrak{g}_0'$, the set $\{f^{+}, f^{-}\}$ spans the odd part $\mathfrak{g}_1'$ and the only non-zero relations among the basis elements are

$$[h, e^{\pm}] = \pm e^{\pm} \qquad [e^{+}, e^{-}] = 2h \tag{5}$$

$$[h, f^{\pm}] = \pm\tfrac{1}{2}f^{\pm} \qquad \{f^{+}, f^{-}\} = \tfrac{1}{2}h \tag{6}$$

$$[e^{\pm}, f^{\mp}] = -f^{\pm} \qquad \{f^{\pm}, f^{\pm}\} = \pm\tfrac{1}{2}e^{\pm}. \tag{7}$$

For $i = 0, \ldots, 2l$, let us introduce operators $E^{\pm} : \bigwedge^{i}(\mathbb{V}^{*})^{\mathbb{C}} \otimes \mathbb{S} \to \bigwedge^{i\pm 2}(\mathbb{V}^{*})^{\mathbb{C}} \otimes \mathbb{S}$ by the prescription

$$E^{\pm} = \pm 2\{F^{\pm}, F^{\pm}\},$$

where $\{,\}$ denotes the anti-commutator in the associative algebra $\mathrm{End}(\mathbb{W})$.

The representation $\sigma : \mathfrak{osp}(1|2) \to \mathrm{End}(\mathbb{W})$ is defined by

$$\sigma(e^{\pm}) = E^{\pm}, \quad \sigma(f^{\pm}) = F^{\pm} \text{ and } \sigma(h) = H$$

and it is extended linearly to the whole algebra $\mathfrak{g}' = \mathfrak{osp}(1|2)$. Let us set $\mathbb{W}_0 = (\bigoplus_{i=0}^{l} \bigwedge^{2i}(\mathbb{V}^{*})^{\mathbb{C}}) \otimes \mathbb{S}$ and $\mathbb{W}_1 = (\bigoplus_{i=0}^{l-1} \bigwedge^{2i+1}(\mathbb{V}^{*})^{\mathbb{C}}) \otimes \mathbb{S}$. The vector space $\mathrm{End}(\mathbb{W})$ will be considered with the super Lie algebra structure inherited from the super vector space structure $\mathbb{W} = \mathbb{W}_0 \oplus \mathbb{W}_1$. We write $\mathrm{End}(\mathbb{W}) = \mathrm{End}_0(\mathbb{W}) \oplus \mathrm{End}_1(\mathbb{W})$.

Theorem 4.1. *The mapping*

$$\sigma : \mathfrak{osp}(1|2) \to \mathit{End}(\mathbb{W})$$

is a super Lie algebra representation.

Proof. First, it is easy to see that $\sigma(\mathfrak{g}_i') \subseteq \mathrm{End}_i(\mathbb{W})$, $i = 0, 1$. Second, we shall check that the operators $\sigma(e^{\pm}), \sigma(h)$ and $\sigma(f^{\pm})$ satisfy the appropriate commutation and anti-commutation relations – namely the ones written in the rows (5), (6), and (7) above. For $i = 0, \ldots, 2l$ and $\alpha \otimes s \in \bigwedge^i(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$, we have

$$\begin{aligned}
[H, F^+](\alpha \otimes s) &= HF^+(\alpha \otimes s) - F^+H(\alpha \otimes s) \\
&= H(\frac{\imath}{2}\sum_{j=1}^{2l} \epsilon^j \wedge \alpha \otimes e_j.s) - F^+\tfrac{1}{2}(i-l)(\alpha \otimes s) \\
&= \sum_{j=1}^{2l}\left[\tfrac{1}{2}\frac{\imath}{2}(i+1-l)\epsilon^j \wedge \alpha \otimes e_j.s - \frac{\imath}{2}\tfrac{1}{2}(i-l)\epsilon^j \wedge \alpha \otimes e_j.s\right] \\
&= \frac{\imath}{4}\sum_{j=1}^{2l} \epsilon^j \wedge \alpha \otimes e_j.s = \tfrac{1}{2}F^+(\alpha \otimes s).
\end{aligned}$$

Thus, we got the $(+)$-version of the first equation written in the row (6) as required. Similarly, one can prove the $(-)$-version of the first equation written in that row. The second relation written in the row (7) and the second relation in the row (6) follow from the definitions of $E^{\pm}$ and H, respectively. The remaining relations, i.e., the ones in the row (5) and the first relation in the row (7), can be proved just using the already derived ones and expanding the commutator and anti-commutator of compositions of endomorphisms. We shall show explicitly, how to prove the first relation in the row (7) only. Using the definitions of the considered mappings only, we may write $[E^+, F^-] = [2\{F^+, F^+\}, F^-] = 4[F^+F^+, F^-] = 4(F^+F^+F^- - F^-F^+F^+) = 4[F^+(-F^-F^+ + \frac{1}{2}H) - F^-F^+F^+] = 4(F^-F^+F^- - \frac{1}{2}HF^+ + \frac{1}{2}F^+H - F^-F^+F^+) = 2[F^+, H] = -F^+$. ∎

Summing up, we have the following

Corollary 4.2. *The representation $\sigma : \mathfrak{osp}(1|2) \to End(\mathbb{W})$ maps the super Lie algebra $\mathfrak{osp}(1|2)$ into the commutant algebra $End_{\mathfrak{g}}(\mathbb{W})$.*

Proof. Follows from Lemma 3.2 and Theorem 4.1 immediately. ∎

Now we define a family $\{\sigma_j\}_{j=0}^l$ of finite dimensional irreducible representations of the (complex) ortho-symplectic super Lie algebra $\mathfrak{g}' = \mathfrak{osp}(1|2)$. For $j = 0, \ldots, l$, let $\mathbb{G}^j$ denote a complex vector space of dimension $2l - 2j + 1$, and let us consider a basis $\{f_i\}_{i=j}^{2l-j}$ of $\mathbb{G}^j$. The super vector space structure on $\mathbb{G}^j$ is defined as follows. For $j = 0, \ldots, l$, we set $(\mathbb{G}^j)_0 = \mathrm{Span}(\{f_i \,|\, i \in \{j, \ldots, 2l-j\} \cap 2\mathbb{N}_0\})$ and $(\mathbb{G}^j)_1 = \mathrm{Span}(\{f_i \,|\, i \in \{j, \ldots, 2l-j\} \cap (2\mathbb{N}_0 + 1)\})$. For convenience, we suppose $f_k = 0$ for $k \in \mathbb{Z} \setminus \{j, \ldots, 2l-j\}$. We will not denote the dependence of the basis elements on the number j explicitly. As a short hand, for each $(i, j) \in \Xi$, we introduce the rational numbers

$$A(l, i, j) = \frac{(-1)^{i-j} + 1}{16}(i - j) + \frac{(-1)^{i-j+1} + 1}{16}(i + j - 2l - 1).$$

Finally, for $j = 0, \ldots, l$, let us define the mentioned representations $\sigma_j : \mathfrak{osp}(1|2) \to$

$\mathrm{End}(\mathbb{G}^j)$ by the formulas

$$\begin{aligned}\sigma_j(f^+)(f_i) &= A(l,i+1,j)f_{i+1},\ i=j,\dots,2l-j\\ \sigma_j(f^-)(f_i) &= f_{i-1},\ i=j,\dots,2l-j\\ \sigma_j(h) &= 2\{\sigma_j(f^+),\sigma_j(f^-)\}\\ \sigma_j(e^\pm) &= \pm 2\{\sigma_j(f^\pm),\sigma_j(f^\pm)\}.\end{aligned}$$

We prove the following

Lemma 4.3. *For $j=0,\dots,l$, the mapping $\sigma_j : \mathfrak{osp}(1|2) \to End(\mathbb{G}^j)$ is an irreducible representation of the super Lie algebra $\mathfrak{osp}(1|2)$.*

Proof. First, we prove that for $j=0,\dots,l$, the mapping σ_j is a representation of the super Lie algebra $\mathfrak{osp}(1|2)$. It is easy to see that whereas the even part of $\mathfrak{g}'$ acts by transforming the even part of $\mathbb{G}^j$ into itself and the odd part into itself as well, the odd part of $\mathfrak{g}'$ acts by interchanging the mentioned two parts of $\mathbb{G}^j$. Now we check whether the relations a the rows (5), (6) and (7) are preserved by the mapping σ_j, $j=0,\dots,l$. The second relation in the row (7) and the second relation in (6) are satisfied due to the definitions of $\sigma_j(e^\pm)$ and $\sigma_j(h)$, respectively. Let us start proving the $(+)$-version of the first relation written in the row (6). For $i=j,\dots,2l-j$, we may write

$$\begin{aligned}[\sigma_j(h)\sigma_j(f^+) &- \sigma_j(f^+)\sigma_j(h)]f_i =\\ &= 2[(\sigma_j(f^+)\sigma_j(f^-)+\sigma_j(f^-)\sigma_j(f^+))\sigma_j(f^+)\\ &\quad -\sigma_j(f^+)(\sigma_j(f^+)\sigma_j(f^-)+\sigma_j(f^-)\sigma_j(f^+))]f_i\\ &= 2[\sigma_j(f^-)\sigma_j(f^+)\sigma_j(f^+)-\sigma_j(f^+)\sigma_j(f^+)\sigma_j(f^-)]f_i\\ &= 2[A(l,i+1,j)\sigma_j(f^-)\sigma_j(f^+)f_{i+1}-\sigma_j(f^+)\sigma_j(f^+)f_{i-1}]\\ &= 2[A(l,i+2,j)A(l,i+1,j)-A(l,i,j)A(l,i+1,j)]f_{i+1}\\ &= 2A(l,i+1,j)[A(l,i+2,j)-A(l,i,j)]f_{i+1}\\ &= \tfrac{1}{2}A(l,i+1,j)f_{i+1}=\tfrac{1}{2}\sigma_j(f^+)f_i.\end{aligned}$$

The $(-)$-version of this relation can be proved in a similar way. To check the relations written in the row (5) and the first relation in the row (7), it is sufficient to use the already derived relations and expand the commutator and anti-commutator of compositions of endomorphisms only.

To prove the irreducibility of the representations σ_j, one proceeds exactly as in the $\mathfrak{sl}(2,\mathbb{C})$ case and its finite dimensional irreducible representations. See, e.g., Samelson [11]. ∎

Now we prove a technical

Lemma 4.4. *For each $k\in\mathbb{N}_0$ and $i=0,\dots,2l$, we have*

$$(F^-)^kF^+ =$$
$$(-1)^kF^+(F^-)^k+\left[\frac{(-1)^k+1}{16}k+\frac{(-1)^{k+1}+1}{16}(2i-2l-k+1)\right](F^-)^{k-1}$$

when acting on $\bigwedge^i(\mathbb{V}^*)^{\mathbb{C}}\otimes\mathbb{S}$.

Proof. We will suppose the operators act on the space $\bigwedge^i(\mathbb{V}^*)^{\mathbb{C}}\otimes\mathbb{S}$ without writing it explicitly and proceed by induction on k.

I. For $k=0$, the lemma holds obviously.

II. a. We suppose the lemma holds for an even integer $k\in\mathbb{N}_0$. We have

$$\begin{aligned}(F^-)^{k+1}F^+ &= F^-(F^-)^kF^+\\ &= F^-[F^+(F^-)^k+\frac{k}{16}((-1)^k+1)(F^-)^{k-1}]\\ &= -(F^+)(F^-)^{k+1}+\tfrac{1}{2}H(F^-)^k+2\frac{k}{16}(F^-)^k\\ &= -F^+(F^-)^{k+1}+\frac{1}{4}(i-k-l)(F^-)^k+\frac{k}{8}(F^-)^k\\ &= -F^+(F^-)^{k+1}+\frac{2}{16}(2i-2l-(k+1)+1)(F^-)^k,\end{aligned}$$

where we have used the induction hypothesis, definition of H and Lemma 3.1 on the values of H. The last written expression coincides with the one in the statement of the lemma for $k+1$ is odd.

b. Now, suppose k is odd. We have

$$\begin{aligned}&(F^-)^{k+1}F^+ = F^-(F^-)^kF^+\\ =\;& F^-[-F^+(F^-)^k+\frac{(-1)^{k+1}+1}{16}(2i-2l-k+1)(F^-)^k]\\ =\;& F^+(F^-)^{k+1}-\tfrac{1}{2}H(F^-)^k+\frac{1}{8}(2i-2l-k+1)(F^-)^k\\ =\;& F^+(F^-)^{k+1}-\frac{1}{8}(2i-2k-2l)(F^-)^k+\\ &+\frac{1}{8}(2i-2l-k+1)(F^-)^k = F^+(F^-)^{k+1}+\frac{2}{16}(k+1)(F^-)^k,\end{aligned}$$

where we have used the same tools as in the previous item. ■

Now, let us define a family $\{\rho_j^\pm\}_{j=0}^l$ of representations $\rho_j^\pm:\mathfrak{g}\to\mathrm{End}(\mathbb{E}_{jj}^\pm)$ of the Lie algebra $\mathfrak{g}=\mathfrak{sp}(\mathbb{V}^{\mathbb{C}},\omega)$ acting on the vector spaces $\mathbb{E}_{jj}^\pm$ by the prescription

$$\rho_j(X)v=\rho(X)v,$$

where $X\in\mathfrak{g}$ and $v\in\mathbb{E}_{jj}^\pm$.

Further, let us introduce a mapping $\mathrm{Sgn}:\{+,-\}\times\mathbb{N}_0\to\{+,-\}$ given by the prescription $\mathrm{Sgn}(\pm,2k)=\pm$ and $\mathrm{Sgn}(\pm,2k+1)=\mp$, $k\in\mathbb{Z}$. Now, for $(i,j)\in\Xi$, we define $\psi_{ij}^\pm:\mathbb{E}_{ij}^\pm\to\mathbb{E}_{jj}^{\mathrm{Sgn}(\pm,i-j)}\otimes\mathbb{G}^j$ by the formula

$$\psi_{ij}^\pm v=(F^-)^{i-j}v\otimes f_i,$$

$v\in\mathbb{E}_{ij}^\pm$. Finally, we set $\psi=\bigoplus_{(i,j)\in\Xi}(\psi_{ij}^+\oplus\psi_{ij}^-)$. In particular,

$$\psi:\bigoplus\nolimits_{(i,j)\in\Xi}(\mathbb{E}_{ij}^+\oplus\mathbb{E}_{ij}^-)\to\bigoplus\nolimits_{j=0}^l[(\mathbb{E}_{jj}^+\oplus\mathbb{E}_{jj}^-)\otimes\mathbb{G}^j].$$

Now, consider $\mathbb{W}=\bigwedge^\bullet(\mathbb{V}^*)^{\mathbb{C}}\otimes\mathbb{S}$ with the action $\rho\otimes\sigma$ and the space

$$\bigoplus\nolimits_{j=0}^l[(\mathbb{E}_{jj}^+\oplus\mathbb{E}_{jj}^-)\otimes\mathbb{G}^j]$$

with the action $\bigoplus_{j=0}^l[(\rho_j^+\oplus\rho_j^-)\otimes\sigma_j]$ – both of the algebra $\mathfrak{g}\times\mathfrak{g}'$. In the next theorem, the aforementioned Howe duality is stated.

Theorem 4.5. *The following* $(\mathfrak{sp}(\mathbb{V}^{\mathbb{C}},\omega)\times\mathfrak{osp}(1|2))$*-module isomorphism*

$$\mathbb{W}\simeq\bigoplus_{j=0}^{l}[(\mathbb{E}^{+}_{jj}\oplus\mathbb{E}^{-}_{jj})\otimes\mathbb{G}^{j}] \text{ holds.}$$

Proof. Due to Theorem 2.3, we know that $\mathbb{W}$ is isomorphic to

$$\bigoplus\nolimits_{(i,j)\in\Xi}(\mathbb{E}^{+}_{ij}\oplus\mathbb{E}^{-}_{ij})$$

as a $\mathfrak{g}$-module. Further, it is evident that ψ is a vector space isomorphism. We prove that for each $(i,j)\in\Xi$, the mapping $\psi^{\pm}_{ij}:\mathbb{E}^{\pm}_{ij}\to\mathbb{E}^{\mathrm{Sgn}(\pm,i-j)}_{jj}\otimes\mathbb{G}^{j}$ is $(\mathfrak{g}\times\mathfrak{g}')$-equivariant. The $\mathfrak{g}$-equivariance follows easily because F^{-} in the definition of $\psi^{\pm}_{ij}$ commutes with the representation ρ of $\mathfrak{g}$ (Lemma 3.2).

We shall prove the $\mathfrak{g}'$-equivariance. For each $(i,j)\in\Xi$ and $v\in\mathbb{E}^{\pm}_{ij}$, we may write $\psi^{\pm}_{ij}\sigma(f^{-})v=\psi^{\pm}_{ij}F^{-}v=(F^{-})^{i-1-j}F^{-}v\otimes f_{i-1}=(F^{-})^{i-j}v\otimes f_{i-1}$. On the other hand, we have

$$\sigma_j(f^{-})(\psi^{\pm}_{ij}v)=\sigma_j(f^{-})((F^{-})^{i-j}v\otimes f_i)=(F^{-})^{i-j}v\otimes f_{i-1}.$$

Now, we check the $\mathfrak{g}'$-equivariance for f^{+}. Using Lemma 4.4, we compute

$$\begin{gathered}\psi^{\pm}_{ij}\sigma(f^{+})v=\psi^{\pm}_{ij}F^{+}v=(F^{-})^{i+1-j}F^{+}v\otimes f_{i+1}=\\ [(-1)^{i+1-j}F^{+}(F^{-})^{i+1-j}v+A(l,i+1,j)(F^{-})^{i-j}v]\otimes f_{i+1}=\\ A(l,i+1,j)(F^{-})^{i-j}v\otimes f_{i+1},\end{gathered}$$

where we have used the fact that $(F^{-})^{i+1-j}v=0$ implied by $v\in\mathbb{E}^{\pm}_{ij}$ (see Remark 3.5). On the other hand, we have $\sigma_j(f^{+})\psi^{\pm}_{ij}v=\sigma_j(f^{+})((F^{-})^{i-j}v\otimes f_i)=(F^{-})^{i-j}v\otimes A(l,i+1,j)f_{i+1}$. Thus, the equivariance with respect to f^{+} is proved. Because the operators H, E^{+} and E^{-} are linear combinations of compositions of the operators F^{+} and F^{-}, the $\mathfrak{g}'$-equivariance of $\psi^{\pm}_{ij}$ follows. ■

Remark 4.6. Due to the fact that the category of Harish-Chandra modules is a full subcategory of the category of $\mathcal{U}(\mathfrak{g})$-modules and due to some basic properties of minimal globalization functors, the results of the paper have their appropriate minimal globalization counterparts. See Kashiwara, Schmid [6].

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Cohomology of the de Rham complex twisted by the oscillatory representation

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ABSTRACT

We introduce a Hilbert A-module structure on the higher oscillatory module, where A denotes the C^*-algebra of continuous endomorphisms of the basic oscillatory module. We also define the notion of an exterior covariant derivative in an A-Hilbert bundle and use it for a construction of an A-elliptic complex of differential operators for certain symplectic manifolds equipped with a flat symplectic connection.

1. Introduction

In mathematical physics, one often uses complexes of exterior forms twisted by finite rank vector bundles. For example, the non-quantum gauge theory of the electromagnetic interaction can be seen as the mathematics of connections on line bundles over four dimensional manifolds. The connections are models of the abelian $U(1)$-gauge fields. Notice also that the Maxwell equations for vacuo can be formulated by taking the line bundle and the line bundle connection to be trivial. Thus, one investigates the de Rham complex over Lorentzian manifolds only. In Riemannian geometry, a similar sequence, namely $(d_k^{\nabla^g}, \Gamma(\bigwedge^k T^*M \otimes TM))_{k\in\mathbb{N}_0}$ can be considered. Here, $d_k^{\nabla^g}$ denotes the k-th exterior covariant derivative induced by the Riemannian connection ∇^g of the Riemannian manifold (M^n, g). This sequence may be used to treat, e.g., the $SO(n)$-invariant parts of the curvature operator and the Bochner–Weitzenböck formulas systematically. However, non-flat

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manifolds, for which the sequence above fails to be a complex, are more in the focus of Riemannian geometers. Roughly speaking, the Riemannian counterpart of the Maxwell equations does not seem to be so widely used.

In symplectic geometry, the role of an adapted connection is not as obvious as in the Riemannian geometry, foremost because of a theorem of Darboux due to which all symplectic manifolds are infinitesimally equivalent. The torsion-free symplectic connections on a symplectic manifold (M, ω) form an infinite dimensional affine space. For it see, e.g., Tondeur [13] or Gelfand et al. [4]. The role of symplectic connections seems to be rather in quantization of mechanics. See Fedosov [1] and notice that the higher oscillatory module studied here is related to the Weyl algebra structure used in the quantization procedure named after him.

In Fomenko and Mishchenko [3], the concept of A-Hilbert bundles and differential operators acting on their sections was established for a unital C^*-algebra A. These bundles have the so-called Hilbert A-modules as fibers. See Solovyov and Troitsky [11] for these notions. The authors of [3] investigate finitely generated projective A-Hilbert bundles over compact manifolds and A-elliptic operators acting between their smooth sections. They prove that such operators have the so-called A-Fredholm property. In particular, their kernels are finitely generated projective Hilbert A-modules. In Krýsl [8], the results of [3] were used in the case of A-elliptic complexes and conclusions for the cohomology groups of these complexes were made.

