

Structure of the curvature tensor on symplectic spinors

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Abstract

We study symplectic manifolds (M^{2l}, ω) equipped with a symplectic torsion-free affine (also called Fedosov) connection ∇ and admitting a metaplectic structure. Let \mathcal{S} be the so called symplectic spinor bundle and let R^S be the curvature tensor field of the symplectic spinor covariant derivative ∇^S associated to the Fedosov connection ∇ . It is known that the space of symplectic spinor valued exterior differential 2-forms, $\Gamma(M, \wedge^2 T^*M \otimes \mathcal{S})$, decomposes into three invariant spaces with respect to the structure group, which is the metaplectic group $Mp(2l, \mathbb{R})$ in this case. For a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, we compute explicitly the projections of $R^S \phi \in \Gamma(M, \wedge^2 T^*M \otimes \mathcal{S})$ onto the three mentioned invariant spaces in terms of the symplectic Ricci and symplectic Weyl curvature tensor fields of the connection ∇ . Using this decomposition, we derive a complex of first order differential operators provided the Weyl tensor of the Fedosov connection is trivial.

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1 Introduction

In the paper, we shall study the action of the symplectic curvature tensor field on symplectic spinors over a symplectic manifold (M^{2l}, ω) with a given metaplectic structure and equipped with a symplectic torsion-free affine connection ∇ . Such connections are usually called Fedosov connections. It is well known that in the case of $l > 1$, the curvature tensor field of the connection ∇ decomposes into two parts, namely into the symplectic Weyl and the symplectic Ricci curvature

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tensor field. In the case $l = 1$, only the symplectic Ricci curvature tensor field appears. See Vaisman [17] for details.

Now, let us say a few words about the metaplectic structure. In the symplectic case, there exists (in a parallel to the Riemannian case) a non-trivial two-fold covering of the symplectic group $Sp(2l, \mathbb{R})$, the so called metaplectic group. We shall denote it by $Mp(2l, \mathbb{R})$. A metaplectic structure on a symplectic manifold (M^{2l}, ω) is a notion parallel to a spin structure on a Riemannian manifold. For a symplectic manifold admitting a metaplectic structure, one can construct the so called *symplectic spinor bundle* \mathcal{S} , introduced by Bertram Kostant in 1974. The symplectic spinor bundle \mathcal{S} is the vector bundle associated to the metaplectic structure on M via the so called Segal-Shale-Weil representation of the metaplectic group $Mp(2l, \mathbb{R})$. See Kostant [11] for details.

The Segal-Shale-Weil representation is an infinite dimensional unitary representation of the metaplectic group $Mp(2l, \mathbb{R})$ on the space of all complex valued square Lebesgue integrable functions $\mathbf{L}^2(\mathbb{R}^l)$. Because of the infinite dimension, the Segal-Shale-Weil representation is not so is to handle. It is known, see, e.g., Kashiwara, Vergne [10], that the underlying Harish-Chandra module of this representation is equivalent to the space $\mathbb{C}[x^1, \dots, x^l]$ of polynomials in l variables, on which the Lie algebra $\mathfrak{sp}(2l, \mathbb{C})$ acts via the so called Chevalley homomorphism,¹ see Britten, Hooper, Lemire [1]. Thus, the infinitesimal structure of the Segal-Shale-Weil representation can be viewed as the complexified *symmetric algebra* $(\bigoplus_{i=0}^{\infty} \odot^i \mathbb{R}^l) \otimes_{\mathbb{R}} \mathbb{C}$ of the Lagrangian subspace \mathbb{R}^l of the canonical symplectic vector space $\mathbb{R}^{2l} \simeq \mathbb{R}^l \oplus \mathbb{R}^l$. This shows that the situation is completely parallel to the complex orthogonal case and the spinor representation, which can be realized as the *exterior algebra* of a maximal isotropic subspace. An interested reader is referred to Weil [19], Kashiwara, Vergne [10] and also to Britten, Hooper, Lemire [1] for more details. For some technical reasons, we shall be using the so called minimal globalization of the Harish-Chandra (\mathfrak{g}, K) -module of the Segal-Shale-Weil representation, which we will call *metaplectic representation* and denote it by \mathbf{S} (the elements of \mathbf{S} will be called *symplectic spinors*). This representation, as well as the Segal-Shale-Weil one, decomposes into two irreducible subrepresentations \mathbf{S}_+ and \mathbf{S}_- .

For any symplectic connection ∇ on a symplectic manifold (M, ω) admitting a metaplectic structure, we can form the associated covariant derivative ∇^S acting on the sections of the symplectic spinor bundle \mathcal{S} . The curvature tensor field $R^S : \Gamma(M, \mathcal{S}) \rightarrow \Gamma(M, \bigwedge^2 TM^* \otimes \mathcal{S})$ of the associated covariant derivative ∇^S is defined by the classical formula. The tensor field R^S decomposes also into two parts, one of which is dependent only on the symplectic Ricci and the second one on the symplectic Weyl tensor. It is known (cf. Krýsl [12]) that the space $\bigwedge^2 \mathbb{R}^{2l} \otimes \mathbf{S}_{\pm}$ decomposes into three irreducible summands wr. to the natural action of $Mp(2l, \mathbb{R})$ on this space. We shall briefly describe this result in the paper. Let us denote the mentioned three summands of the decomposition of $\bigwedge^2 \mathbb{R}^{2l} \otimes \mathbf{S}_{\pm}$ by \mathbf{E}_{\pm}^{20} , \mathbf{E}_{\pm}^{21} and \mathbf{E}_{\pm}^{22} and the corresponding vector bundles

¹The Chevalley homomorphism realizes the complex symplectic Lie algebra as a Lie subalgebra of the algebra of polynomial coefficients differential operators acting on $\mathbb{C}[x^1, \dots, x^l]$.

associated to the chosen metaplectic structure via these modules by \mathcal{E}_\pm^{20} , \mathcal{E}_\pm^{21} and \mathcal{E}_\pm^{22} , respectively. We define $\mathcal{E}^{2j} := \mathcal{E}_+^{2j} \oplus \mathcal{E}_-^{2j}$ for $j = 0, 1, 2$.

