

Complex of twistor operators in symplectic spin geometry

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Abstract

For a symplectic manifold admitting a metaplectic structure (a symplectic analogue of the Riemannian spin structure), we construct a sequence consisting of differential operators using a symplectic torsion-free affine connection. All but one of these operators are of first order. The first order ones are symplectic analogues of the twistor operators known from Riemannian spin geometry. We prove that under the condition the symplectic Weyl curvature tensor field of the symplectic connection vanishes, the mentioned sequence forms a complex. This gives rise to a new complex for the so called Ricci type symplectic manifolds, which admit a metaplectic structure.

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1 Introduction

In the paper, we shall introduce a sequence of differential operators acting on symplectic spinor valued exterior differential forms over a symplectic manifold (M, ω) admitting the so called metaplectic structure. To define these operators, we make use of a symplectic torsion-free affine connection ∇ on (M, ω) . Under certain condition on the curvature of the connection ∇ , described bellow, we prove that the mentioned sequence forms a complex.

Let us say a few words about the metaplectic structure. The symplectic group $Sp(2l, \mathbb{R})$ admits a non-trivial two-fold covering, the so called metaplectic group. We shall denote it by $Mp(2l, \mathbb{R})$ and its Lie algebra by \mathfrak{g} . A metaplectic structure on a symplectic manifold (M^{2l}, ω) is a notion parallel to a spin

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structure on a Riemannian manifold. In particular, one of its part is a principal $Mp(2l, \mathbb{R})$ bundle $(q : \mathcal{Q} \rightarrow M, Mp(2l, \mathbb{R}))$.

For a symplectic manifold admitting a metaplectic structure, one can construct the so called symplectic spinor bundle $\mathcal{S} \rightarrow M$, introduced by Bertram Kostant in 1974. The symplectic spinor bundle \mathcal{S} is the vector bundle associated to the metaplectic structure $(q : \mathcal{Q} \rightarrow M, Mp(2l, \mathbb{R}))$ on M via the so called Segal-Shale-Weil representation of the metaplectic group $Mp(2l, \mathbb{R})$. See Kostant [11] for details.

The Segal-Shale-Weil representation is an infinite dimensional unitary representation of the metaplectic group $Mp(2l, \mathbb{R})$ on the space of all complex valued square Lebesgue integrable functions $\mathbf{L}^2(\mathbb{R}^l)$. Because of the infinite dimension, the Segal-Shale-Weil representation is not so easy to handle. It is known, see, e.g., Kashiwara, Vergne [10], that the $\mathfrak{g}^{\mathbb{C}}$ -module structure of the underlying Harish-Chandra module of this representation is equivalent to the space $\mathbb{C}[x^1, \dots, x^l]$ of polynomials in l variables, on which the Lie algebra $\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sp}(2l, \mathbb{C})$ acts via the so called Chevalley homomorphism,¹ see Britten, Hooper, Lemire [1]. Thus, the infinitesimal structure of the Segal-Shale-Weil representation can be viewed as the complexified *symmetric* algebra $(\bigoplus_{i=0}^{\infty} \odot^i \mathbb{R}^l) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}[x^1, \dots, x^l]$ of the Lagrangian subspace $(\mathbb{R}^l, 0)$ of the canonical symplectic vector space $\mathbb{R}^{2l} \simeq (\mathbb{R}^l, 0) \oplus (0, \mathbb{R}^l)$. This shows that the situation is completely parallel to the complex orthogonal case, where the spinor representation can be realized as the *exterior* algebra of a maximal isotropic subspace. An interested reader is referred to Weil [20], Kashiwara, Vergne [10] and also to Britten, Hooper, Lemire [1] for details. For some technical reasons, we shall be using the so called minimal globalization of the underlying Harish-Chandra module of the Segal-Shale-Weil representation, which we will call *metaplectic representation* and denote it by \mathbf{S} . The elements of \mathbf{S} will be called symplectic spinors.

Now, let us consider a symplectic manifold (M, ω) together with a symplectic torsion-free affine connection ∇ on it. Such connections are usually called Fedosov connections. Because the Fedosov connection is not unique for a choice of (M, ω) (in the contrary to Riemannian geometry), it seems natural to add the connection to the studied symplectic structure and investigate triples (M, ω, ∇) consisting of a symplectic manifold (M, ω) and a Fedosov connection ∇ . Such triples are usually called Fedosov manifolds and they were used in the deformation quantization. See, e.g., Fedosov [6]. Let us recall that in Vaisman [18], the space of the so called symplectic curvature tensors was decomposed wr. to $Sp(2l, \mathbb{R})$. For $l = 1$, the module of symplectic curvature tensors is irreducible, while for $l \geq 2$, it decomposes into two irreducible submodules. These modules are usually called symplectic Ricci and symplectic Weyl modules, respectively. This decomposition translates to differential geometry level giving rise to the symplectic Ricci and symplectic Weyl curvature tensor fields, which add up to the curvature tensor field of ∇ . See Vaisman [18] and also Gelfand, Retakh, Shubin [4] for a comprehensive treatment on Fedosov manifolds.