In this paper, we introduce a sequence of infinite rank vector bundles over a symplectic manifold and differential operators acting between their sections. The symplectic manifold (M, ω) is supposed to admit a *metaplectic structure*, a symplectic analogue of the Riemannian spin structure. The principal group of the metaplectic bundle is the metaplectic group $Mp(2n, \mathbb{R})$. Since the articles of Shale [12] and Weil [14] were published, a faithful unitary representation of the metaplectic group on the space $H = L^2(\mathbb{R}^n)$ was known. This representation is the Segal–Shale–Weil representation. Besides this name, also the names metaplectic representation or symplectic spinor representation are used. We call this representation the *basic oscillatory module* in this text to stress the fact that it is used as a state space of the quantum harmonic oscillator. Associating the basic oscillatory module to the metaplectic structure, we get the so-called basic oscillatory bundle denoted by $\mathcal{H}$. See Kostant [7] and Habermann and Habermann [5]. The sequence of bundles we investigate is formed by the tensor product of the bundles $\bigwedge^k T^*M$ of exterior k-forms on M and of the bundle $\mathcal{H}$. Let ∇ be a symplectic connection on (M, ω). The lift of the symplectic connection ∇ to the sections of the basic oscillatory bundle $\mathcal{H}$ induces the exterior covariant derivatives $d_k^{\nabla^H} : \Gamma(\bigwedge^k T^*M \otimes \mathcal{H}) \to \Gamma(\bigwedge^{k+1} T^*M \otimes \mathcal{H})$. By definition, if the curvature of ∇ is zero, the sequence $d_H^\bullet = (d_k^{\nabla^H}, \Gamma(\bigwedge^k T^*M \otimes \mathcal{H}))_{k \in \mathbb{N}_0}$ forms a complex. This is what we understand under the name 'de Rham complex twisted by the oscillatory module'. We prove that this complex is A-elliptic and use a result from [8] to get an information on the cohomology groups of this complex when M is compact. As far as we know, this is the first non-trivial explicitly constructed A-elliptic complex in infinite rank vector bundles.

In Section 2, we recall the notion of a Hilbert A-module, introduce the higher oscillatory module as a module over the metaplectic group as well as over the unital C^*-algebra A of continuous endomorphisms of $L^2(\mathbb{R}^n)$. We prove that the oscillatory module is a finitely generated projective Hilbert A-module with respect to a natural Hilbert A-product (Theorem 3). In Section 3, a definition of an A-Hilbert bundle is given, the notion of the exterior covariant derivative in such bundles is introduced, and its symbol is computed (Theorem 5). In Section 3.1, we give a definition of an A-elliptic complex, construct the de Rham complex twisted by the basic oscillatory module and state a theorem on properties of the cohomology groups of this complex (Theorem 6).

2. Higher oscillatory modules

Let A be a unital C^*-algebra with involution, norm and unity denoted by $*$, $|\ |_A$ and 1, respectively. Let us recall a definition of the A-Hilbert module. For general C^*-algebras, this notion was first considered by Paschke in [10]. A pre-Hilbert A-module is a left A-module U equipped with a map $(\,,) : U \times U \to A$ satisfying for each $u, v, w \in U$ and $a \in A$

1) $(u, v + w) = (u, v) + (u, w)$,
2) $(a.u, v) = a(u, v)$,
3) $(u, v) = (v, u)^*$,
4) $(u, u) \geqslant 0$ and $(u, u) = 0$ implies $u = 0$.

The relation $a \geqslant b$ holds for $a, b \in A$ if and only if $a - b$ is hermitian and its spectrum lies in $\mathbb{R}_0^+$. Recall that for an element $a \in A$, its spectrum is the set $\{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible}\}$. Notice that from 2) and 3), we get $(u, a.v) = (u, v)a^*$ for any $a \in A$ and $u, v \in U$. A pre-Hilbert A-module is called a *Hilbert A-module* if it is complete with respect to the norm $|\ |_U : U \to \mathbb{R}$ defined by $|u| = \sqrt{|(u, u)|_A}$, $u \in U$. If U is a pre-Hilbert A-module, we speak of $(\,,)$ as of an A-product. When U is a Hilbert A-module, we call the A-valued map $(\,,)$ a *Hilbert A-product*. If necessary, we write $(\,,)_U$ instead of $(\,,)$.

In the category of pre-Hilbert A-modules, the set of morphisms B is formed by continuous A-equivariant maps between the objects, i.e., $B(a.u) = a.B(u)$ for each $a \in A$ and $u \in U$. Continuity is meant with respect to the (possibly non-complete) norms. Declaring the category of Hilbert A-modules to be the full subcategory of the category of pre-Hilbert A-modules, the set of morphisms in this category is defined. Let us notice that adjoints are considered with respect to the A-products, i.e., for a morphism $B : U \to V$ of pre-Hilbert A-modules, its adjoint, denoted by B^*, is a map $B^* : V \to U$ which satisfies $(Bu, v)_V = (u, B^* v)_U$ for each $u \in U$ and $v \in V$. It is known that unlike for Hilbert spaces, morphisms of Hilbert A-modules do not have adjoints in general, but if they exist, it is elementary to prove that they are unique. When we write a direct sum of Hilbert A-modules, we suppose that the summands are mutually orthogonal with respect to $(\,,)$. In general, orthocomplements do not have the "exhaustion property", i.e., there exist a Hilbert A-module U and a (closed) Hilbert A-submodule V of U such that $U \neq V \oplus V^\perp$. (See, e.g., Lance [9] for an example.) But fortunately we have the following

Theorem 1. *Let U, V be Hilbert A-modules and $B : U \to V$ be a Hilbert A-module morphism. If the adjoint of B exists and $\operatorname{Im} B$ is closed, then $U = \operatorname{Ker} B \oplus \operatorname{Im} B^*$.*

Proof. See Lance [9, Theorem 3.2] for a proof. □

Now, we focus our attention to the higher oscillatory module. Let (V, ω) be a real symplectic vector space of dimension $2n$ and $g : V \times V \to \mathbb{R}$ be a scalar product on V. For any $\xi \in V^*$, we define the vector in $\xi^g \in V$ by the formula $\xi(v) = g(\xi^g, v)$ for any $v \in V$. Further, we denote the appropriate extension of g to $\bigwedge^\bullet V^*$ by g as well. (The orientation for (V, g) is induced by $\omega^{\wedge n}$.) Let $\tilde{G}$ be a realization of the metaplectic group associated to the symplectic space (V, ω), and let λ be the covering homomorphism of the symplectic group $Sp(V, \omega)$ by $\tilde{G}$.

Further, we denote the exterior multiplication of exterior k-forms by a 1-form ξ by ext_k^ξ and recall the following lemma. For technical reasons, let us set $\bigwedge^{-1} V^* = 0$.

Lemma 2. *For any $\xi \in V^* \setminus \{0\}$, the complex $\mathrm{ext}_\xi^\bullet = (\mathrm{ext}_k^\xi, \bigwedge^k V^*)_{k=-1}^{2n+1}$ is exact.*

Proof. Suppose $\xi \in V^*$. Because $\xi \wedge \xi \wedge \alpha = 0$ for any $\alpha \in \bigwedge^k V^*$, $\mathrm{ext}_\xi^\bullet$ is a complex. Now, suppose $\xi \wedge \alpha = 0$ for a k-form α. Making the insertion of ξ^g into this equation, we get $(\iota_{\xi^g}\xi)\alpha - \xi \wedge \iota_{\xi^g}\alpha = 0$. From that $\alpha = -(g(\xi, \xi))^{-1}\xi \wedge \iota_{\xi^g}\alpha$ provided $g(\xi, \xi) \neq 0$, which holds if and only if $\xi \neq 0$. Thus $\alpha \in \operatorname{Im}(\mathrm{ext}_{k-1}^\xi)$. □

Let L be a Lagrangian subspace of (V, ω). The oscillatory representation of the metaplectic group $\tilde{G}$ is a faithful unitary representation of $\tilde{G}$ on the complex Hilbert space $H = L^2(L, g_{|L \times L})$. Let us denote it by $\rho_0 : \tilde{G} \to \mathrm{Aut}(H)$ and the scalar product on H by $(\,,\,)_H$, i.e.,

$$(k, l)_H = \int\limits_{x \in L} \overline{k(x)} l(x)\, dx \quad \text{for each } k, l \in H.$$

Notice that the oscillatory representation splits into the irreducible $\tilde{G}$-submodules of even and odd square integrable functions on L. See Shale [12], Weil [14] and Kashiwara and Vergne [6] for more information on ρ_0.

Let us consider the tensor product ρ of the wedge powers of the dual of the representation $\lambda : \tilde{G} \to \mathrm{Aut}(V)$ and of the oscillatory representation ρ_0. In particular, $\rho : \tilde{G} \to \mathrm{Aut}(C^\bullet)$, where $C^\bullet = \bigwedge^\bullet V^* \otimes H$ is considered with the canonical Hilbert space topology. Note that ρ is not unitary unless $V = \mathbb{R}$. The $\tilde{G}$-module $C^\bullet$ is called the *higher oscillatory module* and $C^0 = H$ the *basic oscillatory module*.

Now, we would like to investigate $C^\bullet$ from the A-module point of view, where $A = \mathrm{End}(H)$ is the unital C^*-algebra of continuous endomorphisms of H. The involution $* : A \to A$ in A is given by the adjoint of endomorphisms, i.e., $*a = a^*$ for any $a \in A$. For the norm in A, we take the supremum norm, i.e., for any $a \in A$, we set $|a|_A = \sup_{|k|_H \leqslant 1} |a(k)|_H$, where $|\ |_H$ denotes the norm on H derived from the scalar product $(\,,\,)_H$. Let us remark, that we consider everywhere defined operators only. In particular, the star $*$ is a well defined (and continuous) anti-involutive map in the C^*-algebra A.

The space $C^\bullet$ introduced above is not only a $\tilde{G}$-module, but it is also an A-module with the action given by

$$a.(\alpha \otimes k) = \alpha \otimes a(k), \quad \alpha \otimes k \in C^\bullet \text{ and } a \in A.$$

For any $k \in H$, let $k^* : H \to \mathbb{C}$ denote the $(\,,\,)_H$-dual element to k, i.e., $k^*(l) = (k, l)_H$. Now, let us introduce an A-product $(\,,\,)$ on the higher oscillatory module $C^\bullet$. For any $\alpha \otimes k, \beta \otimes l \in C^\bullet$, we set

$$(\alpha \otimes k, \beta \otimes l) = g(\alpha, \beta) k \otimes l^* \in A$$

where by $k \otimes l^*$, we mean the element of A defined by $(k \otimes l^*)(m) = l^*(m) k \in H$ for all $m \in H$. The product is extended to non-homogeneous elements linearly. Let us make the following observation of which we make use later. For $k, l \in H$, we have

$$\left(k \otimes l^*\right)^* = l \otimes k^*.$$

Indeed, for any $m, n \in H$, we may write $((k \otimes l^*)^* m, n)_H = (m, (k \otimes l^*) n)_H = (m, k(l, n)_H)_H = (l, n)_H (m, k)_H = (\overline{(m, k)_H} l, n)_H = (l(k, m)_H, n)_H = ((l \otimes k^*) m, n)_H$.

Notice that a Hilbert A-module U is called *finitely generated* and *projective* if there exists an integer $n \in \mathbb{N}_0$ and a Hilbert A-submodule $V \subseteq A^n$ such that $U \oplus V \simeq A^n$. Here, A^n denotes the direct sum of n copies of the tautological A-module A. For equivalent definitions, see Solovyov and Troitsky [11].

Theorem 3. *The space $C^\bullet$ together with $(\,,\,)$ is a finitely generated projective Hilbert A-module.*

Proof. Let e_0 be a unit length vector in H and v an arbitrary element of H. The map $b = v \otimes e_0^*$ has the property $b(e_0) = v$ and $|b|_A \leqslant |v|_H$, i.e., b is bounded and thus, continuous. Let $\{e_i\}_{i=1}^{2^{2n}}$ be an orthonormal basis of $\bigwedge^\bullet V^*$. Then obviously, $(e_i \otimes e_0)_{i=1}^{4^n}$ is a set of generators of $C^\bullet$. Thus, $C^\bullet$ is finitely generated over A.

Now, we prove that $C^\bullet$ is a Hilbert A-module.

1) A-linearity of $(\,,)$. For any $a \in A$, $\alpha \otimes k, \beta \otimes l \in C^\bullet$ and $m \in H$, we have

$$\begin{aligned}\big(a.(\alpha \otimes k), \beta \otimes l\big)(m) &= \big(\alpha \otimes a(k), \beta \otimes l\big)(m)\\ &= g(\alpha,\beta)\big(a(k) \otimes l^*\big)(m)\\ &= g(\alpha,\beta)a(k)(l,m)_H.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\big[a(\alpha \otimes k, \beta \otimes l)\big](m) &= g(\alpha,\beta)\big[a\big(k \otimes l^*\big)\big](m)\\ &= a(k)g(\alpha,\beta)(l,m)_H.\end{aligned}$$

2) Hermitian symmetry. For any $\alpha \otimes k, \beta \otimes l \in C^\bullet$, we have

$$\begin{aligned}(\alpha \otimes k, \beta \otimes l)^* = g(\alpha,\beta)\big(k \otimes l^*\big)^* &= g(\alpha,\beta)\big(l \otimes k^*\big)\\ = g(\beta,\alpha)\big(l \otimes k^*\big) &= (\beta \otimes l, \alpha \otimes k).\end{aligned}$$

3) Positive definiteness. Let $c = \sum_{i=1}^{4^n} e_i \otimes k_i^*$ for $k_i \in H$, $i = 1, \ldots, 4^n$. Then $(c,c) = \sum_{i,j=1}^{4^n} g(e_i,e_j)(k_i \otimes k_j^*) = \sum_{i,j}^{4^n} \delta_{ij}(k_i \otimes k_j^*) = \sum_{i=1}^{4^n}(k_i \otimes k_i^*)$. The spectrum of each of the summands consists of the non-negative numbers $(k_i,k_i)_H$ and 0. Thus, $k_i \otimes k_i^* \geqslant 0$. Because the non-negative elements in a C^*-algebra form a cone, $(c,c) \geqslant 0$. Suppose $(c,c) = 0$ and that the summand in $\sum_{i=1}^{4^n}(k_i \otimes k_i^*)$ with index i_0 is non-zero. Writing $-(k_{i_0} \otimes k_{i_0}^*) = \sum_{i \in \{1,\ldots,4^n\}\setminus\{i_0\}} k_i \otimes k_i^*$ gives a contradiction.
4) Completeness is obvious because the normed space $C^\bullet$ is a finite direct sum of copies of the Hilbert A-module $H = L^2(L, g_{|L\times L})$.

Since as we already showed, $C^\bullet$ is a finitely generated Hilbert A-module, it is projective. For it, see Frank and Larsen [2, Theorem 5.9]. □

Let us notice that if we take the compact operators on H for the C^*-algebra, the basic oscillatory module is also a finitely generated Hilbert A-module.

3. Covariant derivatives and the twisted de Rham complex

Let M be a manifold and $p : \mathcal{E} \to M$ be an A-Hilbert bundle, where A is a unital C^*-algebra. This means in particular, that p is a smooth Banach bundle the fibers of which are isomorphic to a fixed Hilbert A-module U. As it is standard, we denote the space of smooth sections of $\mathcal{E}$ by $\Gamma(\mathcal{E})$. For any $m \in M$, the fiber $p^{-1}(\{m\})$ is denoted by $\mathcal{E}_m$, and the Hilbert A-product defined on it by $(\,,)_m$. The morphisms between A-Hilbert bundles $p_i : \mathcal{E}^i \to M$, $i = 1, 2$, are supposed to be smooth bundle maps $S : \mathcal{E}^1 \to \mathcal{E}^2$, i.e., $p_1 = p_2 \circ S$ and for each point $m \in M$, $S_{|(\mathcal{E}^1)_m} : (\mathcal{E}^1)_m \to (\mathcal{E}^2)_m$ is a morphism of A-Hilbert modules. See Solovyov and Troitsky [11] for more information on A-Hilbert bundles.

Let us set $\mathcal{A} = M \times A \to M$ for the trivial bundle, and introduce a map $(\,,)_\mathcal{A} : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{A})$ on $\Gamma(\mathcal{E})$ by the prescription

$$(s,t)_\mathcal{A}(m) = \big(s(m), t(m)\big)_m \in A,$$

where $s, t \in \Gamma(\mathcal{E})$ and $m \in M$. Of course, $\Gamma(\mathcal{A}) \cong \mathcal{C}^\infty(M, A)$.

Now, let us choose a Riemannian metric g on M and denote by $|\mathrm{vol}_g|$ a choice of the volume element associated to g. The extension of g to $\bigwedge^\bullet T^*M$ with respect to the orientation induced by $|\mathrm{vol}_g|$ will still be denoted by g. From now on, we suppose that M is compact. The space $\Gamma(\mathcal{E})$ of smooth sections of $\mathcal{E}$ carries a pre-Hilbert A-module structure. The action of A on $\Gamma(\mathcal{E})$ is defined by $(a.s)(m) = a.(s(m))$, $a \in A$, $s \in \Gamma(\mathcal{E})$ and $m \in M$, and the A-product is given by

$$(s,t)_{\Gamma(\mathcal{E})} = \int_M (s,t)_{\mathcal{A}} |\mathrm{vol}_g|.$$

Let us notice that in the formula for the A-product $(\,,\,)_{\Gamma(\mathcal{E})}$, any absolutely convergent integral for Banach space valued functions may be considered, e.g., the Bochner integral.

Lemma 4. *If (M,g) is a compact Riemannian manifold and $\mathcal{E}$ is an A-Hilbert bundle over M, then $\mathcal{E}' = TM \otimes \mathcal{E}$ and $\mathcal{E}'' = T^*M \otimes \mathcal{E}$ are A-Hilbert bundles as well.*

Proof. Let us set $a.(v \otimes c) = v \otimes a.c$ and $a.(\alpha \otimes c) = \alpha \otimes a.c$ for any $a \in A$, $c \in \mathcal{E}_m$, $v \in T_mM$, $\alpha \in T_m^*M$ and $m \in M$. Further, set $(u \otimes c, v \otimes d)_m = g_m(u,v)(c,d)_m \in A$ and $(\alpha \otimes c, \beta \otimes d)_m = g_m(\alpha,\beta)(c,d)_m$ for $c,d \in \mathcal{E}_m$, $u,v \in T_mM$ and $\alpha,\beta \in T_m^*M$, $m \in M$. It is straightforward to verify that these structures define A-Hilbert bundles. $\square$

In what follows, when given an A-Hilbert bundle $\mathcal{E}$, we always consider the bundles $\mathcal{E}'$ and $\mathcal{E}''$ with the A-Hilbert bundle structure defined in Lemma 4.

Definition 1. Let $p : \mathcal{E} \to M$ be an A-Hilbert bundle. We call a map $\nabla : \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$ *covariant derivative* in the bundle $\mathcal{E}$ if for each function $f \in \mathcal{C}^\infty(M)$ and sections $s_1, s_2 \in \Gamma(\mathcal{E})$, we have

$$\nabla(s_1 + s_2) = \nabla s_1 + \nabla s_2,$$
$$\nabla(fs_1) = df \otimes s_1 + f\nabla s_1.$$

Any covariant derivative ∇ in an A-Hilbert bundle $\mathcal{E}$ induces the exterior covariant derivatives $d_k^\nabla : \Gamma(\bigwedge^k T^*M \otimes \mathcal{E}) \to \Gamma(\bigwedge^{k+1} T^*M \otimes \mathcal{E})$ by the formula

$$d_k^\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s$$

where $\alpha \otimes s \in \Gamma(\bigwedge^k T^*M \otimes \mathcal{E})$ and $k = 0, \ldots, \dim M$. To non-homogeneous elements, the exterior covariant derivative is extended by linearity.

Let $\mathcal{E}, \mathcal{F}$ be A-Hilbert bundles over M. Suppose that $\mathfrak{d} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ is an A-differential operator. Then it is known that the symbol $\sigma : \mathcal{E}'' \to \mathcal{F}$ of $\mathfrak{d}$ and $\mathfrak{d}$ are adjointable A-Hilbert bundle and pre-Hilbert A-module morphisms, respectively. Notice that we consider finite order operators only. Further, for each $r \in \mathbb{N}_0$, one defines the Sobolev type product $(\,,\,)_r$ on $\Gamma(\mathcal{E})$ by the formula

$$(s,s')_r = \int_M \left(s, (1 + \triangle_g)^r s'\right)_{\mathcal{A}} |\mathrm{vol}_g|,$$

where $\triangle_g$ is the (positive definite) Laplace–Beltrami operator on (M,g). The A-modules $\Gamma(\mathcal{E})$ equipped with $(\,,\,)_r$ are pre-Hilbert A-modules. Let us denote the norm associated to $(\,,\,)_r$ by $|\;|_r$. Notice that $(\,,\,)_0 = (\,,\,)_{\Gamma(\mathcal{E})}$. The spaces $W^r(\mathcal{E})$ are defined as completions of $\Gamma(\mathcal{E})$ with respect to the norms $|\;|_r$. Because of the shape of the formula for $(\,,\,)_r$, we call the A-modules $W^r(\mathcal{E})$ the *Sobolev type completions* (of the pre-Hilbert A-module $(\Gamma(E), (\,,\,)_r)$). Let us notice that for any $r \in \mathbb{N}_0$, each differential operator $\mathfrak{d} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$

has a continuous extension to $W^r(\mathcal{E})$ and that this extension is unique. For these notions and results, see Solovyov and Troitsky [3].

We use the symbols ξ^g and ext_k^ξ introduced in Section 2 to denote the g-dual vector field and the exterior multiplication of exterior differential k-forms by a differential 1-form ξ also in the case of a Riemannian manifold (M, g).