In the paper, we shall prove that the symplectic Ricci tensor field maps a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$ into $\Gamma(M, \mathcal{E}^{20} \oplus \mathcal{E}^{21})$ and the symplectic Weyl tensor field maps a symplectic spinor field into $\Gamma(M, \mathcal{E}^{21} \oplus \mathcal{E}^{22})$. For an arbitrary symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, the projections of $R^S \phi$ to the invariant spaces $\Gamma(M, \mathcal{E}^{2j})$ ($j = 0, 1, 2$) are explicitly computed. This describes a structure of the action of the curvature tensor field R^S on the space of symplectic spinor fields in terms of the invariant parts of the underlying connection. In what follows, this result will be called the decomposition result.

The result described above seems to be rather abstract or technical. But actually, knowing the decomposition of $R^S \phi$ makes it possible to derive several conclusions for certain invariant differential operators, which are defined with help of the Fedosov connection.

This is the case of the application we shall mention. Let us briefly describe its context. In 1994, K. Habermann introduced a symplectic analogue of the Dirac operator known from Riemannian geometry, the so called symplectic Dirac operator. The symplectic Dirac operator was introduced with the help of the so called symplectic Clifford multiplication, see Habermann [7]. It is possible to define the same operator (up to a complex scalar multiple) using the de Rham sequence tensored (twisted) by symplectic spinor fields as one usually does in the Riemannian spin geometry to get a definition of the Dirac, twistor and Rarita-Schwinger operator and their further higher spin analogues.

Under the assumption the symplectic Weyl tensor W of the Fedosov connection is trivial, there exists a complex consisting of two differential operators T_0 and T_1 . These operators will also be called symplectic twistor operators and they will be defined using the de Rham sequence tensored by symplectic spinor fields. One of the advantage of the decomposition result is a complete avoidance of possibly lengthy computations in coordinates when proving that T_0 and T_1 form a complex (provided $W = 0$). One can say that the coordinate computations were absorbed into the proof of the decomposition result. Though finding the complex seems to be a rather particular result, there is a strong hope of deriving a longer complex under the same assumption.

The reader interested in applications of this theory in physics is referred to Green, Hull [3], where the symplectic spinors are used in the context of 10 dimensional super string theory. In Reuter [15], symplectic spinors are used in the theory of the so called Dirac-Kähler fields.

In the second section, some basic facts on the symplectic spinor representation and higher symplectic spinors are recalled. In the section 3, basic properties of torsion-free symplectic (Fedosov) connections and their curvature tensor field are mentioned. In the fourth section, the action of the curvature tensor field R^S of the associated symplectic spinor covariant derivative ∇^S acting on the space of symplectic spinor fields described (Corollary 11). In this section, the mentioned complex of the two symplectic twistor operators is presented (Theorem 12).

2 Metaplectic representation, higher symplectic spinors and basic notation

We start with a summary of notions from representation theory, which we shall need in this paper. From the point of view of this article, the notions are rather of a technical character. Let G be a reductive Lie group in the sense of Vogan (see Vogan [18]), \mathfrak{g} be the Lie algebra of G and K be a maximal compact subgroup of G . Typical examples of reductive groups are finite covers of semisimple Lie subgroups of a general linear group of a finite dimensional vector space. Let $\mathcal{R}(G)$ be the category the object of which are complete, locally convex, Hausdorff topological spaces with continuous linear G -action, such that the resulting representation is admissible and of finite length; the morphisms are continuous G -equivariant linear maps between the objects. Let $\mathcal{HC}(\mathfrak{g}, K)$ be the category of Harish-Chandra (\mathfrak{g}, K) -modules and let us consider the forgetful Harish-Chandra functor $HC : \mathcal{R}(G) \rightarrow \mathcal{HC}(\mathfrak{g}, K)$. It is well known that there exists an adjoint functor $mg : \mathcal{HC}(\mathfrak{g}, K) \rightarrow \mathcal{R}(G)$ to the Harish-Chandra functor HC . This functor is usually called the minimal globalization functor and its existence is a deep result in representation theory. For details and for the existence of the minimal globalization functor mg , see Kashiwara, Schmid [9] and/or Vogan [18].

For a representation $\mathbf{E} \in \mathcal{R}(G)$ of G , we shall denote the corresponding Harish-Chandra (\mathfrak{g}, K) -module $HC(\mathbf{E})$ by E . When we will only be considering its \mathfrak{g} -module structure, we shall use the symbol \mathbb{E} for it.

Now, suppose K is connected and two complex modules $E, F \in \mathcal{HC}(\mathfrak{g}, K)$ are given such that both \mathbb{E} and \mathbb{F} are irreducible highest weight $\mathfrak{g}^{\mathbb{C}}$ -modules. Because mg is an adjoint functor to the functor HC , we have $\text{Hom}_G(mg(E), mg(F)) \simeq \text{Hom}_{(\mathfrak{g}, K)}(E, F)$. It is well known that the category of (\mathfrak{g}, K) -modules is a full subcategory of the category of \mathfrak{g} -modules provided K is connected. Due to that, we have $\text{Hom}_{(\mathfrak{g}, K)}(E, F) \simeq \text{Hom}_{\mathfrak{g}}(\mathbb{E}, \mathbb{F})$. Because \mathbb{E} and \mathbb{F} are complex irreducible highest weight modules over $\mathfrak{g}^{\mathbb{C}}$, the Dixmier's version of the Schur lemma implies $\dim \text{Hom}_{\mathfrak{g}}(\mathbb{E}, \mathbb{F}) = 1$ iff $\mathbb{E} \simeq \mathbb{F}$ (see Dixmier [2], Theorem 2.6.5 and Theorem 2.6.6). Summing up, we have $\dim \text{Hom}_G(mg(E), mg(F)) = 1$ iff $\mathbb{E} \simeq \mathbb{F}$. For brevity, we will refer to this simple statement as to the *globalized Schur lemma*.

Further, if $(p : \mathcal{G} \rightarrow M, G)$ is a principal G -bundle, we shall denote the vector bundle associated to this principal bundle via a representation $\sigma : G \rightarrow \text{Aut}(\mathbf{W})$ of G on \mathbf{W} by \mathcal{W} , i.e., $\mathcal{W} = \mathcal{G} \times_{\sigma} \mathbf{W}$. Let us also mention that we shall often use the Einstein summation convention for repeated indices (lower and upper) without mentioning it explicitly.