Theorem 5. *Let (M, g) be a compact Riemannian manifold, $p : \mathcal{E} \to M$ be an A-Hilbert bundle and ∇ be a covariant derivative in $\mathcal{E}$. Then d_k^∇ is a differential operator of order one. For each $\xi \in \Gamma(T^*M)$, the symbol σ_k^ξ of d_k^∇ is given by $\sigma_k^\xi = \mathrm{ext}_k^\xi \otimes \mathrm{Id}_\mathcal{E}$ and its adjoint satisfies $(\sigma_k^\xi)^* = \iota_{\xi^g} \otimes \mathrm{Id}_\mathcal{E}$.*

Proof. For any function f on M and any section ψ of $\bigwedge^k T^*M \otimes \mathcal{E}$, we get $d_k^\nabla(f\psi) - f d_k^\nabla \psi = (df) \wedge \psi + f d_k^\nabla \psi - f d_k^\nabla \psi = df \wedge \psi$ which shows that the exterior covariant derivative d_k^∇ is a differential operator of first order. For $\xi \in \Gamma(T^*M)$, let us compute the symbol σ_k^ξ of d_k^∇. It is sufficient to work locally. Using the previous formula for $\xi = df$ and $\psi = \alpha \otimes s \in \Gamma(\bigwedge^k T^*M \otimes \mathcal{E})$, we obtain $\sigma_k^\xi(\alpha \otimes s) = \xi \wedge \alpha \otimes s$. In particular, the symbol acts on the form part only. Because the adjoint of the wedge multiplication by a differential form ξ is the interior product with the dual vector field ξ^g, we get the formula $(\sigma_i^\xi)^*(\alpha \otimes s) = \iota_{\xi^g} \alpha \otimes s$. $\square$

Remark. From the proof of the previous theorem, we see that the symbol of d_k^∇ is an adjointable homomorphism between the A-Hilbert bundles $(\bigwedge^k T^*M \otimes \mathcal{E}) \otimes T^*M$ and $\bigwedge^{k+1} T^*M \otimes \mathcal{E}$.

3.1. De Rham complex twisted by the oscillatory representation

Let $p_k : \mathcal{E}^k \to M$, $k \in \mathbb{N}_0$, be a set of A-Hilbert bundles and $\mathfrak{d}^\bullet = (\mathfrak{d}_k, \Gamma(\mathcal{E}^k))_{k \in \mathbb{N}_0}$ be a complex of A-differential operators. We call such a complex *A-elliptic* if out of the zero section of the cotangent bundle T^*M, the symbol sequence $\sigma^\bullet$ of $\mathfrak{d}^\bullet$ is an exact complex in the category of A-Hilbert bundles. Let us notice that if the A-Hilbert bundles are vector bundles associated to a principal bundle, it is sufficient to demand the exactness of the symbol sequence at the level of fibers only, i.e., in the category of Hilbert A-modules.

Suppose that for each $k \in \mathbb{N}_0$, $\mathcal{E}^k \to M$ is a *finitely generated projective A-Hilbert bundle* over a compact manifold M, i.e., the fibers of each $\mathcal{E}^k$ are such Hilbert A-modules. For any complex $\mathfrak{d}^\bullet = (\mathfrak{d}_k, \Gamma(\mathcal{E}^k))_{k \in \mathbb{N}_0}$ of differential operators, one may consider the sequence of its associated Laplacians $\triangle_k = \mathfrak{d}_{k-1}\mathfrak{d}_{k-1}^* + \mathfrak{d}_k^*\mathfrak{d}_k$, $k \in \mathbb{N}_0$, $\mathfrak{d}_{-1} = 0$. Let us denote the order of $\triangle_k$ by r_k. In Krýsl [8, Theorem 11], the following implication is proved. If for each $k \in \mathbb{N}_0$, the extension of the Laplacian $\triangle_k$ to $W^{r_k}(\mathcal{E}^k)$ has closed image, then the cohomology groups

$$H^k(\mathfrak{d}^\bullet, A) = \frac{\mathrm{Ker}(\mathfrak{d}_k : \Gamma(\mathcal{E}^k) \to \Gamma(\mathcal{E}^{k+1}))}{\mathrm{Im}(\mathfrak{d}_{k-1} : \Gamma(\mathcal{E}^{k-1}) \to \Gamma(\mathcal{E}^k))}$$

of $\mathfrak{d}^\bullet$ are finitely generated A-modules and Banach spaces. The norm considered on the cohomology groups $H^k(\mathfrak{d}^\bullet, A)$ is the quotient norm derived from the norm $|\ |_0$ on the smooth sections $\Gamma(\mathcal{E}^k)$.

In a similar way as one defines the spin structure in Riemannian geometry, one may introduce the so-called *metaplectic structure* in the case of a symplectic manifold. See Habermann and Habermann [5] for a more explicit definition. Suppose that (M, ω) possesses such a metaplectic structure and denote it by $\tilde{\mathcal{G}}$. In particular, $\tilde{\mathcal{G}}$ is a principal $\tilde{G}$-bundle over M, where $\tilde{G}$ is the metaplectic group. Let $\mathcal{C}^\bullet$ denote the sequence of vector bundles associated to the principal bundle $\tilde{\mathcal{G}}$ via the representation $\rho : \tilde{G} \to \mathrm{Aut}(C^\bullet)$, i.e., $\mathcal{C}^\bullet = \tilde{\mathcal{G}} \times_\rho C^\bullet$. Especially, $\mathcal{C}^0 = \tilde{\mathcal{G}} \times_\rho H$ is the so-called *basic oscillatory bundle* which we denote by $\mathcal{H}$ here. In Habermann and Habermann [5], this bundle is called the symplectic spinor bundle. Notice that in this notation, $\mathcal{C}^k = \bigwedge^k T^*M \otimes \mathcal{H}$.

Now, let ∇ be a symplectic connection on (M, ω), i.e., ∇ is a covariant derivative in $TM \to M$ preserving the symplectic form ω. We allow the connection to have a torsion. Let us denote a lift of this connection to $\tilde{\mathcal{G}}$ by $\tilde{\omega}^{\nabla}$. Associating $\tilde{\omega}^{\nabla}$ to $\mathcal{C}^0$, we get a covariant derivative ∇^H in the sections of the basic oscillatory bundle. This covariant derivative gives rise to the sequence $d^{\bullet}_H = (d^{\nabla^H}_k, \Gamma(\mathcal{C}^k))_{k=-1}^{2n+1}$. Because C^k is a Hilbert A-module (Theorem 3), the bundle $\mathcal{C}^k$ is an A-Hilbert bundle, where $A = \mathrm{End}(H)$. But see also the discussion below Theorem 3.

Theorem 6. *Let (M^{2n}, ω) be a compact symplectic manifold which admits a metaplectic structure, and ∇ be a flat symplectic connection. If the continuous extension to the Sobolev type completions $W^2(\mathcal{C}^k)$ of each of the associated Laplacians $\triangle_k$ has closed image, then the cohomology groups $H^k(d^{\bullet}_H, A)$ are finitely generated A-modules and Banach topological vector spaces.*

Proof. Due to Theorem 3, for each $k \in \mathbb{N}_0$ the bundle $\mathcal{C}^k \to M$ is a finitely generated projective A-Hilbert bundle. Due to Theorem 5, the symbol σ_k of $d^{\nabla^H}_k$ is given by $\sigma^{\xi}_k = \mathrm{ext}^{\xi}_k \otimes \mathrm{Id}_{\mathcal{H}}$. Thus the exactness of $(\sigma^{\xi}_k, \mathcal{C}^k)_{k=-1}^{2n+1}$ is equivalent to the exactness of $\mathrm{ext}^{\bullet}_{\xi}$. Lemma 4 implies that $(\sigma^{\xi}_k, \mathcal{C}^k)_{k=-1}^{2n+1}$ is exact for each $\xi \in T^*M \setminus \{0\}$ and thus, $d^{\bullet}_H$ is an A-elliptic complex. Therefore Theorem 11 in [8], mentioned above, may be applied and the conclusions for the cohomology groups $H^k(d^{\bullet}_H, A)$ follow. □

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Hodge theory for complexes over C^*-algebras with an application to A-ellipticity

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Abstract For a class of co-chain complexes in the category of pre-Hilbert A-modules, we prove that their cohomology groups equipped with the canonical quotient topology are pre-Hilbert A-modules, and derive the Hodge theory and, in particular, the Hodge decomposition for them. As an application, we show that A-elliptic complexes of pseudodifferential operators acting on sections of finitely generated projective A-Hilbert bundles over compact manifolds belong to this class if the images of the continuous extensions of their associated Laplace operators are closed. Moreover, we prove that the cohomology groups of these complexes share the structure of the fibers, in the sense that they are also finitely generated projective Hilbert A-modules.

Keywords Hodge theory · Hilbert C^*-modules · C^*-Hilbert bundles · Elliptic systems of partial differential equations

Mathematics Subject Classification Primary 46M18; Secondary 46L08 · 46M20

1 Introduction

The Hodge theory is known to hold for any co-chain complex in the category of finite dimensional vector spaces and linear maps. This theory holds also for *elliptic complexes* of pseudodifferential operators acting between smooth sections of finite rank vector bundles over compact manifolds. See, e.g., Wells [15] or Palais [10] and the references therein. Let us notice that in this case, the considered co-chain complexes consist of spaces of smooth sections of the bundles, which are infinite dimensional if the manifold contains more than a finite number of points.

Let us remark that in connection with renormalization and regularization of certain quantum theories, Hilbert and Banach bundles of infinite rank enjoy an increasing interest. See,

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e.g., the papers on stochastical quantum mechanics and parallel transport of Prugovečki [11], Drechsler and Tuckey [3], and on spin foams of Denicola et al. [1]. This list of references should not be considered as complete. The theory of indices and the K-theory are well established for a class of the so-called A-Hilbert bundles, and especially for the subclass consisting of the finitely generated projective ones. See, e.g., Fomenko and Mishchenko [6] and the monograph of Solovyov and Troitsky [13].

One of the reasons for writing of this paper is to separate features that are important for proving the Hodge theory for an algebraically defined and fairly general class of complexes (specified below) from the ones which are specific for A-elliptic complexes appearing in differential geometry and analysis on manifolds. A further reason is to describe also the topological properties of the Hodge isomorphism.

Recall that for a C^*-algebra A, a pre-Hilbert A-module U is a left module over A that is equipped with a map $(,)_U : U \times U \to A$ which is sesquilinear over A and positive definite in the sense that firstly, for any $u \in U$, the inequality $(u, u)_U \geq 0$ holds in A, and secondly, if $(u, u)_U = 0$, then $u = 0$. Let us notice that the product $(,)_U$ induces a norm $| |_U$ on U. A pre-Hilbert A-module is called a Hilbert A-module, if it is complete with respect to the norm $| |_U$. Hilbert spaces are particular examples of Hilbert A-modules for $A = \mathbb{C}$. An A-Hilbert bundle is, roughly speaking, a Banach bundle whose fibers are Hilbert A-modules.

Let us consider a co-chain complex $d^\bullet = (C^k, d_k)_{k \in \mathbb{Z}}$, where C^k are pre-Hilbert A-modules and the differentials $d_k : C^k \to C^{k+1}$ are A-linear and continuous maps with respect to the induced norms. We suppose that the differentials are adjointable for to may speak about harmonic and co-exact elements. By a *Hodge theory* for a given complex, we mean the Hodge decomposition and the Hodge isomorphism for this complex. The *Hodge decomposition* is an orthogonal sum decomposition [with respect to of $(,)_{C^k}$] of each pre-Hilbert A-module C^k in the complex into the module of harmonic, the module of exact, and the module of co-exact elements. By a *Hodge isomorphism*, one usually means a linear isomorphism of the vector space of harmonic forms and the appropriate cohomology group. Since the cohomology groups of a complex of pre-Hilbert A-modules may not be finite dimensional, we demand the isomorphism to be a homeomorphism. There is one reason more although connected, why we want the isomorphism to have this additional topological feature. Namely, the cohomology groups are quotients by images of the differentials in the complex. Since the images need not be closed, the cohomology groups need not be Hausdorff spaces. Let us notice that the Hausdorff property is well known to be equivalent to the uniqueness of limits of sequences in the considered space and therefore in physical theories, it seems to be reasonable to demand the “Hausdorffness” for each space of measured quantities.

We prove the Hodge theory for the so-called self-adjoint parametrix possessing complexes of pre-Hilbert A-modules. We start dealing with one operator $L : V \to V$ only and prove that the image, Im L, is closed and that the decomposition $V = \mathrm{Ker}\, L \oplus \mathrm{Im}\, L$ (no closure) holds if L is self-adjoint parametrix possessing. An endomorphism $L : V \to V$ is called self-adjoint parametrix possessing if there exist maps $g, p : V \to V$ satisfying $1 = gL + p = Lg + p$, $Lp = 0$ and $p = p^*$. After that we handle the case of complexes. To each complex $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ of pre-Hilbert A-modules and adjointable differentials, we assign the sequence of self-adjoint endomorphisms $L_i = d_{i-1} d_{i-1}^* + d_i^* d_i : C^i \to C^i$, $i \in \mathbb{N}_0$, called the associated *Laplace operators*. The complexes with self-adjoint parametrix possessing Laplace operators are called *self-adjoint parametrix possessing*. Under the condition that $(C^k, d_k)_{k \in \mathbb{N}_0}$ is self-adjoint parametrix possessing, we show that $C^i = \mathrm{Ker}\, L_i \oplus \mathrm{Im}\, d_i^* \oplus \mathrm{Im}\, d_{i-1}$ (the Hodge decomposition) and that each cohomology group $H^i(d^\bullet, A)$ of $d^\bullet$ is isomorphic to the space $\mathrm{Ker}\, L_i$ of harmonic elements as a pre-Hilbert A-module (the Hodge isomorphism). In particular, the cohomologies of a self-adjoint parametrix possessing complex are Hausdorff spaces being homeomorphic to kernels of continuous maps. Using these abstract considerations, we prove that the Hodge theory holds also for complexes $D^\bullet = (\Gamma(F^k), D_k)_{k\in\mathbb{N}_0}$ of the so-called A-elliptic operators acting on smooth sections of finitely generated projective A-Hilbert bundles F^k, under an assumption on the image of extensions of the Laplacians $\triangle_k = D_{k-1}D_{k-1}^* + D_k^* D_k$. Supposing that A is unital, we prove that the cohomology groups of these complexes are finitely generated and projective. Let us notice that the theory of parametrix possessing operators is more general then the theory of A-elliptic operators. We demonstrate this fact by giving an explicit example.

Two properties of C^*-algebras, they share with the complex numbers, appear to be important for proving the Hodge decomposition at the abstract level. Namely, we use that for any non-negative hermitian elements a, b of A, the inequality $|a+b|_A \geq |a|_A$ holds, as well as that $a+b=0$ implies $a=b=0$, where $|\,|_A$ denotes the norm in the C^*-algebra A. For these theorems see, e.g., Dixmier [2]. In Krýsl [8], the existence of an A-module isomorphism between the cohomology groups and the space of harmonic elements of the so-called parametrix possessing complexes (Definition 2 in [8]) is proved. However, conditions under which this A-module isomorphism is a homeomorphism are not treated there. Without supposing the self-adjointness, the proof of the existence of this isomorphism as given in [8] is rather intricate. On the contrary, in the present paper, the existence of the isomorphism together with determining its topological character are easy consequences of the Hodge decomposition. Let us notice that A-elliptic complexes are treated also in Troitsky [14] in connection with operator indices and K-theory. In the article of Schick [12], one can find a more geometrically oriented approach to a related subject area (twisted de Rham complexes, connections and curvature). The cohomology groups and their topology are not investigated in the two papers mentioned last.

In the second section, we recall notions related to (pre-)Hilbert modules, and derive several simple properties for projections, orthogonal complementability, and norm topologies on quotients of these modules. Then, we prove that for a self-adjoint parametrix possessing endomorphism $L: V \to V$, the decomposition $V = \operatorname{Ker} L \oplus \operatorname{Im} L$ holds (Theorem 6). In the third section, we derive the Hodge decomposition for self-adjoint parametrix possessing complexes (Theorem 11) and the existence of the Hodge isomorphism (Corollary 14). In the fourth section, we recall definitions of A-Hilbert bundles and A-elliptic complexes. In that section, a theorem on the Hodge theory and a specification of the cohomology groups for the mentioned class of A-elliptic complexes is proved (Theorem 18). At the end, we give the example of a self-adjoint parametrix possessing map which is not A-elliptic.

Preamble: All manifolds and bundles (total spaces, base spaces, and bundle projections) are smooth. Base spaces of all bundles are considered to be finite dimensional. The A-pseudodifferential operators are supposed to be of finite order. Further, if an index of a labeled object exceeds its allowed range, it is set to be zero.

2 Parametrix possessing endomorphisms of pre-Hilbert modules

Let A be a unital C^*-algebra. We denote the involution in A, the norm in A, and the partial ordering on hermitian elements in A by *, $|\,|_A$, and $\geq$, respectively.

A *pre-Hilbert A-module* is first a complex vector space U on which A acts. We consider that A acts from the left, and denote the action by a dot. Second, it has to be equipped with a map $(,)_U : U \times U \to A$ such that for all $a \in A$ and $u, v \in U$, the following relations hold

(1) $(a \cdot u, v)_U = a^*(u, v)_U$

(2) $(u, v)_U = (v, u)_U^*$
(3) $(u, u)_U \geq 0$, and
(4) $(u, u)_U = 0$ if and only if $u = 0$.

Any map $(,)_U : U \times U \to A$ with properties 1–4 is called an A-product. If A is the standard normed algebra of complex numbers, properties 3 and 4 are equivalent to the positive definiteness of $(,)_U$. For a pre-Hilbert A-module $(U, (,)_U)$, one defines the norm $||_U : U \to [0, \infty)$ induced by $(,)_U$ by the prescription $U \ni u \mapsto |u|_U = \sqrt{|(u, u)_U|_A}$. By a pre-Hilbert A-submodule U of a pre-Hilbert module V, we mean an A-submodule of V which is also a pre-Hilbert module if equipped with the restriction of the A-product in V to U. In particular, U has to be closed in V with respect to $||_V$. By a *pre-Hilbert A-module homomorphism L* between pre-Hilbert A-modules U and V, we mean an A-linear map, i.e., $L(a \cdot u) = a \cdot L(u)$ for each $a \in A$ and $u \in U$ that is continuous with respect to the norms $||_U$ and $||_V$. We denote the set of pre-Hilbert A-module homomorphisms of U into V by $\mathrm{Hom}_A(U, V)$. As usual, $\mathrm{End}_A(U)$ denotes the space $\mathrm{Hom}_A(U, U)$. An adjoint of a pre-Hilbert A-module homomorphism $L : U \to V$ is a map L^* from V to U satisfying for each $u \in U$ and $v \in V$ the identity $(Lu, v)_V = (u, L^*v)_U$. If the adjoint exists, it is unique, and it is a pre-Hilbert A-module homomorphism as well. See, e.g., Lance [9]. We hope that denoting the adjoint of a homomorphism by the same symbol as the involution in A does not cause a confusion. Quite often in the literature, a pre-Hilbert A-module homomorphism $L : U \to V$ is supposed to be adjointable. We do not follow this convention. Let us notice that when we speak of an A-module, we consider it with its algebraic structure only. Finally, a pre-Hilbert A-module $(U, (,)_U)$ is called a *Hilbert A-module* if it is complete with respect to $||_U$.

Elements $u, v \in U$ are called orthogonal if $(u, v)_U = 0$. When we write a direct sum $V = U \oplus U'$ where U and U' are pre-Hilbert A-submodules of V, we suppose that the summands are mutually orthogonal. For any pre-Hilbert A-submodule U of V, we denote by $U^\perp$ the orthogonal complement of U. It is defined by $U^\perp = \{v \in V \,|(v, u)_V = 0 \text{ for all } u \in U\}$ as one expects. We call U *orthogonally complementable* if there exists a pre-Hilbert A-submodule $U' \subseteq V$ such that $V = U \oplus U'$. It is well known that Hilbert and consequently pre-Hilbert A-submodules need not be complementable. For it, see, e.g., Lance [9]. It is easy to realize that for any pre-Hilbert A-submodules $U \subseteq V$ of a pre-Hilbert A-module W, the operation of taking the orthogonal complement changes the inclusion sign, i.e.,

$$U^\perp \supseteq V^\perp. \tag{1}$$

An element p in $\mathrm{End}_A(V)$ is called a *projection* if $p^2 = p$. Especially, we do not require a projection to be self-adjoint.

2.1 Complementability and quotients

We start with the following simple observation. Let p be a projection and let us denote the A-submodule $\mathrm{Im}\, p$ by U. For each $z \in U$, there exists $x \in V$ such that $z = px$. Thus, $pz = p^2x$ that implies $pz = px = z$. In other words, if p is a projection onto an A-submodule U, then its restriction to U is the identity on U. Further, if $V = U \oplus U'$ and if we set $p(x_U + x_{U'}) = x_U$, where $x_U \in U$ and $x_{U'} \in U'$, then p is a projection. We call this map a projection onto U along U'. We prove the following simple technical lemma which we will need later.

Lemma 1 *Let V be a pre-Hilbert A-module and U be an orthogonally complementable pre-Hilbert A-submodule of V.*

(1) *If* $V = U \oplus U'$ *holds for a pre-Hilbert* A*-module* U'*, then* $U' = U^\perp$*, and the projection* p *onto* U *along* $U^\perp$ *is self-adjoint.*
(2) *If* p *is a projection in* V *which is self-adjoint, then* $\mathrm{Im}\, p$ *is orthogonally complementable by* $(\mathrm{Im}\, p)^\perp$ *and* p *is a projection onto* $\mathrm{Im}\, p$ *along* $(\mathrm{Im}\, p)^\perp$*. Further,* $1-p$ *is a self-adjoint projection onto* $(\mathrm{Im}\, p)^\perp$ *along* $\mathrm{Im}\, p$*.*

Proof Because the sum $U \oplus U'$ is orthogonal, $U' \subseteq U^\perp$. Let $x \in U^\perp$ and let us write it according to the decomposition $U \oplus U'$ as $x = x_U + x_{U'}$. We have $(x_U, x_U)_V = (x - x_{U'}, x_U)_V = (x, x_U)_V - (x_{U'}, x_U)_V = 0$ since $x \in U^\perp$ and since U and U' are mutually orthogonal. Thus $x_U = 0$ and consequently, $x \in U'$ which proves the opposite inclusion. Further, for any $x \in V$ and $y = y_U + y_{U'} \in V, y_U \in U, y_{U'} \in U'$, we may write $(px, y)_V = (x_U, y_U + y_{U'})_V = (x_U, y_U)_V = (x, y_U)_V = (x, py)_V$, i.e., p is self-adjoint.

For the second statement, let us set $U = p(V)$ and $U' = (1-p)(V)$. From $x = px+(x-px)$, which holds for any $x \in V$, we have $V = U+U'$. For $x \in U$ and $y \in U'$, there are $u, v \in V$ such that $x = pu$ and $y = (1-p)v$. We may write $(x, y)_V = (pu, (1-p)v)_V = (pu, v)_V - (pu, pv)_V = (pu, v)_V - (p^*pu, v)_V = (pu, v)_V - (p^2u, v)_V = 0$. Thus, the above written sum $V = U + U'$ is orthogonal. Due to Lemma 1 item 1, $U' = (\mathrm{Im}\, p)^\perp$. Since for any $v \in V$, $p(1-p)v = pv - p^2v = pv - pv = 0$, the projection p kills elements from U'. Summing up, p is a projection onto $\mathrm{Im}\, p$ along $(\mathrm{Im}\, p)^\perp$. Since $(1-p)^2 = 1-p-p+p^2 = 1-p$ and $(1-p)^* = 1-p^* = 1-p$, we see that $1-p$ is a self-adjoint projection. The operator $1-p$ projects onto U' which equals to $(\mathrm{Im}\, p)^\perp$ as already mentioned. Further, since $(1-p)pv = pv - p^2v = pv - pv = 0$ for any $v \in V$, $1-p$ is a projection onto $(\mathrm{Im}\, p)^\perp$ along $\mathrm{Im}\, p$. $\square$

Let us remark that item 1 of the previous lemma expresses the uniqueness for the complements of orthogonally complementable pre-Hilbert A-modules.

Now, we focus our attention to quotients of pre-Hilbert A-modules. Let $U \subseteq V$ be an orthogonally complementable pre-Hilbert A-submodule of a pre-Hilbert A-module V, and p be the projection onto $U^\perp$ along U. When we speak of a quotient V/U, we consider it with the quotient A-module structure, and with the following A-product $(,)_{V/U}$. We set $([u], [v])_{V/U} = (pu, pv)_V$, $u, v \in V$. The map $(,)_{V/U}$ is easily seen to be correctly defined. First, it maps into the set of non-negative elements of A. Second, let us suppose that $([u], [u])_{V/U} = 0$ for an element $u \in V$. Then $(pu, pu)_V = 0$ and consequently, $pu = 0$. Thus $u \in U$ and therefore $[u] = 0$ proving that $(,)_{V/U}$ is an A-product. Summing up, in the case of an orthogonally complementable pre-Hilbert A-submodule U of a pre-Hilbert A-module V, we obtain a pre-Hilbert A-module structure on V/U. We shall call this structure the *canonical quotient structure*. However, let notice that for a normed space $(Y, |\,|_Y)$ and its closed subspace X, one usually considers the quotient space Y/X equipped with the norm $|\,|_q : Y/X \to [0, \infty)$ defined by

$$|[y]|_q = \inf\{|y - x|_Y \,|\, x \in X\},$$

where $y \in Y$ and $[y]$ denotes the equivalence class of y in Y/X. We call $|\,|_q$ the quotient norm. It is well known that if Y is a Banach space, the quotient equipped with the quotient norm is a Banach space as well.

The following lemma is often formulated for complementable closed subspaces of Banach spaces. Since we shall need it for pre-Hilbert spaces and in order to stress that the completeness is inessential, we give a detailed proof.

Lemma 2 *Let* U *be an orthogonally complementable pre-Hilbert* A*-submodule of a pre-Hilbert* A*-module* $(V, (,)_V)$*. Then*

(1) *the quotient norm* $|\,|_q$ *coincides with the norm induced by* $(,)_{V/U}$ *and*
(2) V/U *and* $U^\perp$ *are isomorphic as pre-Hilbert* A*-modules.*

Proof Let $p : V \to V$ be the projection onto $U^\perp$ along U. Then $p' = 1 - p$ is the projection onto U along $U^\perp$ (Lemma 1). For any $v \in V$, we have

$$\begin{aligned}
|[v]|_q^2 &= \inf_{u\in U}|v-u|_V^2\\
&= \inf_{u\in U}|(v-u, v-u)_V|_A\\
&= \inf_{u\in U}|(p'v+pv-u, p'v+pv-u)_V|_A\\
&= \inf_{u\in U}|(p'v-u, p'v+pv-u)_V + (pv, p'v+pv-u)_V|_A\\
&= \inf_{u\in U}|(p'v-u, p'v-u)_V + (pv, pv)_V|_A\\
&= |(pv, pv)_V|_A = |[v]|_{V/U}^2,
\end{aligned}$$

where in the second last step, we used the fact that $|a+b|_A \geq |a|_A$ holds for any non-negative elements $a, b \in A$. This is a direct consequence of the well known fact that $\geq$ is compatible with the vector space structure in A. (See, for instance, Dixmier [2], pp. 18.) Thus, the first assertion is proved.

It is easy to check that $\Phi : V/U \to U^\perp$, $\Phi([v]) = pv$, is a well-defined A-module homomorphism of V/U into $U^\perp$. Further, let us consider the A-module homomorphism $\Psi : U^\perp \to V/U$ defined by $\Psi(u) = [u]$, $u \in U^\perp$. For any $u \in U^\perp$, we have $\Phi(\Psi(u)) = \Phi([u]) = pu = u$ since p is a projection onto $U^\perp$. For each $[v] \in V/U$, we may write $\Psi(\Phi([v])) = \Psi(pv) = [pv]$. Because the difference of v and pv lies in U, we get $\Psi \circ \Phi = 1_{|V/U}$. Thus, Ψ and Φ are mutually inverse and consequently, V/U and $U^\perp$ are isomorphic as A-modules.

Since the topology generated by $|\,|_q$ and the one generated by $|\,|_{V/U}$ coincide, and since Ψ is the quotient map, Ψ is continuous with respect to the induced norm topologies on $(U^\perp, (,)_V)$ and $(V/U, (,)_{V/U})$. Further, let $N \subseteq U^\perp$ be an open subset of $U^\perp$. Then $p^{-1}(N)$ is an open set because p is continuous with respect to $|\,|_V$ and with respect to the restriction of $|\,|_V$ to $U^\perp$, being a projection of V onto $U^\perp$ (along U). The set of all $[x] \in V/U$ such that $x \in p^{-1}(N)$ is an open subset of V/U as follows from the definition of the quotient topology and the fact that $|\,|_q = |\,|_{V/U}$. Thus, Φ is continuous as well. Summing up, V/U and $U^\perp$ are isomorphic as pre-Hilbert A-modules. □

Remark 3 Let U be an orthogonally complementable pre-Hilbert A-module of a pre-Hilbert A-module V. Due to Lemma 2, if $(V/U, |\,|_q)$ is a Banach space, then $(V/U, (,)_{V/U})$ is a Hilbert A-module. Further, if V is a Hilbert A-module, then $(V/U, (,)_{V/U})$ is a Hilbert A-module as well.

2.2 Parametrix possessing endomorphisms

Now, we focus our attention to a relationship of the orthogonal complementability of images of pre-Hilbert A-module endomorphisms and the property described in the following definition.

Definition 4 Let L be an endomorphism of a pre-Hilbert module $(V, (,))_V$. We call L *parametrix possessing* if there exist pre-Hilbert A-module endomorphisms $g, p : V \to V$

such that

$$1 = gL + p$$
$$1 = Lg + p \quad \text{and}$$
$$Lp = 0,$$

where 1 denotes the identity on V. We call a parametrix possessing map L *self-adjoint parametrix possessing* if L and p are self-adjoint.

Remark 5 The first two equations in Definition 4 will be referred to as the *parametrix equations* (for L). Notice that there exist pre-Hilbert A-module endomorphisms which are not parametrix possessing (see Example 8) and also such for which the maps g and p are not uniquely determined. Homomorphisms with the latter property exist already for finite dimensional Hilbert spaces ($A = \mathbb{C}$). The name 'parametrix' is borrowed from the theory of partial differential equations where the operator g is often called the *Green function*.

In the next theorem, we derive the following splitting property for the self-adjoint parametrix possessing endomorphisms.

Theorem 6 *Let $L : V \to V$ be a self-adjoint parametrix possessing endomorphism of a pre-Hilbert A-module $(V, (,)_V)$ with the corresponding maps denoted by g and p. Then*

(1) *p is a projection onto* $\operatorname{Ker} L$ *along* $(\operatorname{Im} p)^\perp$ *and*
(2) $V = \operatorname{Ker} L \oplus \operatorname{Im} L$.

Proof (1) Composing the first parametrix equation with p from the right and using the third equation from the definition of a parametrix possessing endomorphism, we get that $p^2 = p$, i.e., p is a projection. Restricting $1 = gL + p$ to $\operatorname{Ker} L$, we get $1_{|\operatorname{Ker} L} = p_{|\operatorname{Ker} L}$ which implies that $\operatorname{Ker} L \subseteq \operatorname{Im} p$. Further, $Lp = 0$ forces $\operatorname{Im} p \subseteq \operatorname{Ker} L$. Thus, $\operatorname{Im} p = \operatorname{Ker} L$. Using Lemma 1 item 2, p is a projection onto $\operatorname{Im} p = \operatorname{Ker} L$ along $(\operatorname{Im} p)^\perp$.

(2) Since p is a projection onto $\operatorname{Im} p$ along $(\operatorname{Im} p)^\perp$, we have the orthogonal decomposition $V = \operatorname{Im} p \oplus (\operatorname{Im} p)^\perp$. Using the above derived result $\operatorname{Im} p = \operatorname{Ker} L$, we conclude that $V = \operatorname{Im} p \oplus (\operatorname{Im} p)^\perp = \operatorname{Ker} L \oplus (\operatorname{Ker} L)^\perp$. It is thus sufficient to prove the equality

$$(\operatorname{Ker} L)^\perp = \operatorname{Im} L \tag{2}$$

First, we prove that $\operatorname{Im} L \subseteq (\operatorname{Ker} L)^\perp$. Let $y = Lx$ for an element $x \in V$. For any $z \in \operatorname{Ker} L$, we may write $(y, z)_V = (Lx, z)_V = (x, L^* z)_V = (x, Lz)_V = 0$. Thus, $y \in (\operatorname{Ker} L)^\perp$. Now, we prove that $(\operatorname{Ker} L)^\perp \subseteq \operatorname{Im} L$. Let $x \in (\operatorname{Ker} L)^\perp$. Using the second parametrix equation, we obtain $Lgx = (1 - p)x = x$ since $1 - p$ is a projection onto $(\operatorname{Ker} L)^\perp$ (Lemma 1 item 2). Therefore, $x = Lgx \in \operatorname{Im} L$. Summing up, $\operatorname{Im} L = (\operatorname{Ker} L)^\perp$ and the equation $V = \operatorname{Ker} L \oplus \operatorname{Im} L$ follows.

□

Remark 7 Let us notice that due to Theorem 6, the image of a self-adjoint parametrix possessing endomorphism is closed (see also Eq. 2).

Example 8 We give an example of a self-adjoint Hilbert A-module endomorphism which is not self-adjoint parametrix possessing. See, e.g., Lance [9] for this example in a bit different context. Let us consider the commutative C^*-algebra $A = C([0, 1])$ equipped with the supremum norm and the complex conjugation as the involution. Take $V = A = C([0, 1])$ with the action given by the point-wise multiplication, i.e., $(f \cdot g)(x) = f(x)g(x)$, $x \in [0, 1]$,

$f, g \in A = V$ and the A-product $(f, g) = fg \in A$. The operator $L : C([0,1]) \to C([0,1])$ is given by $(Lf)(x) = xf(x)$, $x \in [0,1]$, $f \in C([0,1])$, It is obviously self-adjoint, and thus adjointable. If L were self-adjoint parametrix possessing, we would get that $\mathrm{Im}\, p = \mathrm{Ker}\, L$ according to item 1 in the proof of Theorem 6. The definition $Lf = xf$ implies that $\mathrm{Ker}\, L = \{f \in V | f = 0 \text{ on } (0,1]\}$. Since V consists of continuous functions, we see that $\mathrm{Ker}\, L = \{f \in V | f = 0 \text{ on } [0,1]\} = 0 \in V$. Consequently, $\mathrm{Im}\, p = 0$ and therefore, p is zero. Now, the parametrix equations imply that L is bijective. On the other hand, any non-zero constant function in V is not in the image of L. This is a contradiction. See also Exel [5] for treating a connected matter in the context of (generalized) pseudoinverses.

3 Hodge theory for self-adjoint parametrix possessing complexes

In this section, we focus our attention to co-chain complexes $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ of pre-Hilbert A-modules and adjointable pre-Hilbert A-module homomorphisms, i.e., for each $k \in \mathbb{N}_0$, the morphism $d_k : C^k \to C^{k+1}$ is supposed to be an adjointable pre-Hilbert A-module homomorphism, and $d_{k+1} d_k = 0$. Let us consider the sequence of *Laplace operators* $L_k = d_k^* d_k + d_{k-1} d_{k-1}^*$, $k \in \mathbb{N}_0$, associated to $d^\bullet$. Notice that in concordance with the preamble, L_0 equals $d_0^* d_0$.

Lemma 9 *Let $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ be a co-chain complex of pre-Hilbert A-modules and adjointable pre-Hilbert A-module homomorphisms. Then*

$$\mathrm{Ker}\, L_k = \mathrm{Ker}\, d_k \cap \mathrm{Ker}\, d_{k-1}^*.$$

Proof The inclusion $\mathrm{Ker}\, L_k \supseteq \mathrm{Ker}\, d_k \cap \mathrm{Ker}\, d_{k-1}^*$ follows directly from the definition of the Laplace operator L_k. To prove the opposite one, let us consider an element $x \in \mathrm{Ker}\, L_k$, and let us write $0 = (x, L_k x)_{C^k} = (x, d_k^* d_k x + d_{k-1} d_{k-1}^* x)_{C^k} = (d_k x, d_k x)_{C^{k+1}} + (d_{k-1}^* x, d_{k-1}^* x)_{C^{k-1}}$. It is known that the intersection of the cone of non-negative hermitian elements in A with the opposite cone consists only of the zero element. See, e.g., Dixmier [2], Proposition 1.6.1., pp. 15 and 16. Thus, $(d_k x, d_k x)_{C^{k+1}} = 0$ and $(d_{k-1}^* x, d_{k-1}^* x)_{C^{k-1}} = 0$, and consequently, $d_k x = d_{k-1}^* x = 0$ due to the positive definiteness of the A-products in C^{k+1} and C^{k-1}, respectively. □

As announced earlier, we prove the Hodge theory for complexes introduced in the next definition.

Definition 10 Let $d^\bullet = (C^k, d_k)_{k \in \mathbb{N}_0}$ be a co-chain complex of pre-Hilbert A-modules and adjointable pre-Hilbert A-module homomorphisms. We call $d^\bullet$ a *parametrix possessing complex* if for each $k \in \mathbb{N}_0$, the associated Laplace operator L_k is a parametrix possessing pre-Hilbert A-module endomorphism of C^k. We call $d^\bullet$ a *self-adjoint parametrix possessing complex* if the operators L_k are self-adjoint parametrix possessing pre-Hilbert A-module endomorphisms for all $k \in \mathbb{N}_0$.

Since we suppose that the differentials are pre-Hilbert A-module homomorphisms, the associated Laplace operators are pre-Hilbert A-module endomorphisms as well. Because the associated Laplace operators L_k are self-adjoint by their definitions, we could have demanded the maps L_k to be parametrix possessing and p_k to be self-adjoint in the previous definition only.

In the next theorem, the "abstract" Hodge decomposition is formulated. We use Theorem 6 in its proof.

Theorem 11 *Let $d^\bullet = (C^k, d_k)_{k\in\mathbb{N}_0}$ be a self-adjoint parametrix possessing complex. Then for any $k \in \mathbb{N}_0$, we have the decomposition*

$$C^k = \operatorname{Ker} L_k \oplus \operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1}.$$

Proof (1) Due to Lemma 9, we have $\operatorname{Ker} L_k \subseteq \operatorname{Ker} d_{k-1}^*$. Therefore, using the formulas (1) and (2), we get $(\operatorname{Ker} d_{k-1}^*)^\perp \subseteq (\operatorname{Ker} L_k)^\perp = \operatorname{Im} L_k$. Further, due to Lemma 9 again, we have $\operatorname{Ker} L_k \subseteq \operatorname{Ker} d_k$. Using (1) and (2), we get $(\operatorname{Ker} d_k)^\perp \subseteq (\operatorname{Ker} L_k)^\perp = \operatorname{Im} L_k$. Summing up, $(\operatorname{Ker} d_{k-1}^*)^\perp + (\operatorname{Ker} d_k)^\perp \subseteq \operatorname{Im} L_k$.

(2) The inclusion $\operatorname{Im} d_{k-1} \subseteq (\operatorname{Ker} d_{k-1}^*)^\perp$ holds since for any $x \in C^{k-1}$ and $y \in \operatorname{Ker} d_{k-1}^*$, we have $(d_{k-1}x, y)_{C^k} = (x, d_{k-1}^* y)_{C^{k-1}} = 0$. Similarly, $\operatorname{Im} d_k^* \subseteq (\operatorname{Ker} d_k)^\perp$. Combining these two facts with the result of item 1 of this proof, we get $\operatorname{Im} d_{k-1} + \operatorname{Im} d_k^* \subseteq (\operatorname{Ker} d_{k-1}^*)^\perp + (\operatorname{Ker} d_k)^\perp \subseteq \operatorname{Im} L_k$. Now, we show that the sum $\operatorname{Im} d_k^* + \operatorname{Im} d_{k-1}$ is orthogonal. Let us take two elements $d_k^* x$ and $d_{k-1}z$ (for $x \in C^{k+1}$ and $z \in C^{k-1}$) from $\operatorname{Im} d_k^*$ and $\operatorname{Im} d_{k-1}$, respectively. The computation $(d_k^* x, d_{k-1}z)_{C^k} = (x, d_k d_{k-1} z)_{C^{k+1}} = 0$ shows that $\operatorname{Im} d_k^*$ and $\operatorname{Im} d_{k-1}^*$ are mutually orthogonal. Summing up, $\operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1} \subseteq \operatorname{Im} L_k$.

(3) It is easy to prove that $\operatorname{Im} L_k \subseteq \operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1}$. Indeed, for any $y \in \operatorname{Im} L_k$, there exists $x \in C^k$ such that $y = L_k x = d_k^* d_k x + d_{k-1} d_{k-1}^* x = d_k^*(d_k x) + d_{k-1}(d_{k-1}^* x) \in \operatorname{Im} d_k^* + \operatorname{Im} d_{k-1}$. This observation together with item 2 proves that $\operatorname{Im} L_k = \operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1}$.

(4) Because L_k is a self-adjoint parametrix possessing pre-Hilbert A-module endomorphism of C^k, we get the equality $C^k = \operatorname{Im} L_k \oplus \operatorname{Ker} L_k$ due to Theorem 6. Substituting for $\operatorname{Im} L_k$ from item 3 of this proof, we obtain the decomposition from the statement of the theorem.

□

Remark 12 (1) In item 3 of the proof of the previous theorem, we obtained for a self-adjoint parametrix possessing complex $d^\bullet$ the decomposition

$$\operatorname{Im} L_k = \operatorname{Im} d_k^* \oplus \operatorname{Im} d_{k-1}.$$

(2) Notice that if $d^\bullet = (C^k, d_k)_{k\in\mathbb{N}_0}$ is a co-chain complex, then its adjoint $(C^{k+1}, d_k^*)_{k\in\mathbb{N}_0}$ is a chain complex as follows from $d_k^* d_{k+1}^* = (d_{k+1} d_k)^*$.

Theorem 13 *Let $d^\bullet = (C^k, d_k)_{k\in\mathbb{N}_0}$ be a self-adjoint parametrix possessing complex. Then for any $k \in \mathbb{N}_0$,*

$$\operatorname{Ker} d_k = \operatorname{Ker} L_k \oplus \operatorname{Im} d_{k-1} \quad \text{and}$$
$$\operatorname{Ker} d_k^* = \operatorname{Ker} L_{k+1} \oplus \operatorname{Im} d_{k+1}^*.$$

Proof Due to Theorem 11, we know that the sums at the right hand side in both rows are orthogonal.

The inclusion $\operatorname{Ker} L_k \oplus \operatorname{Im} d_{k-1} \subseteq \operatorname{Ker} d_k$ is an immediate consequence of the definition of a co-chain complex and of Lemma 9. To prove the opposite inclusion, let us consider an element $y \in \operatorname{Ker} d_k$. Due to Theorem 11, there exist elements $y_1 \in \operatorname{Ker} L_k$, $y_2 \in \operatorname{Im} d_{k-1}$, and $y_3 \in \operatorname{Im} d_k^*$ such that $y = y_1 + y_2 + y_3$. It is sufficient to prove that $y_3 = 0$. Let $z_3 \in C^{k+1}$ be such that $y_3 = d_k^* z_3$. We have $0 = (d_k y, z_3) = (d_k y_1 + d_k y_2 + d_k y_3, z_3) = (d_k y_3, z_3) = (y_3, d_k^* z_3) = (y_3, y_3)$ which implies $y_3 = 0$. Thus, the first equality follows.