Now, we shall focus our attention to the studied case, i.e., to the symplectic one. To fix a notation, let us recall some notions from the symplectic linear algebra. Let us consider a real symplectic vector space (\mathbb{V}, ω_0) of dimension $2l$, i.e., \mathbb{V} is a $2l$ dimensional real vector space and ω_0 is a non-degenerate antisymmetric bilinear form on \mathbb{V} . Let us choose two Lagrangian subspaces² $\mathbb{L}, \mathbb{L}' \subseteq \mathbb{V}$

²maximal isotropic wr. to ω_0

such that $\mathbb{L} \oplus \mathbb{L}' = \mathbb{V}$. It follows that $\dim(\mathbb{L}) = \dim(\mathbb{L}') = l$. Throughout this article, we shall use a symplectic basis $\{e_i\}_{i=1}^{2l}$ of \mathbb{V} chosen in such a way that $\{e_i\}_{i=1}^l$ and $\{e_i\}_{i=l+1}^{2l}$ are respective bases of \mathbb{L} and \mathbb{L}' . Because the definition of a symplectic basis is not unique, let us fix one which shall be used in this text. A basis $\{e_i\}_{i=1}^{2l}$ of \mathbb{V} is called symplectic basis of (\mathbb{V}, ω_0) if $\omega_{ij} := \omega_0(e_i, e_j)$ satisfies $\omega_{ij} = 1$ if and only if $i \leq l$ and $j = i+l$; $\omega_{ij} = -1$ if and only if $i > l$ and $j = i-l$ and finally, $\omega_{ij} = 0$ in other cases. Let $\{\epsilon^i\}_{i=1}^{2l}$ be the basis of \mathbb{V}^* dual to the basis $\{e_i\}_{i=1}^{2l}$. For $i, j = 1, \dots, 2l$, we define ω^{ij} by $\sum_{k=1}^{2l} \omega_{ik} \omega^{jk} = \delta_i^j$, for $i, j = 1, \dots, 2l$. Notice that not only $\omega_{ij} = -\omega_{ji}$, but also $\omega^{ij} = -\omega^{ji}$, $i, j = 1, \dots, 2l$.

Let us denote the symplectic group of (\mathbb{V}, ω_0) by G , i.e., $G := Sp(\mathbb{V}, \omega_0) \simeq Sp(2l, \mathbb{R})$. Because the maximal compact subgroup K of G is isomorphic to the unitary group $K \simeq U(l)$ which is of homotopy type \mathbb{Z} , there exists a nontrivial two-fold covering \tilde{G} of G . See, e.g., Habermann, Habermann [8] for details. This two-fold covering is called metaplectic group of (\mathbb{V}, ω_0) and it is denoted by $Mp(\mathbb{V}, \omega_0)$. Let us remark that $Mp(\mathbb{V}, \omega_0)$ is reductive in the sense of Vogan. In the considered case, we have $\tilde{G} \simeq Mp(2l, \mathbb{R})$. For a later use, let us reserve the symbol λ for the mentioned covering. Thus $\lambda : \tilde{G} \rightarrow G$ is a fixed member of the isomorphism class of all nontrivial $2 : 1$ coverings of G . Because $\lambda : \tilde{G} \rightarrow G$ is a homomorphism of Lie groups and G is a subgroup of the general linear group $GL(\mathbb{V})$ of \mathbb{V} , the mapping λ is also a representation of the metaplectic group \tilde{G} on the vector space \mathbb{V} . Let us define $\tilde{K} := \lambda^{-1}(K)$. Then \tilde{K} is a maximal compact subgroup of \tilde{G} . One can easily see that $\tilde{K} \simeq U(l) := \{(g, z) \in U(l) \times \mathbb{C}^\times \mid \det(g) = z^2\}$ and thus, \tilde{K} is connected. The Lie algebra of \tilde{G} is isomorphic to the Lie algebra \mathfrak{g} of G and we will identify them. One has $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}, \omega_0) \simeq \mathfrak{sp}(2l, \mathbb{R})$.

From now on, we shall restrict ourselves to the case $l \geq 2$ without mentioning it explicitly. The case $l = 1$ should be handled separately (though analogously) because the shape of the root system of $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R})$ is different from that one of the root system of $\mathfrak{sp}(2l, \mathbb{R})$ for $l > 1$. As usual, we shall denote the complexification of \mathfrak{g} by $\mathfrak{g}^\mathbb{C}$. Obviously, $\mathfrak{g}^\mathbb{C} \simeq \mathfrak{sp}(2l, \mathbb{C})$. Let us choose a Cartan subalgebra $\mathfrak{h}^\mathbb{C}$ of $\mathfrak{g}^\mathbb{C}$ and an ordering on the set of roots of $(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})$. If \mathbb{E} is an irreducible highest weight $\mathfrak{g}^\mathbb{C}$ -module with a highest weight λ , we shall denote it by the symbol $L(\lambda)$. Let us denote the fundamental weight basis of $\mathfrak{g}^\mathbb{C}$ wr. to the above choices by $\{\varpi_i\}_{i=1}^l$.

2.1 Metaplectic representation and symplectic spinors

There exists a distinguished infinite dimensional unitary representation of the metaplectic group \tilde{G} which does not descend to a representation of the symplectic group G . This representation, called *Segal-Shale-Weil*,³ plays a fundamental role in geometric quantization of Hamiltonian mechanics, see, e.g., Woodhouse [20], and in the theory of modular forms and theta correspondence, see, e.g.,

³The names oscillator or metaplectic representation are also used in the literature. We shall use the name Segal-Shale-Weil in this text, and reserve the name metaplectic for certain representation arising from the Segal-Shale-Weil one.

Howe [6]. We shall not give a definition of this representation here and refer the interested reader to Weil [19] or Habermann, Habermann [8].