The inclusion $\operatorname{Ker} L_{k+1} \oplus \operatorname{Im} d_{k+1}^* \subseteq \operatorname{Ker} d_k^*$ follows from Lemma 9 and from item 2 of Remark 12. To prove the inclusion $\operatorname{Ker} d_k^* \subseteq \operatorname{Ker} L_{k+1} \oplus \operatorname{Im} d_{k+1}^*$, we proceed similarly as in the previous paragraph. For $y \in \operatorname{Ker} d_k^*$, there exist $y_1 \in \operatorname{Ker} L_{k+1}$, $y_2 \in \operatorname{Im} d_k$, and

$y_3 \in \operatorname{Im} d_{k+1}^*$ such that $y = y_1 + y_2 + y_3$ (Theorem 11). Let us consider an element $z_2 \in C^k$ for which $y_2 = d_k z_2$. We have $0 = (d_k^* y, z_2) = (d_k^* y_1 + d_k^* y_2 + d_k^* y_3, z_2) = (d_k^* y_2, z_2) = (y_2, y_2)$. Thus $y_2 = 0$ which proves the equation in the second row. □

Now, for a complex $d^\bullet = (C^k, d_k)_{k\in\mathbb{N}_0}$ of pre-Hilbert A-modules, we consider the cohomology groups

$$H^i(d^\bullet, A) = \frac{\operatorname{Ker}(d_i : C^i \to C^{i+1})}{\operatorname{Im}(d_{i-1} : C^{i-1} \to C^i)},$$

$i \in \mathbb{N}_0$. Notice that in general, the A-module $Z^i(d^\bullet, A) = \operatorname{Im}(d_{i-1} : C^{i-1} \to C^i)$ of co-boundaries need not be orthogonally complementable or even not a closed subspace of the pre-Hilbert A-module of boundaries $B^i(d^\bullet, A) = \operatorname{Ker} d_i$. Consequently, the appropriate cohomology group need not be a Hausdorff space (with respect to the quotient topology). Nevertheless, for self-adjoint parametrix possessing complexes, we derive the following corollary.

Corollary 14 *If $d^\bullet = (C^k, d_k)_{k\in\mathbb{N}_0}$ is a self-adjoint parametrix possessing complex of pre-Hilbert A-modules, then for each i the cohomology group $H^i(d^\bullet, A)$ and the space* $\operatorname{Ker} L_i \subseteq C^i$ *are isomorphic as pre-Hilbert A-modules. If $d^\bullet$ is a self-adjoint parametrix possessing complex of Hilbert A-modules, then for each i, the cohomology group $H^i(d^\bullet, A)$ is a Hilbert A-module and in particular, a Banach space.*

Proof Because of Theorem 13, $U = \operatorname{Im} d_{i-1}$ is an orthogonally complementable submodule of $V = \operatorname{Ker} d_i$. Thus we may use Lemma 2 item 2 to conclude that the cohomology group $H^i(d^\bullet, A) = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$ equipped with the canonical quotient structure is a pre-Hilbert A-module isomorphic to the orthogonal complement of $\operatorname{Im} d_{i-1}$ in $\operatorname{Ker} d_i$. This complement equals $\operatorname{Ker} L_i$ thanks to Theorem 13 and the uniqueness for orthogonal complements (Lemma 1 item 1). The second statement follows in the same way using Remark 3. □

Remark 15 The isomorphism $H^i(d^\bullet, A) \cong \operatorname{Ker} L_i$ is the Hodge isomorphism mentioned in the Introduction.

4 Application to A-elliptic complexes

Let M be a finite dimensional manifold and $p : F \to M$ be a Banach bundle over M with a differentiable bundle structure $\mathfrak{S}$. Recall that each Banach bundle has to be equipped with a Banach structure $|| \, || : F \to [0, +\infty)$. As it is standard, we denote the fiber $p^{-1}(m)$ in m by F_m and the restriction of $|| \, ||$ to F_m by $|| \, ||_m$. A Banach structure is a smooth map from F to $\mathbb{R}_0^+$ such that for each $m \in M$, $(F_m, || \, ||_m)$ is a Banach space.

We call a Banach bundle $p : F \to M$ with a differentiable bundle structure $\mathfrak{S}$ an *A-Hilbert bundle* if there exists a Hilbert A-module $(S, (,)_S)$ and a bundle atlas $\mathcal{A}$ in the differentiable bundle structure $\mathfrak{S}$ such that

(1) for each $m \in M$, the fiber F_m is equipped with a Hilbert A-product, denoted by $(,)_m$, such that the Banach spaces $(F_m, | \, |_m)$ and $(F_m, || \, ||_m)$ are isomorphic as normed spaces,
(2) for each $m \in M$ and each chart $(\phi_U, U) \in \mathcal{A}$, $M \supset U \ni m$, the map $\phi_{U|F_m}$: $(F_m, (,)_m) \to (S, (,)_S)$ is a Hilbert A-module isomorphism, and
(3) the transition maps between all charts in the bundle atlas $\mathcal{A}$ are maps into the group $\operatorname{Aut}_A(S)$ of Hilbert A-module automorphisms of S.

The first condition is set in order the norm $|\,|_m$ varies smoothly with respect to $m \in M$ as the Banach structure $||\,||$ has to do due to its definition.

Let us recall that for two bundle charts $\phi_U : p^{-1}(U) \to U \times S$ and $\phi_V : p^{-1}(V) \to V \times S$, their transition map $\phi_{UV} : U \cap V \to \mathrm{Aut}(S)$ (the group of homeomorphisms of $(S, |\,|_S)$) is defined by the formula $(\phi_U \circ \phi_V^{-1})(m, v) = (m, \phi_{UV}(m)v)$, where $m \in U \cap V$ and $v \in S$. A *homomorphism of A-Hilbert bundles* $p_1 : F_1 \to M$ and $p_2 : F_2 \to M$ is a map $R : F_1 \to F_2$ between the total spaces of p_1 and p_2, such that $p_2 \circ R = p_1$ and such that R is a Hilbert A-module homomorphism in each fiber, i.e., for any $m \in M$, $R_{|p_1^{-1}(m)} : (F_1)_m \to (F_2)_m$ is a Hilbert A-module homomorphism. An A-Hilbert bundle is called *finitely generated projective* if the typical fiber, the Hilbert A-module $(S, (,)_S)$, is a finitely generated and projective Hilbert A-module. See, e.g., Solovyov and Troitsky [13] for these notions.

The space $\Gamma(F)$ of smooth sections of an A-Hilbert bundle $p : F \to M$ carries a left A-module structure given by $(a \cdot s)(m) = a \cdot (s(m))$ for $a \in A$, $s \in \Gamma(F)$ and $m \in M$. From now on, let us suppose that M is compact and equipped with a Riemannian metric g. We choose a volume element $|\mathrm{vol}_g|$ on the Riemannian manifold (M, g). For each $t \in \mathbb{N}_0$, one then defines an A-product $(,)_t$ of Sobolev type on $\Gamma(F)$. The Sobolev completion $W^t(F)$ is the completion of the space of smooth sections $\Gamma(F)$ of F with respect to the norm induced by $(,)_t$. The Sobolev completion together with the continuous extension of $(,)_t$ forms a Hilbert A-module. See Solovyov and Troitsky [13] or Fomenko and Mishchenko [6] for these constructions. For a different metric or a different choice of the volume element, one may get different Sobolev completions. However, they are isomorphic as Hilbert A-modules (see Schick [12]). By definition, the A-product $(,)_{\Gamma(F)}$ on $\Gamma(F)$ equals to the restriction of the Hilbert A-product $(,)_0$ on $W^0(F)$ to $\Gamma(F)$.

For a definition of an A-pseudodifferential operator we refer to Solovyov, Troitsky [13], pp. 79 and 80. For any A-pseudodifferential operator $D : \Gamma(F_1) \to \Gamma(F_2)$, we have the order $\mathrm{ord}(D) \in \mathbb{Z}$ of D, the adjoint $D^* : \Gamma(F_2) \to \Gamma(F_1)$ of D (Theorem 2.1.37 in [13]), and the continuous extension $D_t : W^t(F_1) \to W^{t-\mathrm{ord}(D)}(F_2)$ of D (Theorem 2.1.60, p. 89 in [13]) at our disposal. Only finite order A-pseudodifferential operators are considered. Note that the adjoint is an A-pseudodifferential operator and a pre-Hilbert A-module homomorphism, and that the continuous extension D_t is a Hilbert A-module homomorphism.

Let us denote the cotangent bundle $T^*M \to M$ by π. For an A-pseudodifferential operator D, one defines the notion of its symbol $\sigma(D) : \pi^*(F_1) \to F_2$. See Solovyov, Troitsky [13] pp. 79 and 80 for a definition which generalizes the classical one. Notice that the cotangent bundle T^*M is considered with the trivial A-Hilbert bundle structure, i.e., we set $a \cdot \alpha_m = \alpha_m$ for each $a \in A$, $\alpha_m \in T_m^*M$, and $m \in M$. It is known that $\sigma(D) : \pi^*(F_1) \to F_2$ is an adjointable A-Hilbert bundle homomorphism.

Let $(p_k : F^k \to M)_{k \in \mathbb{N}_0}$ be a sequence of A-Hilbert bundles over M and let $D^\bullet = (\Gamma(F^k), D_k)_{k \in \mathbb{N}_0}$ be a complex of A-pseudodifferential operators in F^k, i.e., $D_k : \Gamma(F^k) \to \Gamma(F^{k+1})$ is an A-pseudodifferential operator and $D_{k+1}D_k = 0$, $k \in \mathbb{N}_0$. For each $\xi \in T^*M$, the sequence $\sigma^\bullet(\xi) = (F^k, \sigma(D_k)(\xi, -))_{k \in \mathbb{N}_0}$ is easily seen to be a complex in the category of A-Hilbert bundles.

Definition 16 A complex $D^\bullet = (\Gamma(F^k), D_k)_{k \in \mathbb{N}_0}$ of A-pseudodifferential operators in A-Hilbert bundles is called *A-elliptic* if $\sigma^\bullet(\xi)$ is an exact complex in the category of A-Hilbert bundles for each $\xi \in T^*M \setminus \{(m, 0) \in T^*M | m \in M\}$, i.e., outside the image of the zero section of T^*M.

In accordance with classical conventions, we denote the Laplace operators L_k associated to a complex $D^\bullet = (\Gamma(F^k), D_k)_{k \in \mathbb{N}_0}$ of A-pseudodifferential operators by $\triangle_k$. Their orders, $\mathrm{ord}(\triangle_k)$, will be denoted by r_k for brevity.

Remark 17 (1) A single A-pseudodifferential operator $D : \Gamma(E) \to \Gamma(F)$ may be considered as the complex

$$0 \to \Gamma(E) \xrightarrow{D} \Gamma(F) \to 0.$$

In this case, the definition of an A-elliptic complex coincides with the definition of an A-elliptic operator as given, e.g., in Solovyov and Troitsky [13].

(2) If $D^\bullet$ is an A-elliptic complex, then for each $i \in \mathbb{N}_0$, the Laplace operator $\triangle_i$ is an A-elliptic operator. See Corollary 10 in Krýsl [8] for a proof.

Next, we prove that certain specified A-elliptic complexes are self-adjoint parametrix possessing and that, consequently, the Hodge theory holds for them. We use results from Section 3 and Theorems 8 and 11 from [8] in the proof.

Theorem 18 *Let A be a unital C^*-algebra and $D^\bullet = (\Gamma(F^k), D_k)_{k\in\mathbb{N}_0}$ be an A-elliptic complex in finitely generated projective A-Hilbert bundles F^k over a compact manifold M. Let us suppose that for each $k \in \mathbb{N}_0$, the image of the continuous extension $(\triangle_k)_{r_k} : W^{r_k}(F^k) \to W^0(F^k)$ of the Laplace operator $\triangle_k$ is closed in $W^0(F^k)$. Then for any $i \in \mathbb{N}_0$*

(1) *$H^i(D^\bullet, A)$ is a finitely generated projective Hilbert A-module isomorphic to* $\mathrm{Ker}\,\triangle_i$ *as a Hilbert A-module*
(2) $\Gamma(F^i) = \mathrm{Ker}\,\triangle_i \oplus \mathrm{Im}\,D_i \oplus \mathrm{Im}\,D^*_{i-1}$
(3) $\mathrm{Ker}\,D_i = \mathrm{Ker}\,\triangle_i \oplus \mathrm{Im}\,D^*_i$, *and*
(4) $\mathrm{Ker}\,D^*_i = \mathrm{Ker}\,\triangle_{i+1} \oplus \mathrm{Im}\,D_i$.

Proof For a self-adjoint A-elliptic operator $K : \Gamma(F) \to \Gamma(F)$ of order r such that $\mathrm{Im}\,K_r$ is closed in $W^0(F)$, two maps denoted by G and P are constructed in the proof of Theorem 8 in Krýsl [8]. They satisfy the parametrix equations (for K) and the equation $KP = 0$. In the terminology of the current paper, K is a parametrix possessing pre-Hilbert A-module endomorphism of the pre-Hilbert A-module $(\Gamma(F), (,)_{\Gamma(F)})$. The construction of P goes as follows. For $K_r : W^r(F) \to W^0(F)$, one considers the adjoint $(K_r)^* : W^0(F) \to W^r(F)$ and the projection $p_{\mathrm{Ker}\,(K_r)^*}$ of $W^0(F)$ onto the kernel $\mathrm{Ker}\,(K_r)^*$ along the closed Hilbert A-module $\mathrm{Im}\,K_r$. Thus, according to Lemma 1 item 2, the projection $p_{\mathrm{Ker}\,(K_r)^*}$ is self-adjoint. The operator P is defined as the restriction of $p_{\mathrm{Ker}\,(K_r)^*}$ to $\Gamma(F) \subseteq W^0(F)$. Restricting $p_{\mathrm{Ker}\,(K_r)^*}$ to $\Gamma(F)$ does not change its property of being an idempotent and keeps the operator self-adjoint because the A-product $(,)_{\Gamma(F)}$ coincides with the restriction of $(,)_0$ to $\Gamma(F)$. Summing up, P is a projection and a self-adjoint pre-Hilbert A-module endomorphism. Since K is supposed to be self-adjoint, it is a self-adjoint parametrix possessing pre-Hilbert A-module endomorphism according to Definition 4.

Now, we prove the theorem. Since $\triangle_i = D_{i-1}D^*_{i-1} + D^*_i D_i$ is self-adjoint and A-elliptic (Remark 17 item 2) and since we suppose that $\mathrm{Im}(\triangle_i)_{r_i}$ is closed in $W^0(F^i)$, we may use the conclusion of the previous paragraph for $K = \triangle_i$, $F = F^i$ and $r = r_i$. Thus, $\triangle_i$ is a self-adjoint parametrix possessing pre-Hilbert A-module endomorphism. Consequently, $D^\bullet$ is a self-adjoint parametrix possessing complex (Definition 10). Using Theorems 11 and 13, one obtains the statements in parts 2, 3 and 4.

Due to Corollary 14, the cohomology group $H^i(D^\bullet, A)$ is a pre-Hilbert A-module isomorphic to the kernel of the Laplace operator $\triangle_i$. According to Theorem 11 in [8], $H^i(D^\bullet, A)$ is a finitely generated A-module and a Banach space (with respect to the quotient norm $|\,|_q$). Consequently (Remark 3), $H^i(D^\bullet, A)$ equipped with the canonical quotient structure is a Hilbert A-module. It is known that a finitely generated Hilbert A-module over a unital C^*-algebra is projective. For this, see Theorem 5.9 in Frank, Larson [7]. Thus, also item 1 is proved. □

Remark 19 Notice that the decompositions and the adjoints of the maps contained in items 2, 3 and 4 of the previous theorem are meant with respect to the A-product $(,)_{\Gamma(F^i)}$ on the pre-Hilbert A-module $\Gamma(F^i)$. Instead for pre-Hilbert modules we could have formulated Sects. 2 and 3 for Hilbert A-modules only and then derive a theorem parallel to Theorem 18 for the spaces $W^0(F^k)$ and for the appropriate "L^2-cohomology" groups.

Remark 20 Let us remark that there are holomorphic Banach bundles whose Čech cohomology groups are known to be non-Hausdorff. See Erat [4]. We should mention that the fact that the Čech cohomology groups are considered in that text makes the situation different from the case of cohomology of complexes which we study.

In the future, we would like to find a convenient class of Hilbert A-modules and A-pseudodifferential operators for which the condition on the image of (the extension of) $\triangle_k$ in Theorem 18 is automatically satisfied.

Remark 21 Non-elliptic and parametrix possessing operator

In this example, we show that the notion of a self-adjoint parametrix possessing operator is more general than the one of an A-elliptic operator. (We will not always indicate that we speak about homomorphisms or endomorphisms of Hilbert A-modules and omit the expression "Hilbert A-module".) Let U be an infinite dimensional separable Hilbert space considered as a Hilbert A-module for $A = \mathbb{C}$ and let $l : U \to U$ be the orthogonal projection onto a finite dimensional subspace V of U. For a compact manifold M, we consider the trivial A-Hilbert bundle $q : \mathcal{U} = M \times U \to M$. The projection l can be lifted to the operator L in the space of smooth sections $\Gamma(\mathcal{U}) : L(s)(m) = (m, l(s(m)))$, where $s \in \Gamma(\mathcal{U})$ and $m \in M$. It is of order zero, and thus it equals to its symbol. More precisely, its symbol is the map $\pi^*(\mathcal{U}) \ni (\xi, \tau) \mapsto (q(\tau), l(\mathrm{pr}_2\tau))$, where $\mathrm{pr}_2 : M \times U \to U$ is the projection onto the second component of the product and $\xi \in T^*_{q(\tau)}M$. This map is obviously not an isomorphism (in any fiber) of $\mathcal{U}$ (out of the zero section of T^*M). We set $g = L$ on $\Gamma(\mathcal{U})$ and $(ps)(m) = (m, (1-l)(s(m)))$. It is trivial to verify that $1 = Lg + p$, $1 = gL + p$, and $p = p^*$.

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Elliptic complexes over C^*-algebras of compact operators

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For a C^*-algebra A of compact operators and a compact manifold M, we prove that the Hodge theory holds for A-elliptic complexes of pseudodifferential operators acting on smooth sections of finitely generated projective A-Hilbert bundles over M. For these C^*-algebras and manifolds, we get a topological isomorphism between the cohomology groups of an A-elliptic complex and the space of harmonic elements of the complex. Consequently, the cohomology groups appear to be finitely generated projective C^*-Hilbert modules and especially, Banach spaces. We also prove that in the category of Hilbert A-modules and continuous adjointable Hilbert A-module homomorphisms, the property of a complex of being self-adjoint parametrix possessing characterizes the complexes of Hodge type.

1. Introduction

This paper is devoted to the Hodge theory for Hilbert and pre-Hilbert A-modules, and to an application of this theory to A-elliptic complexes of operators acting on sections of specific A-Hilbert bundles over compact manifolds if A is a C^*-algebra of compact operators. It is a continuation of papers [1] and [2] the main result of which we recall in this article. Let A be a C^*-algebra and M be a compact manifold. In [2], the Hodge theory is proved to hold for an arbitrary A-elliptic complex of operators acting on smooth sections of finitely generated projective A-Hilbert bundles over M under the condition that the images of the extensions (to the appropriate Sobolev spaces) of the Laplacians of the complex are closed. This condition seems to be difficult to verify in particular cases. One of the main results achieved in this paper is that one can omit the assumption on the images if A is a C^*-algebra of compact operators.

We define what it means that *the Hodge theory holds* for a complex in a general additive and dagger category and study this concept in more detail in categories of pre-Hilbert and Hilbert A-modules and continuous adjointable A-equivariant maps. These categories constitute a special class of the so-called R-module categories which are in addition, equipped with an involution on the morphisms spaces. Let us notice that these categories enjoy an interest in the so called categorical quantum mechanics. See, e.g., Selinger [3], Abramsky, Heunen [4] and Abramsky, Coecke [5] for instance. However, we are foremost interested in their occurrence in differential geometry and global analysis. We say that *the Hodge theory holds* for a complex $d^\bullet = (U^i, d_i : U^i \to U^{i+1})_{i\in\mathbb{Z}}$ in an additive and dagger category $\mathfrak{C}$ or that $d^\bullet$ is of *Hodge type* if for each $i \in \mathbb{Z}$, we have

$$U^i = \mathrm{Im}\, d_{i-1} \oplus \mathrm{Im}\, d_i^* \oplus \mathrm{Ker}\, \Delta_i,$$

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where $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$, and d_i^* and d_{i-1}^* are the adjoints of d_i and d_{i-1}, respectively. The operators Δ_i, $i \in \mathbb{Z}$, are called the *Laplace operators* of $d^\bullet$. Notice that no closures are present in the above decomposition since spaces are supposed to be closed whenever we write a direct sum between them.

For a C^*-algebra A, we consider the category PH_A^* of right pre-Hilbert A-modules and continuous adjointable A-equivariant maps. The full subcategory of PH_A^*, the object of which are right Hilbert A-modules is denoted by H_A^* and it is called the category of Hilbert A-modules. See Kaplansky [6], Paschke [7], Lance [8] and Manuilov, Troitsky [9] for information on (pre-)Hilbert C^*-modules. Recall that each object in PH_A^* inherits a norm derived from the inner product with values in the algebra A defined on the object. The categories PH_A^* and H_A^* are additive and dagger with respect to the orthogonal direct sum and the involution defined by the inner product. In Krýsl [2], the so-called self-adjoint parametrix possessing complexes in PH_A^* are introduced. According to results in that paper, any self-adjoint parametrix possessing complex in PH_A^* is of Hodge type and its cohomology groups are pre-Hilbert A-modules isomorphic to the kernels of the Laplace operators as pre-Hilbert A-modules. Especially, the cohomology groups are normed spaces. Note that in general, the cohomology groups of a complex of Hilbert spaces need not be even Hausdorff in the quotient topology. In the present paper, we prove the opposite implication for the category $\mathfrak{C} = H_A^*$, i.e., that if the Hodge theory holds for a complex in the category H_A^*, the complex is already self-adjoint parametrix possessing. Thus, in H_A^* the condition of being self-adjoint parametrix possessing characterizes the Hodge type complexes.

Let us recall that the Hodge theory is well known to hold for elliptic complexes of pseudodifferential operators acting on smooth sections of finite rank vector bundles over compact manifolds. Classical examples of such complexes are deRham and Dolbeault complexes over compact manifolds equipped with appropriate geometric structures. See, e.g., Palais [10] and Wells [11].

Fomenko, Mishchenko prove in [12], that the continuous extensions of an A-elliptic operator to the Sobolev section spaces are A-Fredholm. In [13], Bakić and Guljaš prove that any A-Fredholm endomorphism $F : U \to U$ in H_A^* has closed image if A is a C^*-algebra of compact operators. We generalize this result to the case of an A-Fredholm morphism $F : U \to V$ acting between Hilbert A-modules U and V. For C^*-algebras of compact operators, we further derive a "transfer" theorem which, roughly speaking, enables to deduce properties of certain pre-Hilbert A-module maps from appropriate properties of their continuous Hilbert A-module extensions. Applying the theorem generalizing the result of Bakić and Guljaš, we get that the images of the Sobolev extensions of Laplace operators of an A-elliptic complex are closed. The transfer theorem enables us to prove the main theorem of the article. Namely, that in the case of compact manifolds, C^*-algebras A of compact operators, and A-elliptic complexes, the Laplace operators themselves have closed images, they are self-adjoint parametrix possessing, and consequently as follows from [1], the complexes are of Hodge type. We prove a further characterizations of the cohomology groups as well.

The motivation for our research comes from quantum field theories which aim to include constraints—especially, from the Becchi, Rouet, Stora and Tyutin or simply BRST quantization. See Henneaux, Teitelboim [14], Horuzhy, Voronin [15], Carchedi, Roytenberg [16] and the references there. Let us explain their connection to our paper in more detail. In the BRST quantization, one constructs complexes whose cohomology groups represent state spaces of a given physical system. Because the state spaces in quantum theories are usually formed by infinite dimensional vector spaces, the co-cycle spaces for the cohomology groups have to be infinite dimensional as well. It is agreed that the state spaces shall be equipped with a topology because of a testing of the proposed theory by measurements. Since the measurements do not give a precise value of the measured observable (a result of a measurement is always a value together with an error estimate), the state spaces should have a good behavior of limits of converging sequences. Especially, it is desired that the limit (of a converging sequence) to be unique. On the other hand, it is well known that the uniqueness of limits in a topological space forces the space to be T1. However, the T1 separation axiom in a topological vector space implies that the topological vector space is already Hausdorff. (For it, see, e.g., Theorem 1.12 in Rudin [17].) The quotient of a topological vector space is non-Hausdorff in the quotient topology if and only if the space by which one divides is not closed. If we insist that the state spaces are cohomology groups, we shall be able to assure that the spaces of co-boundaries are closed.

For an explanation of the requirements on a physical theory considered above, we refer to Ludwig [18] and to the first, general, part of the still appealing paper of von Neumann [19]. We hope that our work can be relevant for physicists at least in the case when a particular BRST complex appears to be self-adjoint parametrix possessing in the categories PH_A^* or H_A^* for an arbitrary C^*-algebra A, or an A-elliptic complex in finitely generated projective A-Hilbert bundles over a compact manifold if A is a C^*-algebra of compact operators. Let us mention a further inspiring topic from physics—namely, the parallel transport in Hilbert bundles considered in a connection with quantum theory. See, e.g., Drechsler, Tuckey [20].

Let us notice that in Troitsky [21], indices of A-elliptic complexes are investigated. In that paper the operators are, quite naturally, allowed to be changed by an A-compact perturbation in order the index of the operator is an element of the appropriate K-group. See also Schick [22]. In this paper, we do not follow this approach and do not perturb the operators. If the reader is interested in a possible application of the Hodge theory for A-elliptic complexes, we refer to Krýsl [23].

In the second chapter, we give a definition of the Hodge type complex, recall definitions of a pre-Hilbert and a Hilbert C^*-module, and give several examples of them. We prove that complexes in the category of Hilbert spaces and continuous maps are of Hodge type if the images of their Laplace operators are closed (Lemma 1). Further, we recall the definition of a self-adjoint parametrix possessing complex in PH_A^* and some of its properties including the fact which is important for us—namely, that they are of Hodge type (Theorem 2). We prove that if a complex in H_A^* is of Hodge type, it is already self-adjoint parametrix possessing (Theorem 3). At the end of the second section (Example 3), we give examples of complexes in $H^*_{\mathbb{C}}$ whose cohomology groups are not Hausdorff spaces, and consequently they are neither self-adjoint parametrix possessing nor of Hodge type. In the third chapter, we recall the result of Bakić and Guljaš (Theorem 4), give the mentioned generalization of it (Corollary 5), and prove the transfer theorem (Theorem 6). In the fourth section, basic facts on differential operators acting on sections of A-Hilbert bundles over compact manifolds are recalled. In this chapter, the theorem on the Hodge theory for C^*-algebras of compact operators and A-elliptic complexes in finitely generated projective A-Hilbert bundles over compact manifolds is proved (Theorem 9).

Preamble: All manifolds and bundles are assumed to be smooth. Base manifolds of bundles are assumed to be finite dimensional. We do not suppose the Hilbert spaces to be separable.

2. Self-adjoint maps and complexes possessing a parametrix

Let us recall that a category $\mathfrak{C}$ is called a *dagger category* if there is a contra-variant functor $* : \mathfrak{C} \to \mathfrak{C}$ which is the identity on the objects and involutive on morphisms. Thus, for any objects U and V and any morphism $F : U \to V$, we have the relations $*F : V \to U$, $*\mathrm{Id}_U = \mathrm{Id}_U$ and $*(*F) = F$. The functor $*$ is called the involution or the dagger. The morphism $*F$ is denoted by F^*, and it is called the adjoint of F. See Burgin [24] or Brinkmann, Puppe [25].

Let us give some examples of categories which are additive and dagger.

Example 1. (1) The category of finite dimensional inner product spaces over $\mathbb{R}$ or $\mathbb{C}$ and linear maps is an example of an additive and of a dagger category. The addition (product) of objects is given by the orthogonal sum and the addition of morphism is the addition of linear maps. The involution is defined as the adjoint of maps with respect to the inner products. The existence of the adjoint to any linear map is based on the Gram–Schmidt process which guarantees the existence of an orthonormal basis. The matrix of the adjoint of a morphism with respect to orthonormal bases in the domain and target spaces is given by taking the transpose or the transpose and the complex conjugate of the matrix of the original map with respect to these bases.

(2) The category of Hilbert spaces and continuous maps equipped with the addition of objects and maps, and with the involution given as in item 1 is an example of an additive and dagger category. For the existence of the adjoints, see Meise, Vogt [26]. (The proof is based on the Riesz representation theorem for Hilbert spaces.)

Definition 1. Let $\mathfrak{C}$ be an additive and a dagger category. We say that the *Hodge theory holds* for a complex $d^\bullet = (U^i, d_i : U^i \to U^{i+1})_{i\in\mathbb{Z}}$ in $\mathfrak{C}$ or that $d^\bullet$ is of *Hodge type* if for each $i \in \mathbb{Z}$, we have

$$U^i = \mathrm{Im}\, d_{i-1} \oplus \mathrm{Im}\, d_i^* \oplus \mathrm{Ker}\, \Delta_i$$

where $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$, and d_i^* and d_{i-1}^* are the adjoints of d_i and d_{i-1}, respectively. We call the morphism Δ_i the ith *Laplace operator* of $d^\bullet$, $i \in \mathbb{Z}$. We say that the Hodge theory holds for a subset $\mathfrak{K} \subseteq \mathcal{K}(\mathfrak{C})$ of complexes in $\mathfrak{C}$ if it holds for each element $d^\bullet \in \mathfrak{K}$.

Remark 1. (1) In Definition 1, we demand no compatibility of the involution with the additive structure. However, in the categories of pre-Hilbert and Hilbert A-modules that we consider mostly, the relations $(F+G)^* = F^* + G^*$ and $(zF)^* = z^*F^*$ are satisfied for each objects U, V, morphisms $G, F : U \to V$, and complex number $z \in \mathbb{C}$.

(2) The existence of the Laplace operators of $d^\bullet$ is guaranteed by the definitions of the additive and of the dagger category. If the dagger structure is compatible with the additive structure in the sense of item 1, we see that the Laplace operators are self-adjoint, i.e., $\Delta_i^* = \Delta_i$, $i \in \mathbb{Z}$.

Lemma 1. *Let $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}}$ be a complex in the category of Hilbert spaces and continuous maps. If the images of the Laplace operators of $d^\bullet$ are closed, the Hodge theory holds for $d^\bullet$.*

Proof. On the level of symbols, we do not distinguish the dependence of the inner products on the Hilbert spaces and denote each of them by $(,)$. It is easy to realize that $\mathrm{Ker}\, \Delta_i = \mathrm{Ker}\, d_{i-1}^* \cap \mathrm{Ker}\, d_i$. Namely, the inclusion $\mathrm{Ker}\, \Delta_i \supseteq \mathrm{Ker}\, d_i \cap \mathrm{Ker}\, d_{i-1}^*$ is immediate due to the definition of Δ_i. The opposite one can be seen as follows. For any $u \in \mathrm{Ker}\, \Delta_i$, we have $0 = (\Delta_i u, u) = (d_i^* d_i u + d_{i-1} d_{i-1}^* u, u) = (d_i u, d_i u) + (d_{i-1}^* u, d_{i-1}^* u)$. Since inner products are positive definite, we have $d_i u = 0$ and $d_{i-1}^* u = 0$.

Because we assume the image of Δ_i to be closed, taking the orthogonal complement of $\mathrm{Ker}\, \Delta_i = \mathrm{Ker}\, d_i \cap \mathrm{Ker}\, d_{i-1}^*$, we get $(\mathrm{Ker}\, d_{i-1}^*)^\perp \subseteq (\mathrm{Ker}\, \Delta_i)^\perp = \overline{\mathrm{Im}\, \Delta_i} = \mathrm{Im}\, \Delta_i$ and $(\mathrm{Ker}\, d_i)^\perp \subseteq (\mathrm{Ker}\, \Delta_i)^\perp = \overline{\mathrm{Im}\, \Delta_i} = \mathrm{Im}\, \Delta_i$. Summing-up,

$$(\mathrm{Ker}\, d_{i-1}^*)^\perp + (\mathrm{Ker}\, d_i)^\perp \subseteq \mathrm{Im}\, \Delta_i.$$

Further, it is immediate to see that $\mathrm{Im}\, d_{i-1} \subseteq (\mathrm{Ker}\, d_{i-1}^*)^\perp$ and $\mathrm{Im}\, d_i^* \subseteq (\mathrm{Ker}\, d_i)^\perp$. Indeed, for any $u \in \mathrm{Im}\, d_{i-1}$ there exists an element $u' \in U^{i-1}$ such that $u = d_{i-1}u'$. For each $v \in \mathrm{Ker}\, d_{i-1}^*$, we have $(u, v) = (d_{i-1}u', v) = (u', d_{i-1}^* v) = 0$. Thus, the inclusion follows. The other inclusion can be seen similarly. Using the result of the previous paragraph, we obtain

$$\mathrm{Im}\, d_{i-1} + \mathrm{Im}\, d_i^* \subseteq (\mathrm{Ker}\, d_{i-1}^*)^\perp + (\mathrm{Ker}\, d_i)^\perp \subseteq \mathrm{Im}\, \Delta_i. \tag{1}$$

We prove that the sum $\mathrm{Im}\, d_{i-1} + \mathrm{Im}\, d_i^*$ is direct. For it, we take $u = d_{i-1}u'$ and $v = d_i^* v'$ for $u' \in U^{i-1}$ and $v' \in U^{i+1}$, and compute $(u, v) = (d_{i-1}u', d_i^* v') = (d_i d_{i-1} u', v') = 0$ which holds since $d^\bullet$ is a complex. Therefore, we have $\mathrm{Im}\, d_i^* \oplus \mathrm{Im}\, d_{i-1}$ $(\subseteq \mathrm{Im}\, \Delta_i)$.

The inclusion $\operatorname{Im} \Delta_i \subseteq \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$ is immediate. Thus, we conclude that $\operatorname{Im} \Delta_i = \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$. Since for each $i \in \mathbb{Z}$, Δ_i is self-adjoint and its image is closed, we have $U^i = \operatorname{Im} \Delta_i \oplus \operatorname{Ker} \Delta_i$. Substituting the equation for $\operatorname{Im} \Delta_i$ found at end of the previous paragraph, we get $U^i = \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1} \oplus \operatorname{Ker} \Delta_i$ proving that the Hodge theory holds for $d^\bullet$. □

Remark 2. By Lemma 1, the Hodge theory holds for any complex in the category $\mathfrak{C} = V_{\text{fin}}$ of finite dimensional inner product spaces over real or complex numbers and linear maps since any linear subspace of a finite dimensional vector space is closed. However, it is possible to prove that the Hodge theory holds for $\mathfrak{K} = \mathcal{K}(\mathfrak{C})$ in a simpler way. The relation $\operatorname{Ker} \Delta_i = \operatorname{Ker} d_i \cap \operatorname{Ker} d_{i-1}^*$ is proved in the same way as in the proof of Lemma 1. Since for any $A, B \subseteq U^i$, the equation $(A \cap B)^\perp = A^\perp + B^\perp$ holds, we have $(\operatorname{Ker} d_i \cap \operatorname{Ker} d_{i-1}^*)^\perp = (\operatorname{Ker} d_i)^\perp + (\operatorname{Ker} d_{i-1}^*)^\perp$. Due to the finite dimension, we can write $(\operatorname{Ker} d_i)^\perp = \operatorname{Im} d_i^*$ and $(\operatorname{Ker} d_{i-1}^*)^\perp = \operatorname{Im} d_{i-1}$, and thus $(\operatorname{Ker} \Delta_i)^\perp = (\operatorname{Ker} d_{i-1} \cap \operatorname{Ker} d_i^*)^\perp = \operatorname{Im} d_{i-1} + \operatorname{Im} d_i^*$. The sum is direct as follows from $0 = (d_i d_{i-1} u, v) = (d_{i-1} u, d_i^* v)$, $u \in U^{i-1}$, $v \in U^{i+1}$ – in the same way as in the proof of Lemma 1. Substituting $(\operatorname{Ker} \Delta_i)^\perp = \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$ into $U_i = \operatorname{Ker} \Delta_i \oplus (\operatorname{Ker} \Delta_i)^\perp$, we get $U^i = \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1} \oplus \operatorname{Ker} \Delta_i$. Let us notice that in Lemma 1, we *proved* that the images of d_i and d_{i-1}^* are closed.

Next we recall the definitions of the Hilbert and pre-Hilbert modules over C^*-algebras. (For C^*-algebras, we refer to Dixmier [27].)

Definition 2. For a C^*-algebra A, a *pre-Hilbert A-module* is a complex vector space U, which is a right A-module (the action is denoted by a dot) and which is in addition, equipped with a map $(,) : U \times U \to A$ such that for each $z \in \mathbb{C}$, $a \in A$ and $u, v, w \in U$ the following relations hold

(1) $(u, zv + w) = z(u, v) + (u, w)$
(2) $(u, v \cdot a) = (u, v)a$
(3) $(u, v) = (v, u)^*$
(4) $(u, u) \geq 0$, and $(u, u) = 0$ implies $u = 0$

where a^* denotes the conjugation of the element $a \in A$. A pre-Hilbert A-module $(U, (,))$ is called a *Hilbert A-module* if U is a Banach space with respect to the norm $U \ni u \mapsto |u| = \sqrt{|(u, u)|_A} \in [0, +\infty)$. We call the map $(,) : U \times U \to A$ the *inner product* (on U), or an A-inner product if we would like to stress the target.

Note that if A is the algebra of complex numbers, Definition 2 coincides with the one of pre-Hilbert and of Hilbert spaces.

Morphisms of pre-Hilbert A-modules $(U, (,)_U)$ and $(V, (,)_V)$ are assumed to be continuous, A-linear and adjointable maps. Recall that a map $L : U \to V$ is called A-linear if the equivariance condition $L(u) \cdot a = L(u \cdot a)$ holds for any $a \in A$ and $u \in U$. An adjoint $L^* : V \to U$ of a pre-Hilbert A-module morphism $L : U \to V$ is a map which satisfies $(Lu, v)_V = (u, L^* v)_U$ for any $u \in U$ and $v \in V$. It is known that the adjoint need not exist in general, and that if it exists, it is unique and a pre-Hilbert A-module homomorphism, i.e., continuous and A-linear. Morphisms of Hilbert A-modules have to be morphisms of these modules considered as pre-Hilbert A-modules. The category the objects of which are pre-Hilbert A-modules and the morphisms of which are continuous, A-linear and adjointable maps will be denoted by PH_A^*. The category H_A^* of Hilbert A-modules is defined to be the full subcategory of PH_A^* the object of which are Hilbert A-modules. If we drop the condition on the adjointability of morphisms, we denote the resulting categories by PH_A and H_A. See Manuilov, Troitsky [9] for more information on these objects. By an *isomorphism* $F : U \to V$ in PH_A^* or H_A^*, we mean a morphism which is right and left invertible by a morphism in PH_A^* or H_A^*, respectively. In particular, we demand an isomorphism in these categories neither to preserve the appropriate inner products nor the induced norms.

Submodules of a (pre-)Hilbert A-module V have to be closed subspaces and (pre-)Hilbert A-modules with respect to the restrictions of the algebraic and the inner product structures defined on V. Further, if U is a submodule of the (pre-)Hilbert A-module V, we can construct the space $U^\perp = \{v \in V, (v, u) = 0 \text{ for all } u \in U\}$ which is a (pre-)Hilbert A-module. We call U orthogonally complemented in V if $V = U \oplus U^\perp$. There are Hilbert A-submodules which are not orthogonally complemented. (See Lance [8].) For the convenience of the reader, we give several examples of Hilbert A-modules and an example of a pre-Hilbert A-module. For further examples, see Solovyov, Troitsky [28], Manuilov, Troitsky [9], Lance [8], and Wegge-Olsen [29].

Example 2. (1) Let H be a Hilbert space with the Hilbert inner product denoted by $(,)_H$. The inner product is supposed to be hermitian conjugate in the first (left) variable. The right action of the C^*-algebra $A = B(H)$ of bounded linear operators on the continuous dual H^* of H is given by $l \cdot a = l \circ a$ for any $l \in H^*$ and $a \in B(H)$. Let us denote the unique vector from H representing element $k \in H^*$ by k_*, i.e., $(k_*, w) = k(w)$ for any $w \in H$. Its existence is guaranteed by the Riesz representation theorem. The inner product $(k, l) \in B(H)$ of two elements $k, l \in H^*$ is defined by $(k, l)(v) = l(v) k_*$, where $v \in H$. In this case, the product takes values in the C^*-algebra $K(H)$ of compact operators on H. In fact, the inner product maps into the algebra of finite rank operators.

(2) For a locally compact topological space X, consider the C^*-algebra $A = \mathcal{C}_0(X)$ of continuous functions vanishing at infinity with the product given by the point-wise multiplication, the complex conjugation as the involution, and the supremum norm $|\,|_A : \mathcal{C}_0(X) \to [0, +\infty)$, i.e.,

$$|f|_A = \sup\{|f(x)|, x \in X\}$$

where $f \in A$. For U, we take the C^*-algebra $\mathcal{C}_0(X)$ itself with the module structure defined by the point-wise multiplication, i.e., $(f \cdot g)(x) = f(x)g(x), f \in U, g \in A$ and $x \in X$. The inner product is defined by $(f, g) = \overline{f}g$. Note that this is a particular example of a Hilbert A-module with $U = A$, right action $a \cdot b = ab$ for $a \in U = A$ and $b \in A$, and inner product $(a, b) = a^*b, a, b \in U$.

(3) If $(U, (,))$ is a Hilbert A-module, the orthogonal direct sums of a finite number of copies of U form a Hilbert A-module in a natural way. One can also construct the space $\ell^2(U)$, i.e., the space consisting of sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in U, n \in \mathbb{N}$, for which the series $\sum_{i=1}^{\infty}(a_i, a_i)$ converges in A. The inner product is given by $((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) = \sum_{i=1}^{\infty}(a_i, b_i)$, where $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \ell^2(U)$. See Manuilov, Troitsky [9].

(4) Let A be a C^*-algebra. For a compact manifold M^n, pick a Riemannian metric g and choose a volume element $|\mathrm{vol}_g| \in \Gamma(M, |\bigwedge^n T^*M|)$. Then for any A-Hilbert bundle $\mathcal{E} \to M$ with fiber a Hilbert A-module E, one defines a *pre-Hilbert A-module* $\Gamma(M, \mathcal{E})$ of smooth sections of $\mathcal{E} \to M$ by setting $(s \cdot a)_m = s_m \cdot a$ for $a \in A, s \in \Gamma(M, \mathcal{E})$, and $m \in M$. One sets

$$(s', s) = \int_{m\in M} (s'_m, s_m)_m |\mathrm{vol}_g|_m$$

where $s, s' \in \Gamma(M, \mathcal{E})$, $(,)_m$ denotes the inner product in fiber $\mathcal{E}_m$, and $m \in M$. Taking the completion of $\Gamma(M, \mathcal{E})$ with respect to the norm induced by the A-inner product $(,)$ (as given in Definition 2), we get the Hilbert A-module $(W^0(M, \mathcal{E}), (,)_0)$. Further Hilbert A-modules $(W^t(M, \mathcal{E}), (,)_t)$, $t \in \mathbb{Z}$, are derived from the space $\Gamma(M, \mathcal{E})$ by mimicking the construction of Sobolev spaces for finite rank bundles. See Wells [11] for the finite rank case and Solovyov, Troitsky [28] for the case of A-Hilbert bundles.

Let us turn our attention to the so-called self-adjoint parametrix possessing morphisms in the category $\mathfrak{C} = PH_A^*$.