The Segal-Shale-Weil representation, which we shall denote by U here, is a complex infinite dimensional unitary representation of \tilde{G} on the space of complex valued square Lebesgue integrable functions defined on the Lagrangian subspace \mathbb{L} , i.e.,

$$U : \tilde{G} \rightarrow \mathcal{U}(\mathbf{L}^2(\mathbb{L})),$$

where $\mathcal{U}(\mathbf{W})$ denotes the group of unitary operators on a Hilbert space \mathbf{W} . In order to be precise, let us refer to the space $\mathbf{L}^2(\mathbb{L})$ as to the Segal-Shale-Weil module. It is known that the Segal-Shale-Weil module belongs to the category $\mathcal{R}(\tilde{G})$. (See Kashiwara, Vergne [10] for details and Segal-Shale-Weil representation in general.) It is easy to see that this representation splits into two irreducible modules $\mathbf{L}^2(\mathbb{L}) \simeq \mathbf{L}^2(\mathbb{L})_+ \oplus \mathbf{L}^2(\mathbb{L})_-$. The first module consists of even and the second one of odd complex valued square Lebesgue integrable functions on the Lagrangian subspace \mathbb{L} . Let us remark that a typical construction of the Segal-Shale-Weil representation is based on the so called Schrödinger representation of the Heisenberg group of $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega_0)$ and a use of the Stone-von Neumann theorem.

For technical reasons, we shall need the minimal globalization of the underlying $(\mathfrak{g}, \tilde{K})$ -module $HC(\mathbf{L}^2(\mathbb{L}))$ of the introduced Segal-Shale-Weil module. We shall call this minimal globalization *metaplectic representation* and denote it by *meta*, i.e.,

$$meta : \tilde{G} \rightarrow \text{Aut}(mg(HC(\mathbf{L}^2(\mathbb{L})))),$$

where mg is the minimal globalization functor (see this section and the references therein). For our convenience, let us denote the module $mg(HC(\mathbf{L}^2(\mathbb{L})))$ by \mathbf{S} . Similarly we define \mathbf{S}_+ and \mathbf{S}_- to be the minimal globalizations of the underlying Harish-Chandra modules of the modules $\mathbf{L}^2(\mathbb{L})_+$ and $\mathbf{L}^2(\mathbb{L})_-$ introduced above. Accordingly to $\mathbf{L}^2(\mathbb{L}) \simeq \mathbf{L}^2(\mathbb{L})_+ \oplus \mathbf{L}^2(\mathbb{L})_-$, we have $\mathbf{S} \simeq \mathbf{S}_+ \oplus \mathbf{S}_-$. We shall call the $Mp(\mathbb{V}, \omega)$ -module \mathbf{S} the symplectic spinor module and its elements *symplectic spinors*. For the name "spinor", see Kostant [11] or the Introduction.

Further notion related to the symplectic vector space $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega_0)$ is the so called symplectic Clifford multiplication of elements of \mathbf{S} by vectors from \mathbb{V} . For a symplectic spinor $f \in \mathbf{S}$, we define

$$(e_i \cdot f)(x) := \iota x^i f(x),^4$$

$$(e_{i+l} \cdot f)(x) := \frac{\partial f}{\partial x^i}(x), x = \sum_{i=1}^l x^i e_i \in \mathbb{L}, i = 1, \dots, l.$$

Extending this multiplication \mathbb{R} -linearly, we get the mentioned symplectic Clifford multiplication. Let us mention that the multiplication and the differentiation make sense for any $f \in \mathbf{S}$ because of the interpretation of the minimal

⁴The symbol ι denotes the imaginary unit, $\iota = \sqrt{-1}$.

globalization. See Vogan [18] for details. Let us remark that in the physical literature, the symplectic Clifford multiplication is usually called the Schrödinger quantization prescription.

The following lemma is an easy consequence of the definition of the symplectic Clifford multiplication.

Lemma 1: For $v, w \in \mathbb{V}$ and $s \in \mathbf{S}$, we have

$$v.w.s - w.v.s = -i\omega_0(v, w)s.$$

Proof. See Habermann, Habermann [8], pp. 11. \square

Sometimes, we shall write $v.w.s$ instead of $v.(w.s)$ for $v, w \in \mathbb{V}$ and a symplectic spinor $s \in \mathbf{S}$ and similarly for higher number of multiplying elements. Instead of $e_i.e_j.s$, we shall write $e_{ij}.s$ simply and similarly for expressions with higher number of multiplying elements, e.g., $e_{ijk}.s$ abbreviates $e_i.e_j.e_k.s$.

2.2 Higher symplectic spinors

In this subsection, we shall present a result on a decomposition of the tensor product of the metaplectic representation with the first and with the second wedge power of the representation $\lambda^* : \tilde{G} \rightarrow GL(\mathbb{V}^*)$ of \tilde{G} (dual to the representation λ) into irreducible summands. Let us reserve the symbol ρ for the mentioned tensor product representation of \tilde{G} , i.e.,

$$\rho : \tilde{G} \rightarrow \text{Aut}\left(\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}\right)$$

and

$$\rho(g)(\alpha \otimes s) := \lambda(g)^{* \wedge r} \alpha \otimes \text{meta}(g)s$$

for $r \in \{0, \dots, 2l\}$, $g \in \tilde{G}$, $\alpha \in \bigwedge^r \mathbb{V}^*$, $s \in \mathbf{S}$ and extend it linearly. For definiteness, let us equip the tensor product $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ with the so called Grothendieck tensor product topology. See Vogan [18] and Treves [16] for details on this topological structure. In a parallel to the Riemannian case, we shall call the elements of $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ *higher symplectic spinors*.

In the next theorem, the modules of the exterior 1-forms and 2-forms with values in the module \mathbf{S} of symplectic spinors are decomposed into irreducible summands.

Theorem 2: For $\frac{1}{2}\dim(\mathbb{V}) =: l > 2$, the following isomorphisms

$$\mathbb{V}^* \otimes \mathbf{S} \simeq \mathbf{E}_{\pm}^{10} \oplus \mathbf{E}_{\pm}^{11} \quad \text{and}$$

$$\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}_{\pm} \simeq \mathbf{E}_{\pm}^{20} \oplus \mathbf{E}_{\pm}^{21} \oplus \mathbf{E}_{\pm}^{22}$$

hold. For $j_1 = 0, 1$ and $j_2 = 0, 1, 2$ the modules $\mathbf{E}_{\pm}^{1j_1}$ and $\mathbf{E}_{\pm}^{2j_2}$ are uniquely determined by the conditions that first, they are submodules of the corresponding

tensor products and second,

$$\begin{aligned}\mathbb{E}_+^{10} &\simeq \mathbb{E}_-^{20} \simeq \mathbb{S}_- \simeq L(\varpi_{l-1} - \frac{3}{2}\varpi_l), \quad \mathbb{E}_-^{10} \simeq \mathbb{E}_+^{20} \simeq \mathbb{S}_+ \simeq L(-\frac{1}{2}\varpi_l), \\ \mathbb{E}_+^{11} &\simeq \mathbb{E}_-^{21} \simeq L(\varpi_1 - \frac{1}{2}\varpi_l), \quad \mathbb{E}_-^{11} \simeq \mathbb{E}_+^{21} \simeq L(\varpi_1 + \varpi_{l-1} - \frac{3}{2}\varpi_l), \\ \mathbb{E}_+^{22} &\simeq L(\varpi_2 - \frac{1}{2}\varpi_l) \quad \text{and} \quad \mathbb{E}_-^{22} \simeq L(\varpi_2 + \varpi_{l-1} - \frac{3}{2}\varpi_l).\end{aligned}$$