Definition 3. A pre-Hilbert A-module endomorphism $F : U \to U$ is called *self-adjoint parametrix possessing* if F is self-adjoint, i.e., $F^* = F$, and there exist a pre-Hilbert A-module homomorphism $G : U \to U$ and a self-adjoint pre-Hilbert A-module homomorphism $P : U \to U$ such that

$$1_U = GF + P$$
$$1_U = FG + P$$
$$FP = 0.$$

Remark 3. (1) Definition 3 makes sense in an arbitrary additive and dagger category as well.

(2) The map G from Definition 3 is called a *parametrix* or a *Green operator* and the first two equations in this definition are called the *parametrix equations*.

(3) Composing the first parametrix equation from the right with P and using the third equation, we get that $P^2 = P$.

(4) If $F : U \to U$ is a self-adjoint parametrix possessing endomorphism in PH_A^*, then $U = \operatorname{Ker} F \oplus \operatorname{Im} F$ (see Theorem 6 in Krýsl [2]). In particular, the image of F is closed. Note that we do not assume that U is complete.

(5) A self-adjoint morphism in H_A^* is self-adjoint parametrix possessing if its image is closed. Indeed, the Mishchenko theorem (Theorem 3.2 on pp. 22 in Lance [8]) enables us to write for such a self-adjoint morphism $F : U \to U$ with closed image, the orthogonal decomposition $U = \operatorname{Ker} F \oplus \operatorname{Im} F$. Thus, we can define the projection onto $\operatorname{Ker} F$ along $\operatorname{Im} F$. It is immediate that the projection is self-adjoint. Inverting F on its image and defining it by zero on the kernel of F, we get a map G which satisfies the parametrix equations. It is continuous due to the open map theorem. Thus as follows from item 4, a self-adjoint map F from H_A^* is self-adjoint parametrix possessing if and only if its image is closed.

Let us remind the reader that if $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}}$ is a co-chain complex in the category $\mathfrak{C} = PH_A^*$, each of the Laplace operators $\Delta_i = d_{i-1}d_{i-1}^* + d_i^*d_i$ is self-adjoint since the category is not only additive and dagger, but these structures are also compatible (Remark 1 item 1).

Definition 4. A co-chain complex $d^\bullet \in \mathcal{K}(PH_A^*)$ is called *self-adjoint parametrix possessing* if all of its Laplace operators are self-adjoint parametrix possessing maps.

Remark 4. (1) Since $\Delta_{i+1}d_i = (d_{i+1}^*d_{i+1} + d_id_i^*)d_i = d_id_i^*d_i = d_id_i^*d_i + d_id_{i-1}d_{i-1}^* = d_i(d_i^*d_i + d_{i-1}d_{i-1}^*) = d_i\Delta_i$, the **Laplace operators are co-chain endomorphisms** of $d^\bullet$. Similarly, one derives that the Laplace operators are chain endomorphisms of the chain complex $(U^i, d_i^* : U^i \to U^{i-1})_{i\in\mathbb{Z}}$ "dual" to $d^\bullet$.

(2) Let us assume that the Laplace operators Δ_i of a complex $d^\bullet$ in PH_A^* satisfy equations $\Delta_iG_i + P_i = G_i\Delta_i + P_i = 1_{U_i}$ and that the identity $\Delta_iP_i = 0$ holds. Notice that we do not suppose that the idempotent P_i is self-adjoint. Still, we can prove that the Green operators G_i satisfy $G_{i+1}d_i = d_iG_i$, i.e., that the **Green operators are co-chain endomorphisms** of the complex $d^\bullet$ we consider. For it, see Theorem 3 in Krýsl [1].

(3) In the following picture, facts from the previous two items are summarized in a diagrammatic way.

$$\begin{tikzcd}
U_{i-1} \arrow[r, shift left, "d_{i-1}"] \arrow[d, shift left, "\Delta_{i-1}"] & U_i \arrow[l, shift left, "d_{i-1}^*"] \arrow[r, shift left, "d_i"] \arrow[d, shift left, "\Delta_i"] & U_{i+1} \arrow[l, shift left, "d_i^*"] \arrow[d, shift left, "\Delta_{i+1}"] \\
U_{i-1} \arrow[r, shift left, "d_{i-1}"] \arrow[u, shift left, "G_{i-1}"] & U_i \arrow[l, shift left, "d_{i-1}^*"] \arrow[r, shift left, "d_i"] \arrow[u, shift left, "G_i"] & U_{i+1} \arrow[l, shift left, "d_i^*"] \arrow[u, shift left, "G_{i+1}"]
\end{tikzcd}$$

Let us consider the cohomology groups $H^i(d^\bullet) = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$ of a complex $d^\bullet \in \mathcal{K}(PH_A^*)$, $i \in \mathbb{Z}$. If $\operatorname{Im} d_{i-1}$ is orthogonally complemented in $\operatorname{Ker} d_i$, one can define an inner product in $H^i(d^\bullet)$ by $([u],[v])_{H^i(d^\bullet)} = (p_i u, p_i v)$, where $u, v \in U^i$ and p_i is the projection along $\operatorname{Im} d_{i-1}$ onto the orthogonal complement $(\operatorname{Im} d_{i-1})^\perp$ in $\operatorname{Ker} d_i$. Let us call this inner product the *canonical quotient product*. For facts on inner products on quotients in PH_A^*, see [2].

In the next theorem, we collect results on self-adjoint parametrix complexes from [2].

Theorem 2. *Let A be a C^*-algebra. If $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}} \in \mathcal{K}(PH_A^*)$ is self-adjoint parametrix possessing complex, then for any $i \in \mathbb{Z}$,*

(1) $U^i = \operatorname{Ker} \Delta_i \oplus \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1}$, *i.e., $d^\bullet$ is a Hodge type complex.*
(2) $\operatorname{Ker} d_i = \operatorname{Ker} \Delta_i \oplus \operatorname{Im} d_{i-1}$.
(3) $\operatorname{Ker} d_i^* = \operatorname{Ker} \Delta_{i+1} \oplus \operatorname{Im} d_{i+1}^*$.
(4) $\operatorname{Im} \Delta_i = \operatorname{Im} d_i^* \oplus \operatorname{Im} d_{i-1}$.
(5) *$H^i(d^\bullet)$ is a pre-Hilbert A-module with respect to the canonical quotient product $(,)_{H^i(d^\bullet)}$.*
(6) *The spaces $\operatorname{Ker} \Delta_i$ and $H^i(d^\bullet)$ are isomorphic as pre-Hilbert A-modules. Moreover, if $d^\bullet$ is self-adjoint parametrix possessing complex in $\mathcal{K}(H_A^*)$, then $H^i(d^\bullet)$ is an A-Hilbert module and $\operatorname{Ker} \Delta_i \simeq H^i(d^\bullet)$ are isomorphic as A-Hilbert modules.*

Proof. See Theorem 11 in Krýsl [2] for item 1; Theorem 13 in [2] for items 2 and 3; Remark 12(1) in [2] for item 4; and Corollary 14 in [2] for items 5 and 6. □

Next we prove that in the category $\mathfrak{C} = H_A^*$, the property of a complex to be self-adjoint parametrix possessing characterizes the complexes of Hodge type.

Theorem 3. *Let A be a C^*-algebra. If the Hodge theory holds for a complex $d^\bullet \in \mathcal{K}(H_A^*)$, then $d^\bullet$ is self-adjoint parametrix possessing.*

Proof. Because the Hodge theory holds for $d^\bullet$, we have the decomposition of U^i into Hilbert A-modules

$$U^i = \operatorname{Ker} \Delta_i \oplus \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$$

$i \in \mathbb{Z}$. In particular, the ranges of d_{i-1} and d_i^* are closed topological vector spaces. It is immediate to verify that

$$\operatorname{Ker} d_i^* d_i = \operatorname{Ker} d_i, \qquad \operatorname{Ker} d_{i-1}^* = \operatorname{Ker} d_{i-1} d_{i-1}^*.$$

Since the ranges of d_{i-1} and d_i^* are closed, we get

$$\operatorname{Im} d_i^* d_i = \operatorname{Im} d_i^*, \qquad \operatorname{Im} d_{i-1} d_{i-1}^* = \operatorname{Im} d_{i-1}$$

using the theorem of Mishchenko (Theorem 3.2 in Lance [8]) for d_i^* and d_{i-1}. In particular, we see that the images of $d_i^* d_i$ and $d_{i-1} d_{i-1}^*$ are closed as well.

For $i \in \mathbb{Z}$ and $u \in U^i$, we have

$$(\Delta_i u, \Delta_i u) = (d_i^* d_i u, d_i^* d_i u) + (d_{i-1} d_{i-1}^* u, d_{i-1} d_{i-1}^* u)$$

since $(d_i^* d_i u, d_{i-1} d_{i-1}^* u) = (d_i u, d_i d_{i-1} d_{i-1}^* u) = 0$ for any $u \in U^i$. Due to the definition of the Laplace operators and the positive definiteness of the A-Hilbert product, we have $\operatorname{Ker} \Delta_i = \operatorname{Ker} d_i \cap \operatorname{Ker} d_{i-1}^*$ (as in the proof of Lemma 1). For $u \in (\operatorname{Ker} \Delta_i)^\perp = (\operatorname{Ker} d_i)^\perp + (\operatorname{Ker} d_{i-1}^*)^\perp$, there exist $u_1 \in (\operatorname{Ker} d_i)^\perp = \operatorname{Im} d_i^*$ and $u_2 \in (\operatorname{Ker} d_{i-1}^*)^\perp = \operatorname{Im} d_{i-1}$ such that $u = u_1 + u_2$. Consequently,

$$\begin{aligned}
(\Delta_i u, \Delta_i u) &= (d_i^* d_i (u_1 + u_2), d_i^* d_i (u_1 + u_2)) + (d_{i-1} d_{i-1}^* (u_1 + u_2), d_{i-1} d_{i-1}^* (u_1 + u_2)) \\
&= (d_i^* d_i u_1, d_i^* d_i u_1) + (d_i^* d_i u_2, d_i^* d_i u_2) + (d_i^* d_i u_1, d_i^* d_i u_2) + (d_i^* d_i u_2, d_i^* d_i u_1) \\
&\quad + (d_{i-1} d_{i-1}^* u_1, d_{i-1} d_{i-1}^* u_1) + (d_{i-1} d_{i-1}^* u_2, d_{i-1} d_{i-1}^* u_2) \\
&\quad + (d_{i-1} d_{i-1}^* u_1, d_{i-1} d_{i-1}^* u_2) + (d_{i-1} d_{i-1}^* u_2, d_{i-1} d_{i-1}^* u_1) \\
&= (d_i^* d_i u_1, d_i^* d_i u_1) + (d_{i-1} d_{i-1}^* u_2, d_{i-1} d_{i-1}^* u_2)
\end{aligned}$$

since $(d_i^* d_i u_2, d_i^* d_i u_2) = (d_i^* d_i u_1, d_i^* d_i u_2) = 0$ due to $u_2 \in \operatorname{Im} d_{i-1}$, and $(d_{i-1} d_{i-1}^* u_1, d_{i-1} d_{i-1}^* u_1) = (d_{i-1} d_{i-1}^* u_1, d_{i-1} d_{i-1}^* u_2) = 0$ due to $u_1 \in \operatorname{Im} d_i^*$. Because both summands at the right-hand side of

$$(\Delta_i u, \Delta_i u) = (d_i^* d_i u_1, d_i^* d_i u_1) + (d_{i-1} d_{i-1}^* u_2, d_{i-1} d_{i-1}^* u_2)$$

are non-negative, we obtain $(\Delta_i u, \Delta_i u) \geq (d_i^* d_i u_1, d_i^* d_i u_1)$ and $(\Delta_i u, \Delta_i u) \geq (d_{i-1} d_{i-1}^* u_2, d_{i-1} d_{i-1}^* u_2)$. Consequently

$$|\Delta_i u| \geq |d_i^* d_i u_1| \tag{2}$$

$$|\Delta_i u| \geq |d_{i-1} d_{i-1}^* u_2|. \tag{3}$$

(See paragraph 1.6.9 on pp. 18 in Dixmier [27].) Notice that $d_i^* d_i$ and $d_{i-1}^* d_{i-1}$ are injective on $(\operatorname{Ker} d_i^* d_i)^\perp = (\operatorname{Ker} d_i)^\perp$ and $(\operatorname{Ker} d_{i-1} d_{i-1}^*)^\perp = (\operatorname{Ker} d_{i-1}^*)^\perp$, respectively, and zero on the complements of the respective spaces. Due to an equivalent characterization of closed image maps on Banach spaces, there are positive real numbers α, β such that $|d_i^* d_i u_1| \geq \alpha |u_1|$ and $|d_{i-1} d_{i-1}^* u_2| \geq \beta |u_2|$ hold for any $u_1 \in (\operatorname{Ker} d_i)^\perp$ and $u_2 \in (\operatorname{Ker} d_{i-1}^*)^\perp$ (see Theorem 2.5 in Abramovich, Aliprantis [30]). Substituting these inequalities into (2) and (3) and adding the resulting ones, we see that $2|\Delta_i u| \geq \alpha |u_1| + \beta |u_2|$. Thus $|\Delta_i u| \geq \frac{1}{2}\min\{\alpha, \beta\}(|u_1| + |u_2|) \geq \frac{1}{2}\min\{\alpha, \beta\}|u_1 + u_2| = \frac{1}{2}\min\{\alpha, \beta\}|u|$ by the triangle inequality. Using the characterization of closed image maps again, we get that the image of Δ_i is closed. This implies that $d^\bullet$ is self-adjoint parametrix possessing using Remark 3 item 5. □

Remark 5. From Theorems 2, 3 and Remark 3 item 5, we get that a complex in H_A^* is of Hodge type if and only if the images of its Laplace operators are closed if and only if it is self-adjoint parametrix possessing.

Example 3. We give examples of complexes which are not self-adjoint parametrix possessing.

(1) For a compact manifold M of positive dimension, let us consider the Sobolev spaces $W^{k,l}(M)$ for k, l non-negative integers. For $l = 2$, these spaces are complex Hilbert spaces. Due to the Rellich–Kondrachov embedding theorem and the fact that the dimension of $W^{k,2}(M)$ is infinite, the canonical embedding $i : W^{k,2}(M) \hookrightarrow W^{l,2}(M)$ has a non-closed image for $k > l$. We take

$$d^\bullet = \ 0 \longrightarrow W^{k,2}(M) \xrightarrow{\ i\ } W^{l,2}(M) \longrightarrow 0.$$

Labeling the first element in the complex by zero, the second cohomology $H^2(d^\bullet) = \operatorname{Ker} 0/\operatorname{Im} i = W^{l,2}(M)/i(W^{k,2}(M))$ is non-Hausdorff in the quotient topology. The complex is not self-adjoint parametrix possessing due to Theorem 2 item 5. Consequently, it is not of Hodge type (Theorem 3).

(2) This example shows a simpler construction of a complex in $\mathcal{K}(H_{\mathbb{C}}^*)$ which is not of Hodge type. Without any reference to a manifold, we can define mapping $i : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by setting $i(e_n) = e_n/n$, where $(e_n)_{n=1}^{+\infty}$ denotes the canonical orthonormal system of $\ell^2(\mathbb{N})$. It is easy to check that i is continuous. Further, the set $i(\ell^2(\mathbb{N}))$ is not closed. For it, the sequence $(1, 1/2, 1/3, \ldots) \in \ell^2(\mathbb{N})$ is not in the image. Indeed, the preimage of this element had to be the sequence $(1, 1, 1, \ldots)$ which is not in $\ell^2(\mathbb{N})$. On the other hand, $(1, 1/2, 1/3, \ldots)$ lies in the closure of $i(\ell^2(\mathbb{N}))$ since it is the limit of the sequence $i((1, 0 \ldots)), i((1, 1, 0, \ldots)), i((1, 1, 1, 0 \ldots)), \ldots$. The complex $0 \to \ell^2(\mathbb{N}) \xrightarrow{i} \ell^2(\mathbb{N}) \to 0$ is not of Hodge type and it is not self-adjoint parametrix possessing by similar reasons as those given in the example above.

3. C^*-Fredholm operators over C^*-algebras of compact operators

In this section, we focus on complexes over C^*-algebras of compact operators, and study C^*-Fredholm maps acting between Hilbert modules over such algebras. For the convenience of the reader, we recall some known notions.

Definition 5. Let $(U, (,)_U)$ and $(V, (,)_V)$ be Hilbert A-modules.

(1) For any $u \in U$ and $v \in V$, the operator $F_{u,v} : U \to V$ defined by $U \ni u' \mapsto F_{u,v}(u') = u \cdot (v, u')$ is called an *elementary operator*. A morphism $F : U \to V$ in H_A^* is called of *A-finite rank* if it can be written as a finite $\mathbb{C}$-linear combination of the elementary operators.

(2) The set $K_A(U, V)$ of *A-compact* operators on U is defined to be the closure of the vector space of the A-finite rank morphisms in the operator norm on $\operatorname{Hom}_{H_A^*}(U, V)$, induced by the norms $|\ |_U$ and $|\ |_V$.

(3) We call $F \in \operatorname{Hom}_{H_A^*}(U, V)$ *A-Fredholm* if there exist Hilbert A-module homomorphisms $G_V : V \to U$ and $G_U : U \to V$ and A-compact homomorphisms $P_U : U \to U$ and $P_V : V \to V$ such that

$$G_U F = 1_U + P_U$$

$$F G_V = 1_V + P_V$$

i.e., if F is left and right invertible modulo A-compact operators.

Remark 6. (1) *Equivalent definition of A-compact operators.* The A-finite rank operators are easily seen to be adjointable. Suppose for a moment that we define the "A-compact" operators as morphisms in the category H_A of Hilbert A-modules and continuous A-module homomorphisms that lie in the operator norm closure in $\mathrm{Hom}_{H_A}(U, V) \supseteq \mathrm{Hom}_{H_A^*}(U, V)$ of the A-finite rank operators. One can prove that these operators are adjointable. Thus, the set of these "A-compact" elements coincides with the set of the A-compact operators defined above (Definition 5). For it see, e.g., Corollary 15.2.4 in Wegge-Olsen [29].

(2) *A-compact vs. compact.* It is well known that in general, the notion of an A-compact operator does not coincide with the notion of a compact operator in a Banach space. Indeed, let us consider an infinite dimensional *unital* C^*-algebra A ($1 \in A$), and take $U = A$ with the right action given by the multiplication in A and the inner product $(a, b) = a^*b$ for $a, b \in A$. Then the identity $1_U : U \to U$ is A-compact since it is equal to $F_{1,1}$. But it is not a compact operator in the classical sense since U is infinite dimensional.

Example 4. (1) *A-Fredholm operator with non-closed image.* Let us consider the space $X = [0, 1] \subseteq \mathbb{R}$, the C^*-algebra $A = \mathcal{C}([0, 1])$ and the tautological Hilbert A-module $U = A = \mathcal{C}([0, 1])$ (second paragraph of Example 2). We give a simple proof of the fact (well known in other contexts) that there exists an endomorphism on U which is A-Fredholm but the image of which is not closed. Let us take an arbitrary map $T \in \mathrm{End}_{H_A^*}(U)$. Writing $f = 1 \cdot f$, we have $T(1 \cdot f) = T(1) \cdot f = T(1)f$. Thus, T can be written as the elementary operator $F_{f_0,1}$ where $f_0 = T(1)$. Since T is arbitrary, we see that $K_A(U, U) = \mathrm{End}_{H_A^*}(U)$. Consequently, any endomorphism $T \in \mathrm{End}_{H_A^*}(U)$ is A-Fredholm since $T1_U = 1_U T = 1_U + (T - 1_U)$. Let us consider operator $Ff = xf, f \in U$. This operator satisfies $F = F^*$, and it is clearly a morphism of the Hilbert A-module U. It is immediate to realize that $\mathrm{Ker}\, F = 0$. Suppose that the image of $F = F^*$ is closed. Using Theorem 3.2 in Lance [8], we obtain $\mathcal{C}([0, 1]) = \mathrm{Im}\, F^* \oplus \mathrm{Ker}\, F = \mathrm{Im}\, F$. Since the constant function $1 \notin \mathrm{Im}\, F$, we get a contradiction. Therefore $\mathrm{Im}\, F$ is not closed although F is an A-Fredholm operator. Let us recall that the image of a Fredholm operator on a Banach space, in the classical sense, is closed.

(2) *Hilbert space over its compact operators.* Let $(H, (,)_H)$ be a Hilbert space and let us take $A = K(H)$ and $U = H^*$ with the action and the inner product as in Example 2 item 1. We have $F_{k,l}(m) = k \cdot (l, m)$ for any $k, l, m \in H^*$. An easy computation shows that $F_{k,l}(m) = k(l_*)m$ where l_* is defined in Example 2. Thus, $F_{k,l}$ are scalar operators and their finite linear combinations are scalar operators as well. Therefore, their closure $K_{K(H)}(H^*) = \mathbb{C}1_{H^*}$, where 1_{H^*} denotes the identity on H^*.

Remark 7. Let us remark that the definition of an A-Fredholm operator on pp. 841 in Mishchenko, Fomenko [12] is different from the definition of an A-Fredholm operator given in item 3 of Definition 5 of our paper. However, an A-Fredholm operator in the sense of Fomenko and Mishchenko is necessarily invertible modulo an A-compact operator (see Theorem 2.4 in Fomenko, Mishchenko [12]), i.e., it is an A-Fredholm operator in our sense.

Definition 6. A C^*-algebra is called a *C^*-algebra of compact operators* if it is a C^*-subalgebra of the C^*-algebra of compact operators $K(H)$ on a Hilbert space H.

If A is a C^*-algebra of compact operators, an analog of an *orthonormal system* in a Hilbert space is introduced for the case of Hilbert A-modules in the paper of Bakić, Guljaš [13]. For a fixed Hilbert A-module, the cardinality of any of its orthonormal systems does not depend on the choice of such a system. We denote the cardinality of an orthonormal system of a Hilbert A-module U over a C^*-algebra A of compact operators by $\dim_A U$. Let us note that in particular, an orthonormal system forms a set of generators of the module (see [13]).