Proof. See Krýsl [13] or Krýsl [14]. \square

Remark: In this paper, the multiplicity freeness of the previous two decompositions will be used substantially. One can show that the decompositions are multiplicity-free also in the case $l = 2$. (One only needs to modify the prescription for the highest weights of the summands in the decompositions. See Krýsl [14] for this case). Let us also mention, that the Theorem 2 is a simple consequence of a theorem of Britten, Hooper, Lemire [1].

Let us set $\mathbf{E}^{ij_i} := \mathbf{E}_+^{ij_i} \oplus \mathbf{E}_-^{ij_i}$, for $i = 1, 2$, $j_1 = 0, 1$ and $j_2 = 0, 1, 2$. For the mentioned i, j_i , let us consider the projections $p^{ij_i} : \bigwedge^i \mathbb{V}^* \otimes \mathbf{S} \rightarrow \mathbf{E}_{ij_i}$. The definition is correct because of the multiplicity freeness of the decomposition of the appropriate tensor products. In the paper, we shall need explicit formulas for these projections. In order to find them, let us introduce the following mappings.

For $r = 0, \dots, 2l$ and $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, we set

$$\begin{aligned}X &: \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \rightarrow \bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S}, \quad X(\alpha \otimes s) := - \sum_{i=1}^{2l} \epsilon^i \wedge \alpha \otimes e_i \cdot s; \\ Y &: \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \rightarrow \bigwedge^{r-1} \mathbb{V}^* \otimes \mathbf{S}, \quad Y(\alpha \otimes s) := \sum_{i=1}^{2l} \omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot s \quad \text{and} \\ H &: \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \rightarrow \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}, \quad H := \{X, Y\} = XY + YX.\end{aligned}$$

In order to be able to use these operators in a geometric setting, we shall need the following lemma.

Lemma 3: The homomorphisms X, Y, H are \tilde{G} -equivariant with respect to the representation ρ of \tilde{G} .

Proof. This can be verified by a direct computation. See Krýsl [13] or Krýsl [14] for a proof. \square

In the next lemma, the values of H on homogeneous components of $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ are computed.

Lemma 4: Let (\mathbb{V}, ω_0) be a $2l$ dimensional symplectic vector space. Then for $r = 0, \dots, 2l$, we have

$$H|_{\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}} = \iota(r-l)\text{Id}|_{\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}}.$$

Proof. This can be verified by a direct computation as well. See Krýsl [13] or Krýsl [14] for a proof. \square

In the next lemma, the projections p^{2j} , $j = 0, 1, 2$, are computed explicitly with help of the operators X and Y .

Lemma 5: For $l > 1$, the following equalities hold on $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}$.

$$p^{20} = \frac{1}{l} X^2 Y^2, \quad (1)$$

$$p^{21} = \frac{l}{1-l} (XY - \frac{l}{l} X^2 Y^2) \text{ and} \quad (2)$$

$$p^{22} = \text{Id}_{\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}} - \frac{l}{1-l} XY + \frac{1}{1-l} X^2 Y^2. \quad (3)$$

Proof.

1. From the definition of Y , the fact that it is \tilde{G} -equivariant (Lemma 3) and the Theorem 2, we know that Y^2 maps $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}_{\pm}$ into \mathbf{S}_{\pm} . Because X^2 is \tilde{G} -equivariant (Lemma 3), it maps \mathbf{S}_{\pm} into a submodule of $\bigwedge^2 \mathbb{V}^* \otimes (\mathbf{S}_+ \oplus \mathbf{S}_-)$ which is a (possibly empty) direct sum of submodules isomorphic to \mathbf{S}_{\pm} . Regarding the multiplicity-free decomposition structure of $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}_{\pm}$ (Theorem 2), we see that $p' := X^2 Y^2$ maps $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}_{\pm}$ into \mathbf{E}_{\pm}^{20} . Computing the value of p' on the element $\psi := \omega_{ij} \epsilon^i \wedge \epsilon^j \otimes s$ for an $s \in \mathbf{S}$, we find that $p' \psi = l \psi$. Using the globalized Schur lemma (see the section 2), we have $p^{20} = \frac{1}{l} X^2 Y^2$.
2. As in the 1st item, it is easy to see that $p'' := XY(\text{Id}_{\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}} - \frac{1}{l} X^2 Y^2)$ maps $\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}$ into \mathbf{E}^{21} . Let us consider a symmetric 2-vector $\sigma \in \odot^2 \mathbb{V}$ and denote its (i, j) -th component wr. to the basis $\{e_i\}_{i=1}^{2l}$ by σ^{ij} . Computing the value $p'' \psi$ for $\psi := \sigma^{ij} \epsilon^k \wedge \epsilon^l \omega_{jl} \otimes e_{ik} \cdot s$, $s \in \mathbf{S}$, we get $p'' \psi = l(l-1) \psi$. Using the globalized Schur lemma again, we have $p^{21} = \frac{l}{1-l} p''$. Using the defining identity $H = XY + YX$ and the Lemma 4, we get the formula for p^{21} written in the statement of the lemma.
3. The third equation follows from the fact $p^{20} + p^{21} + p^{22} = \text{Id}_{\bigwedge^2 \mathbb{V}^* \otimes \mathbf{S}}$ and the preceding two items.

□

3 Symplectic curvature tensor field

After we have finished the algebraic part of this paper, we shall recall some results of Vaisman in [17] and of Gelfand, Retakh and Shubin [4]. Let (M, ω) be a symplectic manifold and ∇ be a symplectic torsion-free affine connection. Such connections are also called Fedosov connections and were used, e.g., in the so called Fedosov quantization. See Fedosov [5] for this use. Let us remark, that there is no uniqueness result for Fedosov connections, which one has in the case of Riemannian manifolds and Riemannian connections. By symplectic and torsion-free, we mean $\nabla \omega = 0$ and $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$ for

all $X, Y \in \mathfrak{X}(M)$, respectively. The triple (M, ω, ∇) will be called a *Fedosov manifold*.

To fix our notation, let us recall the classical definition of the curvature tensor R^∇ of the connection ∇ , we shall be using here. Let

$$R^\nabla(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$$

for $X, Y, Z \in \mathfrak{X}(M)$. Let us choose a local symplectic frame $\{e_i\}_{i=1}^{2l}$ on a fixed open subset $U \subseteq M$. We shall use the following convention. For $i, j, k, l = 1, \dots, 2l$, we set

$$R_{ijkl} := \omega(R^\nabla(e_k, e_l)e_j, e_i). \quad (4)$$

Let us remark that the convention is different from that one used in Habermann, Habermann [8]. We shall often write expressions in which indices i, j, k or l e.t.c. occur. We will implicitly mean i, j, k or l are running from 1 to the dimension of the manifold M without mentioning it explicitly.

Obviously, one has

$$R_{ijkl} = -R_{ijlk} \text{ and} \quad (5)$$

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \text{ (1}^{st} \text{ Bianchi identity)}. \quad (6)$$

One can also prove the identity

$$R_{ijkl} = R_{jikl}. \quad (7)$$

See Gelfand, Retakh, Shubin [4] for the proof.

For a symplectic manifold with a Fedosov connection, one has also the following simple consequence of the 1st Bianchi identity:

$$R_{ijkl} + R_{jkli} + R_{klji} + R_{lijk} = 0 \text{ (extended 1}^{st} \text{ Bianchi identity)}. \quad (8)$$

From the symplectic curvature tensor field R^∇ , we can build the symplectic Ricci curvature tensor field σ^∇ defined by the classical formula

$$\sigma^\nabla(X, Y) := \text{Tr}(V \mapsto R^\nabla(V, X)Y)$$

for each $X, Y \in \mathfrak{X}(M)$ (the variable V denotes a vector field on M). For the chosen frame and $i, j = 1, \dots, 2l$, we define

$$\sigma_{ij} := \sigma^\nabla(e_i, e_j).$$

Further, let us define

$$\begin{aligned} \tilde{\sigma}_{ijkl}^\nabla &:= \frac{1}{2(l+1)}(\omega_{il}\sigma_{jk} - \omega_{ik}\sigma_{jl} + \omega_{jl}\sigma_{ik} - \omega_{jk}\sigma_{il} + 2\sigma_{ij}\omega_{kl}), \\ \tilde{\sigma}^\nabla(X, Y, Z, V) &:= \tilde{\sigma}_{ijkl}X^iY^jZ^kV^l \text{ and} \\ W^\nabla &:= R^\nabla - \tilde{\sigma}^\nabla. \end{aligned}$$

for local vector fields $X = X^i e_i$, $Y = Y^j e_j$, $Z = Z^k e_k$ and $V = V^l e_l$. We will call the tensor field W^∇ the symplectic Weyl curvature tensor field. These tensor fields were already introduced in Vaisman [17]. We shall often drop the index ∇ in the previous expressions. Thus we shall often write W , σ and $\tilde{\sigma}$ instead of W^∇ , σ^∇ and $\tilde{\sigma}^\nabla$, respectively.

In the next lemma, a symmetry of σ and an equivalent definition of σ are stated.

Lemma 6: The symplectic Ricci curvature tensor field σ is symmetric and

$$R^{ijkl} \omega_{kl} = 2\sigma^{ij}.$$

Proof. The proof follows from the definition of the symplectic Ricci curvature tensor field and the equation (7). See Vaisman [17] for a proof. \square

Remark: As in the Riemannian geometry, we would like to raise and lower indices. Because the symplectic form ω is antisymmetric, we should be more careful in this case. For coordinates $K_{ab\dots c\dots d}{}^{rs\dots t\dots u}$ of a tensor field on the considered symplectic manifold (M, ω) , we denote the expression $\omega^{ic} K_{ab\dots c\dots d}{}^{rs\dots t}$ by $K_{ab\dots}{}^i{}_{\dots d}{}^{rs\dots t}$ and $K_{ab\dots c}{}^{rs\dots t\dots u} \omega_{ti}$ by $K_{ab\dots c}{}^{rs\dots}{}_{\dots i}{}^{\dots u}$ (similarly for other types of tensor fields).

Remark: In Vaisman [17], one can find a proof of a statement saying that the space of tensors $R \in \mathbb{V}^{\otimes 4}$ ($\dim \mathbb{V} = 2l$) satisfying the relations (5), (6) and (7) is an $Sp(\mathbb{V}, \omega)$ -irreducible module if $l = 1$ and decomposes into a direct sum of two irreducible $Sp(\mathbb{V}, \omega)$ -submodules if $l > 1$.

In the next lemma, two properties of the symplectic Weyl tensor field are described.

Lemma 7: The symplectic Weyl curvature tensor field is totally trace-free, i.e.,

$$\begin{aligned} W^{ijkl} \omega_{ij} &= W^{ijkl} \omega_{ik} = W^{ijkl} \omega_{il} = \\ W^{ijkl} \omega_{jk} &= W^{ijkl} \omega_{jl} = W^{ijkl} \omega_{kl} = 0 \end{aligned}$$

and the following equation

$$W_{ijkl} + W_{lijk} + W_{klji} + W_{jkli} = 0 \quad (\text{extended 1}^{st} \text{ Bianchi identity for } W) \quad (9)$$

holds.

Proof. The proof is straightforward and can be done just using the definitions of the symplectic Weyl curvature tensor field W , the tensor field $\tilde{\sigma}$ and the Lemma 6. \square

4 Metaplectic structure and the curvature tensor on symplectic spinors fields

Let us start describing the geometric structure with help of which the action of the symplectic curvature tensor field on symplectic spinors, and the symplectic twistor operators are defined. This structure, called metaplectic, is a

R^S on the symplectic spinor bundle is given by the classical formula

$$R^S := d^{\nabla^S} \nabla^S,$$

where d^{∇^S} is the associated exterior covariant derivative.

In the next lemma, the action of R^S on the sapce of symplectic spinors is described using just the symplectic curvature tensor field R .