Theorem 4. *Let A be a C^*-algebra of compact operators, U be a Hilbert A-module, and $F \in \mathrm{End}_{H_A^*}(U)$. Then F is A-Fredholm, if and only if its image is closed and $\dim_A \mathrm{Ker}\, F$ and $\dim_A(\mathrm{Im}\, F)^\perp$ are finite.*

Proof. Bakić, Guljaš [13], pp. 268. □

Corollary 5. *Let A be a C^*-algebra of compact operators, U and V be Hilbert A-modules, and $F \in \mathrm{Hom}_{H_A^*}(U, V)$. Then F is an A-Fredholm operator, if and only if its image is closed and $\dim_A \mathrm{Ker}\, F$ and $\dim_A(\mathrm{Im}\, F)^\perp$ are finite.*

Proof. Let $F : U \to V$ be an A-Fredholm operator and G_U, P_U and G_V, P_V be the corresponding left and right inverses and projections, respectively, i.e., $G_U F = 1_U + P_U$ and $FG_V = 1_V + P_V$.

Let us consider the element $\mathfrak{F} = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix} \in \mathrm{End}_{H_A^*}(U \oplus V)$. For this element, we can write

$$\begin{pmatrix} 0 & G_U \\ G_V^* & 0 \end{pmatrix} \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix} = \begin{pmatrix} 1_U + P_U & 0 \\ 0 & 1_V + P_V^* \end{pmatrix} = \begin{pmatrix} 1_U & 0 \\ 0 & 1_V \end{pmatrix} + \begin{pmatrix} P_U & 0 \\ 0 & P_V^* \end{pmatrix}.$$

Since the last written matrix is an A-compact operator in $\mathrm{End}_{H_A^*}(U \oplus V)$, $\mathfrak{F}$ is left invertible modulo an A-compact operator on $U \oplus V$. The right invertibility is proved in a similar way. Summing-up, $\mathfrak{F}$ is A-Fredholm. According to Theorem 4, $\mathfrak{F}$ has closed image. This implies that F has closed image as well due to the orthogonality of the modules U and V in $U \oplus V$. Let us denote the orthogonal projections of $U \oplus V$ onto U and V by proj_U and proj_V, respectively. Due to Theorem 4, $\dim_A(\mathrm{Ker}\,\mathfrak{F})$ and $\dim_A(\mathrm{Im}\,\mathfrak{F})^\perp$ are finite. Since $\mathrm{Ker}\,F = \mathrm{proj}_U(\mathrm{Ker}\,\mathfrak{F})$ and $\mathrm{Im}\,F = \mathrm{proj}_V(\mathrm{Im}\,\mathfrak{F})$, the finiteness of $\dim_A \mathrm{Ker}\,F$ and $\dim_A(\mathrm{Im}\,F)^\perp$ follows from the finiteness of $\dim_A(\mathrm{Ker}\,\mathfrak{F})$ and $\dim_A(\mathrm{Im}\,\mathfrak{F})^\perp$.

If $\dim_A \mathrm{Ker}\,F$ and $\dim_A(\mathrm{Im}\,F)^\perp$ are finite, we deduce that $\dim_A \mathrm{Ker}\,\mathfrak{F}$ and $\dim_A(\mathrm{Im}\,\mathfrak{F})^\perp$ are finite as well using $\mathrm{Ker}\,F^* = (\mathrm{Im}\,F)^\perp$ and $\mathrm{Ker}\,F = (\mathrm{Im}\,F^*)^\perp$. If $\mathrm{Im}\,F$ is closed, the image $\mathrm{Im}\,\mathfrak{F} = \mathrm{Im}\,F^* \oplus \mathrm{Im}\,F$ is closed by Theorem 3.2 in [8]. Thus, $\mathfrak{F}$ satisfies the assumptions of Theorem 4, and we conclude that $\mathfrak{F}$ is A-Fredholm. Consequently, there exists a map $\mathfrak{G} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{End}_{H_A^*}(U \oplus V)$ such that $\mathfrak{F}\mathfrak{G} = 1_{U\oplus V} + P_{U\oplus V}$ for an A-compact operator $P_{U\oplus V}$ in $U \oplus V$. Expanding this equation, we get $FB = 1_V + \mathrm{proj}_V P_{U\oplus V|V}$. It is immediate to realize that $\mathrm{proj}_V P_{U\oplus V|V}$ is an A-compact operator in V. Thus, F is right invertible modulo an A-compact operator in V. Similarly, one proceeds in the case of the left inverse. Summing-up, F is an A-Fredholm morphism. □

Now, we state and prove the theorem on the "transfer" as promised in the introduction.

Theorem 6. *Let A be a C^*-algebra of compact operators, $(V, (,)_V)$ and $(W, (,)_W)$ be Hilbert A-modules, and $(U, (,)_U)$ be a pre-Hilbert A-module which is a vector subspace of V and W such that the norms $|\,|_W$ and $|\,|_U$ coincide on U, and $|\,|_V$ restricted to U dominates $|\,|_U$. Suppose that $D \in \mathrm{End}_{PH_A^*}(U)$ is a self-adjoint morphism having a continuous adjointable extension $\widetilde{D} : V \to W$ such that*

(i) *$\widetilde{D}$ is A-Fredholm,*
(ii) *$\widetilde{D}^{-1}(U), \widetilde{D}^{*-1}(U) \subseteq U$ and*
(iii) *$\mathrm{Ker}\,\widetilde{D}$ and $\mathrm{Ker}\,\widetilde{D}^*$ are subsets of U.*

Then D is a self-adjoint parametrix possessing operator in U.

Proof. We construct the parametrix and the projection.

(1) Using assumption (i), $\widetilde{D}$ has closed image by Corollary 5. By Theorem 3.2 in Lance [8], the image of $\widetilde{D}^* : W \to V$ is closed as well, and the following decompositions

$$V = \mathrm{Ker}\,\widetilde{D} \oplus \mathrm{Im}\,\widetilde{D}^*,$$
$$W = \mathrm{Ker}\,\widetilde{D}^* \oplus \mathrm{Im}\,\widetilde{D}$$

hold. Restricting $\widetilde{D}$ to the Hilbert A-module $\mathrm{Im}\,\widetilde{D}^*$, we obtain a continuous bijective Hilbert A-module homomorphism $\mathrm{Im}\,\widetilde{D}^* \to \mathrm{Im}\,\widetilde{D}$.
Let us set

$$\widetilde{G}(x) = \begin{cases} (\widetilde{D}_{|\mathrm{Im}\,\widetilde{D}^*})^{-1}(x) & x \in \mathrm{Im}\,\widetilde{D} \\ 0 & x \in \mathrm{Ker}\,\widetilde{D}^*. \end{cases}$$

The operator $\widetilde{G} : W \to V$ is continuous by the open map theorem. Due to its construction, $\widetilde{G}$ is a morphism in the category H_A. Because of the adjointability of $\widetilde{D}$, and the definition of $\widetilde{G}$, $\widetilde{G}$ is adjointable as well. Summing-up, $\widetilde{G} \in \mathrm{Hom}_{H_A^*}(W, V)$. Note that $\widetilde{G} : W \to \mathrm{Im}\,\widetilde{D}^*$.

(2) It is easy to see that the decomposition $V = \mathrm{Ker}\,\widetilde{D} \oplus \mathrm{Im}\,\widetilde{D}^*$ restricts to U in the sense that $U = \mathrm{Ker}\,D \oplus (\mathrm{Im}\,\widetilde{D}^* \cap U)$. Indeed, let $u \in U$. Then $u \in V$ and thus $u = v_1 + v_2$ for $v_1 \in \mathrm{Ker}\,\widetilde{D}$ and $v_2 \in \mathrm{Im}\,\widetilde{D}^*$. Since $\mathrm{Ker}\,\widetilde{D} \subseteq U$ (assumption (iii)) and $\mathrm{Ker}\,D \subseteq \mathrm{Ker}\,\widetilde{D}$, we have $\mathrm{Ker}\,\widetilde{D} = \mathrm{Ker}\,D$. Similarly, one proves that $\mathrm{Ker}\,\widetilde{D}^* = \mathrm{Ker}\,D$. In particular, $v_1 \in \mathrm{Ker}\,D$. Since U is a vector space, $v_2 = u - v_1$ and $u, v_1 \in U$, v_2 is an element of U as well. Thus, $U \subseteq \mathrm{Ker}\,D \oplus (\mathrm{Im}\,\widetilde{D}^* \cap U)$. Since $\mathrm{Ker}\,D, \mathrm{Im}\,\widetilde{D}^* \cap U \subseteq U$, the decomposition holds.

(3) Further, we have $\mathrm{Im}\,\widetilde{D}^* \cap U = \mathrm{Im}\,D$. Indeed, if $u \in U$ and $u = \widetilde{D}^* w$ for an element $w \in W$ then $w \in U$ due to item (ii) and consequently, $u = \widetilde{D}^* w = D^* w = Dw$ that implies $\mathrm{Im}\,\widetilde{D}^* \cap U \subseteq \mathrm{Im}\,D$. The opposite inclusion is immediate. (Similarly, one proves that $\mathrm{Im}\,\widetilde{D} \cap U = \mathrm{Im}\,D$.) Putting this result together with the conclusion of item 2 of this proof, we obtain $U = \mathrm{Ker}\,D \oplus \mathrm{Im}\,D$.

(4) It is easy to realize that $\widetilde{G}_{|U}$ is into U. Namely, if $v = \widetilde{G}u$ for an element $u \in U \subseteq V$, we have $u = u_1 + u_2$ for $u_1 \in \mathrm{Ker}\,\widetilde{D}$ and $u_2 \in \mathrm{Im}\,\widetilde{D}^*$ according to the decomposition of V above. Since $u_2 = u - u_1$ and $u_1 \in U$ (due to (iii)), we see that u_2 is an element of U as well. Consequently, we may write $v = \widetilde{G}_{|U} u = \widetilde{G}u_1 + \widetilde{G}u_2 = \widetilde{D}^{-1}_{|\mathrm{Im}\,\widetilde{D}^*} u_2$. Since $\widetilde{D}^{-1}(U) \subseteq U$ (item (ii)), we obtain that $v \in U$ proving that $\widetilde{G}_{|U}$ is into U. Let us set $G = \widetilde{G}_{|U}$. Due to the assumptions on the norms $|\,|_U$, $|\,|_V$, and $|\,|_W$ and the continuity of $\widetilde{G} : (W, |\,|_W) \to (V, |\,|_V)$, it is easy to see that $G : U \to U$ is continuous as well.

(5) Defining P to be the projection of U onto $\mathrm{Ker}\,D$ along the $\mathrm{Im}\,D$, we get a self-adjoint projection on the pre-Hilbert module U due to the decomposition $U = \mathrm{Ker}\,D \oplus \mathrm{Im}\,D$ derived in item 3 of this proof. The relations $DP = 0$ and $1_U = GD + P = DG + P$ are then easily verified using the relation $\mathrm{Ker}\,\widetilde{D}^* = \mathrm{Ker}\,D$. □

Remark 8. In the assumptions of the preceding theorem, specific properties are generalized which hold for self-adjoint elliptic operators acting on smooth sections of vector bundles over compact manifolds. For instance, assumption (ii) corresponds to the smooth regularity and (iii) is a generalization of the fact that differential operators are of finite order. See, e.g., Palais [10] or Wells [11]. As we will see in the next chapter, these properties remain true in the case of elliptic complexes on sections of finitely generated projective Hilbert C^*-bundles over compact manifolds.

4. Complexes of pseudodifferential operators in C^*-Hilbert bundles

For a definition of a C^*-Hilbert bundle, bundle atlae and differential structures of bundles, we refer to Mishchenko, Fomenko, [12], Schick [22] or Krýsl [31]. For definitions of the other notions used in this and in the next paragraph, see Solovyov, Troitsky [28]. Let us recall that for an A-pseudodifferential operator $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F})$ acting between smooth sections of A-Hilbert bundles $\mathcal{E}$ and $\mathcal{F}$ over a manifold M, we have the order $\mathrm{ord}(D) \in \mathbb{Z}$ of D and the symbol map $\sigma(D) : \pi^*(\mathcal{E}) \to \pi^*(\mathcal{F})$ of D at our disposal. Here, the map $\pi : T^*M \to M$ denotes the projection of the cotangent bundle. Moreover, if M is compact, then for A-Hilbert bundles $\mathcal{E} \to M$ and $\mathcal{F} \to M$, an A-pseudodifferential operator $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F})$, and an integer $t \in \mathbb{Z}$, we can form

(1) the so called Sobolev type completions $(W^t(M, \mathcal{E}), (,)_t)$ of $(\Gamma(M, \mathcal{E}), (,))$
(2) the adjoint $D^* : \Gamma(M, \mathcal{F}) \to \Gamma(M, \mathcal{E})$ of D and
(3) the continuous extensions $D_t : W^t(M, \mathcal{E}) \to W^{t-\mathrm{ord}(D)}(M, \mathcal{F})$ of D.

Smooth sections $(\Gamma(M, \mathcal{G}), (,))$ of an A-Hilbert bundle $\mathcal{G} \to M$ form a pre-Hilbert A-module and spaces $(W^t(\mathcal{G}), (,)_t)$ are Hilbert A-modules. See Example 2 item 4 for a definition of the inner product $(,)$. The adjoint D^* of an A-pseudodifferential operator D is considered with respect to the inner products $(,)$ on the pre-Hilbert A-modules of smooth sections of the appropriate bundles. Operators D and D^* are pre-Hilbert A-module morphisms, extensions D_t are adjointable Hilbert A-module morphisms, and the symbol map $\sigma(D)$ is a morphism of A-Hilbert bundles.

The definition of A-ellipticity we give below, is a straightforward generalization of the ellipticity of differential operators and differential complexes that act on bundles with finite dimensional fibers. The first part of the definition appears already in Solovyov, Troitsky [28].

Definition 7. Let $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F})$ be an A-pseudodifferential operator. We say that D is *A-elliptic* if $\sigma(D)(\xi, -) : \mathcal{E} \to \mathcal{F}$ is an isomorphism of A-Hilbert bundles for any non-zero $\xi \in T^*M$. Let $(p_i : \mathcal{E}^i \to M)_{i\in\mathbb{Z}}$ be a sequence of A-Hilbert bundles and $(\Gamma(M, \mathcal{E}^i), d_i : \Gamma(M, \mathcal{E}^i) \to \Gamma(M, \mathcal{E}^{i+1}))_{i\in\mathbb{Z}}$ be a complex of A-pseudodifferential operators. We say that $d^\bullet$ is *A-elliptic* if and only if the complex of symbol maps $(\mathcal{E}^i, \sigma(d_i)(\xi, -))_{i\in\mathbb{Z}}$ is exact for each non-zero $\xi \in T^*M$.

Remark 9. One can show that the Laplace operators $\Delta_i = d_{i-1}d_{i-1}^* + d_i^*d_i$, $i \in \mathbb{Z}$, of an A-elliptic complex are A-elliptic operators in the sense of Definition 7. For a proof in the C^*-case, see Lemma 9 in Krýsl [1]. Let us notice that the assumption on unitality of A is inessential in the proof of Lemma 9 in [1].

Recall that an A-Hilbert bundle $\mathcal{G} \to M$ is called *finitely generated projective* if its fibers are finitely generated and projective Hilbert A-modules. See Manuilov, Troitsky [9] for information on projective Hilbert A-modules. Let us recall a theorem of Fomenko and Mishchenko on a relation of the A-ellipticity and the A-Fredholm property.

Theorem 7. *Let A be a C^*-algebra, M a compact manifold, $\mathcal{E} \to M$ a finitely generated projective A-Hilbert bundle over M, and $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{E})$ an A-elliptic operator. Then the continuous extension*

$$D_t : W^t(M, \mathcal{E}) \to W^{t-\mathrm{ord}(D)}(M, \mathcal{E})$$

is an A-Fredholm morphism for any $t \in \mathbb{Z}$.

Proof. See Fomenko, Mishchenko [12] and Remark 7. □

Corollary 8. *Under the assumptions of Theorem 7, $\mathrm{Ker}\, D_t = \mathrm{Ker}\, D$ for any $t \in \mathbb{Z}$. If moreover D is self-adjoint, then also $\mathrm{Ker}\, {D_t}^* = \mathrm{Ker}\, D$ for any $t \in \mathbb{Z}$.*

Proof. See Theorem 7 in Krýsl [1] for the first claim, and the formula (5) in [1] for the second one. □

Let us notice that the first assertion in Corollary 8 appears as Theorem 2.1.145 on pp. 101 in Solovyov, Troitsky [28]. Now, we use Theorems 2, 6 and a part of Corollary 5 to derive the "main theorem" in which the Hodge theory for A-elliptic complexes is established for the case of algebras of compact operators, compact manifolds and finitely generated projective C^*-Hilbert bundles.

Theorem 9. *Let A be a C^*-algebra of compact operators, M be a compact manifold, $(p_i : \mathcal{E}^i \to M)_{i\in\mathbb{Z}}$ be a sequence of finitely generated projective A-Hilbert bundles over M and $d^\bullet = (\Gamma(M, \mathcal{E}^i), d_i : \Gamma(M, \mathcal{E}^i) \to \Gamma(M, \mathcal{E}^{i+1}))_{i\in\mathbb{Z}}$ be a complex of A-differential operators. If $d^\bullet$ is A-elliptic, then for each $i \in \mathbb{Z}$*

(1) *$d^\bullet$ is of Hodge type, i.e.,* $\Gamma(M, \mathcal{E}^i) = \mathrm{Ker}\, \Delta_i \oplus \mathrm{Im}\, d_i^* \oplus \mathrm{Im}\, d_{i-1}$.
(2) $\mathrm{Ker}\, d_i = \mathrm{Ker}\, \Delta_i \oplus \mathrm{Im}\, d_{i-1}$.
(3) $\mathrm{Ker}\, d_i^* = \mathrm{Ker}\, \Delta_{i+1} \oplus \mathrm{Im}\, d_{i+1}^*$.
(4) $\mathrm{Im}\, \Delta_i = \mathrm{Im}\, d_{i-1} \oplus \mathrm{Im}\, d_i^*$.
(5) *The cohomology group $H^i(d^\bullet)$ is a finitely generated projective Hilbert A-module that is isomorphic to the A-Hilbert module* $\mathrm{Ker}\, \Delta_i$.

Proof. Since $d^\bullet$ is an A-elliptic complex, the associated Laplace operators are A-elliptic (Remark 9) and self-adjoint. According to Theorem 7, the extensions $(\Delta_i)_t$ are A-Fredholm for any $t \in \mathbb{Z}$.

Let us set $D = \Delta_i$, $U = \Gamma(M, \mathcal{E}^i)$, $V = W^{\mathrm{ord}(\Delta_i)}(M, \mathcal{E}^i)$, and $W = W^0(M, \mathcal{E}^i)$ considered with the appropriate inner products. Then U is a vector subspace of $V \cap W$. Since Δ_i is A-elliptic, $\mathrm{Ker}\,\Delta_i = \mathrm{Ker}\,(\Delta_i)_t = \mathrm{Ker}\,(\Delta_i)_t^*$ due to Corollary 8. Because the operator D is of finite order, $\widetilde{D}^{-1}(\Gamma(M, \mathcal{E}^i))$, $\widetilde{D}^{*-1}(\Gamma(M, \mathcal{E}^i)) \subseteq \Gamma(M, \mathcal{E}^i)$. The norm on $U = \Gamma(M, \mathcal{E}^i)$ coincides with the norm on $W = W^0(M, \mathcal{E}^i)$ restricted to U and the norm $|\,|_U$ on U is dominated by the norm $|\,|_V$ on $V = W^{\mathrm{ord}(D)}(M, \mathcal{E}^i)$ restricted to U. Thus, the assumptions of Theorem 6 are satisfied and we may conclude, that Δ_i is a self-adjoint parametrix possessing morphism, and thus, $d^\bullet$ is self-adjoint parametrix possessing according to the definition. The assertions in items 1–4 follow from the corresponding assertions of Theorem 2.

Using Theorem 2 item 5, $H^i(d^\bullet) \simeq \mathrm{Ker}\,\Delta_i$ as pre-Hilbert modules. Let us notice again that the norm on $W^0(M, \mathcal{E}^i)$ restricted to $\Gamma(M, \mathcal{E}^i)$ coincides with the norm on $\Gamma(M, \mathcal{E}^i)$. Since $\mathrm{Ker}\,\Delta_i = \mathrm{Ker}\,(\Delta_i)_0$ (Corollary 8) and the latter space is a Hilbert A-module, the cohomology group is a Hilbert A-module as well and the isomorphism is a Hilbert A-module isomorphism.

Because $(\Delta_i)_0$ is A-Fredholm and A is a C^*-algebra of compact operators, $\dim_A \mathrm{Ker}(\Delta_i)_0 < \infty$ by Corollary 5. It follows by Corollary 8 that the kernel of Δ_i is finitely generated. Hence, the cohomology group $H^i(d^\bullet)$ is finitely generated as well.

Since the image of $(\Delta_i)_0$ is closed (Corollary 5), we have $W^0(M, \mathcal{E}^i) = \mathrm{Ker}\,(\Delta_i)_0 \oplus \mathrm{Im}\,(\Delta_i)_0^*$ and the image $(\Delta_i)_0^*$ is closed both due to the Mishchenko theorem (Theorem 3.2 in Lance [8]). Consequently, $\mathrm{Ker}\,\Delta_i = \mathrm{Ker}\,(\Delta_i)_0$ is a projective A-Hilbert module by Theorems 3.1 and 1.3 in Fomenko, Mishchenko [12]. □

Remark 10. Let us notice that if the assumptions of Theorem 9 are satisfied, the cohomology groups share the properties of the fibers in the sense that they are finitely generated projective A-Hilbert modules.

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