Lemma 8: Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Then for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, we have

$$R^S \phi = \frac{l}{2} R^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \otimes e_i \cdot e_j \cdot \phi.$$

Proof. See Habermann, Habermann [8] pp. 42. \square .

Let us define the tensor fields σ^S and W^S by the formulas

$$\sigma^S \phi := \frac{l}{2} \sigma^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \otimes e_i \cdot e_j \cdot \phi \quad \text{and}$$

$$W^S \phi := \frac{l}{2} W^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \otimes e_i \cdot e_j \cdot \phi$$

for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$.

Theorem 9: Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Then for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, we have

$$\sigma^S \phi \in \Gamma(M, \mathcal{E}^{20} \oplus \mathcal{E}^{21}).$$

Proof. Using the definition of $\tilde{\sigma}$ and the Lemma 1 repeatedly we have for $\phi \in \Gamma(M, \mathcal{S})$,

$$\begin{aligned} \frac{2}{l} \sigma^S \phi &= \tilde{\sigma}^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi \\ &= (\omega^i{}_l \sigma^j{}_k - \omega^i{}_k \sigma^j{}_l + \omega^j{}_l \sigma^i{}_k - \omega^j{}_k \sigma^i{}_l + 2\sigma^{ij} \omega_{kl}) \epsilon_k \wedge \epsilon_l \otimes e_{ij} \cdot \phi \\ &= (-\sigma^j{}_k \epsilon^k \wedge \epsilon^i + \sigma^j{}_k \epsilon^i \wedge \epsilon^k - \sigma^i{}_k \epsilon^k \wedge \epsilon^j + \sigma^i{}_k \epsilon^j \wedge \epsilon^k + \\ &\quad + 2\sigma^{ij} \omega_{kl} \epsilon^k \wedge \epsilon^l) \otimes e_{ij} \cdot \phi \\ &= 2\sigma^j{}_k \epsilon^i \wedge \epsilon^k \otimes e_{ij} \cdot \phi - 2\sigma^i{}_k \epsilon^k \wedge \epsilon^j \otimes e_{ij} \cdot \phi + 2\sigma^{ij} \omega_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi \\ &= 2\sigma^j{}_k \epsilon^i \wedge \epsilon^k \otimes (e_{ij} \cdot + e_{ji} \cdot) \phi + 2\sigma^{ij} \omega_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi \\ &= 2\sigma^{jl} \omega_{lk} \epsilon^i \wedge \epsilon^k \otimes (e_{ij} \cdot + e_{ji} \cdot) \phi + 2\sigma^{ij} \omega_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi \\ &= 4\sigma^{jl} \omega_{lk} \epsilon^i \wedge \epsilon^k \otimes e_{ij} \cdot \phi + 2\sigma^{ij} \omega_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi \end{aligned}$$

It is straightforward but tedious to verify the next identities:

$$\begin{aligned} X^2 Y^2 (2\sigma^{ij} \omega_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi) &= 2l \sigma^{ij} \omega_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi, \\ X^2 Y^2 (4\sigma^{jl} \omega_{lk} \epsilon^i \wedge \epsilon^k \otimes e_{ij} \cdot \phi) &= 2\sigma^{ik} \omega_{jm} \epsilon^j \wedge \epsilon^m \otimes e_{ik} \cdot \phi, \\ XY (2\sigma^{ij} \omega_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi) &= -2l \sigma^{ij} \omega_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi \quad \text{and} \\ XY (4\sigma^{jl} \omega_{lk} \epsilon^i \wedge \epsilon^k \otimes e_{ij} \cdot \phi) &= 4l(l-1) \sigma^{jl} \omega_{lk} \epsilon^m \wedge \epsilon^k \otimes e_{mj} \cdot \phi - \\ &\quad 2l \sigma^{jl} \omega_{mi} \epsilon^m \wedge \epsilon^i \otimes e_{lj} \cdot \phi. \end{aligned}$$

Using the formulas (1) and (2), we get:

$$p^{20}\sigma^S\phi = \iota\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes (e_{ij}\cdot + \frac{1}{l}e_{ij}\cdot)\phi \text{ and} \quad (10)$$

$$p^{21}\sigma^S\phi = \iota\sigma^{ij}\epsilon^k \wedge \epsilon^l (2\omega_{il} \otimes e_{kj}\cdot - \frac{1}{l}\omega_{kl} \otimes e_{ij}\cdot)\phi. \quad (11)$$

Adding these two formulas and comparing them with the result of the computation of $\frac{2}{i}\sigma^S\phi$, we get $(p^{20} + p^{21})\sigma^S\phi = \sigma^S\phi$. Now, the statement follows. \square

Theorem 10: Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Then for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, we have

$$W^S\phi \in \Gamma(M, \mathcal{E}^{21} \oplus \mathcal{E}^{22}).$$

Proof. Let us compute $Y^2W^S\phi$ for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$.

$$\begin{aligned} \frac{2}{i}Y^2W^S\phi &= Y(\omega^{nm}W^{ij}_{kl}e_n(\epsilon^k \wedge \epsilon^l) \otimes e_{mij}\cdot\phi) \\ &= Y(\omega^{nm}W^{ij}_{kl}(\delta_n^k\epsilon^l - \delta_n^l\epsilon^k) \otimes e_{mij}\cdot\phi) \\ &= Y((\omega^{nm}W^{ij}_{nl}\epsilon^l - W^{ij}_{kn}\epsilon^k) \otimes e_{mij}\cdot\phi) \\ &= 2\omega^{nm}Y(W^{ij}_{nl}\epsilon^l \otimes e_{mij}\cdot\phi) \\ &= 2\omega^{pk}\omega^{nm}W^{ij}_{nl}e_p\epsilon_l \otimes e_{kmij}\cdot\phi \\ &= 2\omega^{pk}\omega^{nm}W^{ij}_{np}e_{kmij}\cdot\phi = 2W^{ijkl}e_{lkij}\cdot\phi. \end{aligned}$$

Now, let us use the extended 1st Bianchi identity for the symplectic Weyl curvature tensor field, Eq. (9), i.e.,

$$W^{ijkl} + W^{jkli} + W^{klij} + W^{lijk} = 0.$$

Multiplying this identity by the operator $e_{lkij}\cdot$, using the relation $e_{ij}\cdot - e_{ji}\cdot = -\omega_{ij}$ (Lemma 1) and the fact that the symplectic Weyl tensor field is totally trace free (Lemma 7), we get the following chain of equations.

$$\begin{aligned} W^{ijkl}e_{lkij}\cdot + W^{jkli}e_{lkij}\cdot + W^{klij}e_{lkij}\cdot + W^{lijk}e_{lkij}\cdot &= 0, \\ W^{ijkl}e_{lkij}\cdot + W^{jkli}e_l(e_{ik}\cdot - \omega_{ki})e_j\cdot + W^{klij}(e_{lik}\cdot - \omega_{ki}e_{lj}\cdot) + \\ + W^{lijk}(e_{klij}\cdot - \omega_{lk}e_{ij}\cdot) &= 0, \\ W^{ijkl}e_{lkij}\cdot + W^{jkli}(e_{ilk}\cdot - \omega_{li}e_{kj}\cdot) + W^{klij}(e_{ilk}\cdot - \omega_{li}e_{kj}\cdot) + \\ + W^{lijk}(e_{klji}\cdot - \omega_{ij}e_{kl}\cdot) &= 0, \\ W^{ijkl}e_{lkij}\cdot + W^{jkli}(e_{iljk}\cdot - \omega_{kj}e_{il}\cdot) + W^{klij}(e_{iklj}\cdot - \omega_{lk}e_{ij}\cdot) + \\ + W^{lijk}(e_{kjl i}\cdot - \omega_{lj}e_{ki}\cdot) &= 0 \text{ and} \\ 3W^{ijkl}e_{lkij}\cdot + W^{klij}(e_{ikjl}\cdot - \omega_{lj}e_{ik}\cdot) &= 0. \end{aligned}$$

Continuing in a similar way, we get $4W^{ijkl}e_{lkij}\cdot = 0$. Summing up, we have $Y^2W^S = 0$. Using the relation (1) for p^{20} , we have $p^{20}W^S\phi = 0$. Hence the statement follows. \square

Let us consider a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$. By a straightforward way, we get $XYW^S\phi = 2W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}\cdot\phi$. Using this result, Theorem 10, definition of W^S and the relations (2) and (3) for p^{21} and p^{22} , we get

$$p^{21}W^S\phi = \frac{2l}{1-l}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}\cdot\phi \text{ and} \quad (12)$$

$$p^{22}W^S\phi = \frac{l}{2}W^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}\cdot\phi - \frac{2l}{1-l}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mnij}\cdot\phi. \quad (13)$$

Summing up the preceding two theorems, we have the

Corollary 11: In the situation described in the formulation of the Theorem 10, we have for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$

$$\begin{aligned} p^{20}R^S\phi &= \iota\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes (e_{ij}\cdot + \frac{1}{l}e_{ij}\cdot)\phi, \\ p^{21}R^S\phi &= \iota\sigma^{ij}\epsilon^k \wedge \epsilon^l (2\omega_{il} \otimes e_{kj}\cdot - \frac{1}{l}\omega_{kl} \otimes e_{ij}\cdot)\phi + \\ &\quad + \frac{2l}{1-l}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}\cdot\phi \text{ and} \\ p^{22}R^S\phi &= \frac{l}{2}W^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}\cdot\phi - \frac{2l}{1-l}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}\cdot\phi. \end{aligned}$$

Proof. The equations follow from the equations (10), (11), (12) and (13) and the definitions of σ^S and W^S . \square

Now, let us turn our attention to the mentioned application of the decomposition result (Corollary 11). Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure (\mathcal{Q}, Λ) . Then we have the associated bundles $\mathcal{E}^{ij} \rightarrow M$ ($i = 1, 2, j_1 = 0, 1$ and $j_2 = 0, 1, 2$) and the symplectic spinor covariant derivative ∇^S as well as the associated exterior covariant derivative d^{∇^S} at our disposal. Let us introduce the following first order $Mp(2l, \mathbb{R})$ -invariant differential operators:

$$\begin{aligned} T_0 &: \Gamma(M, \mathcal{S}) \rightarrow \Gamma(M, \mathcal{E}^{11}), \quad T_0 := p^{11}\nabla^S \text{ and} \\ T_1 &: \Gamma(M, \mathcal{E}^{11}) \rightarrow \Gamma(M, \mathcal{E}^{22}), \quad T_1 := p^{22}d^{\nabla^S}_{|\Gamma(M, \mathcal{E}^{11})}. \end{aligned}$$

We shall call these operators *symplectic twistor operators*. These definitions are symplectic counterparts of the definitions of twistor operators in Riemannian spin-geometry. Using the Corollary 11, we get

Theorem 12: Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Suppose the symplectic Weyl tensor field $W = 0$. Then

$$0 \longrightarrow \Gamma(M, \mathcal{S}) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^{11}) \xrightarrow{T_1} \Gamma(M, \mathcal{E}^{22})$$

is a complex of first order differential operators.

Proof. Let us suppose $W = 0$. Then $p^{22}R^S = 0$ (due to the Corollary 11). Using the definition of R^S , we have $0 = p^{22}R^S = p^{22}(d^{\nabla^S}\nabla^S) = p^{22}d^{\nabla^S}(p^{11} +$

$p^{10})\nabla^S = p^{22}d^{\nabla^S}p^{11}\nabla^S + p^{22}d^{\nabla^S}p^{10}\nabla^S$. According to Krýsl [12], $p^{22}d^{\nabla^S}p^{10} = 0$. Thus we have $0 = p^{22}d^{\nabla^S}p^{11}\nabla^S = T_1T_0$, giving the statement. \square

Remark: In Krýsl [12], the $Mp(2l, \mathbb{R})$ -module $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ was decomposed into irreducible summands. Let us denote these irreducible summands by \mathbf{E}^{ij} (the specification of the indices i, j can be found in the mentioned article or in Krýsl [14]). Similarly as above, we can introduce the projections $p^{ij} : \bigwedge^i \mathbb{V}^* \otimes \mathbf{S} \rightarrow \mathbf{E}^{ij}$. In the mentioned article, we proved that $p^{i+1, j}d_{\Gamma(M, \mathcal{E}^{ik})}^{\nabla^S} = 0$ for all appropriate i, k and $j > k+1$ or $j < k-1$. In the proof of the preceding theorem, we used this information in the case of $i = 1, k = 0$ and $j = 2$.

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