¹The Chevalley homomorphism is a Lie algebra monomorphism of the complex symplectic Lie algebra $\mathfrak{sp}(2l, \mathbb{C})$ into the Lie algebra of the associative algebra of polynomial coefficients differential operators acting on $\mathbb{C}[x^1, \dots, x^l]$.

Now, let us suppose that a Fedosov manifold (M, ω, ∇) admits a metaplectic structure $(q : \mathcal{Q} \rightarrow M^{2l}, Mp(2l, \mathbb{R}))$. Let $\mathcal{S} \rightarrow M$ be the symplectic spinor bundle associated to $(q : \tilde{\mathcal{Q}} \rightarrow M, Mp(2l, \mathbb{R}))$ and let us consider the space $\Omega^\bullet(M, \mathcal{S})$ of exterior differential forms with values in \mathcal{S} , i.e., $\Omega^\bullet(M, \mathcal{S}) := \Gamma(M, \mathcal{Q} \times_\rho (\bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S}))$, where ρ is the obvious tensor product representation of $Mp(2l, \mathbb{R})$ on $\bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S}$. In Krýsl [14], the $Mp(2l, \mathbb{R})$ -module $\bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S}$ was decomposed into irreducible submodules. The elements of $\bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S}$ are specific examples of the so called higher symplectic spinors. For $i = 0, \dots, 2l$, let us denote the so called Cartan component of the tensor product $\bigwedge^i(\mathbb{R}^{2l})^* \otimes \mathbf{S}$ by \mathbf{E}^{im_i} . (For $i = 0, \dots, 2l$, the numbers m_i will be specified in the text.) For $i = 0, \dots, 2l - 1$, we introduce an operator T_i acting between the sections of the vector bundle \mathcal{E}^{im_i} associated to \mathbf{E}^{im_i} and the sections of the vector bundle $\mathcal{E}^{i+1, m_{i+1}}$ associated to $\mathbf{E}^{i+1, m_{i+1}}$. In a parallel to the Riemannian case, we shall call these operators symplectic twistor operators. These operators are first order differential operators and they are defined using the symplectic torsion-free affine connection ∇ as follows. First, the connection ∇ induces a covariant derivative $\nabla^{\mathcal{S}}$ on the bundle $\mathcal{S} \rightarrow M$ in the usual way. Second, the covariant derivative $\nabla^{\mathcal{S}}$ determines the associated exterior covariant derivative, which we denote by $d^{\nabla^{\mathcal{S}}}$. For $i = 0, \dots, 2l - 1$, we define the symplectic twistor operator T_i as the restriction of $d^{\nabla^{\mathcal{S}}}$ to $\Gamma(M, \mathcal{E}^{im_i})$ composed with the projection to $\Gamma(M, \mathcal{E}^{i+1, m_{i+1}})$, $i = 0, \dots, 2l - 1$.

Because we would like to derive a condition under which $T_{i+1}T_i = 0$, $i = 0, \dots, 2l - 1$, we should focus our attention to the curvature tensor $R^{\Omega^\bullet(M, \mathcal{S})} := d^{\nabla^{\mathcal{S}}} d^{\nabla^{\mathcal{S}}}$ of $d^{\nabla^{\mathcal{S}}}$ acting on the space $\Omega^\bullet(M, \mathcal{S})$. The curvature $R^{\Omega^\bullet(M, \mathcal{S})}$ depends only on the curvature of the symplectic connection ∇ , which consists of the symplectic Ricci and symplectic Weyl curvature tensor fields. In the paper, we will analyze the action of the symplectic Ricci curvature tensor field on symplectic spinor valued exterior differential forms and especially on $\Gamma(M, \mathcal{E}^{im_i})$, $i = 0, \dots, 2l - 2$. We shall prove that the symplectic Ricci curvature tensor field when restricted to $\Gamma(M, \mathcal{E}^{im_i})$ maps this submodule into at most three $Mp(2l, \mathbb{R})$ -submodules sitting in symplectic spinor valued forms of degree $i + 2$, $i = 0, \dots, 2l - 2$. These submodules will be explicitly described. This will help us to prove that $T_{i+1}T_i = 0$ ($i = 0, \dots, l - 2$) and $T_{i+1}T_i = 0$ ($i = l, \dots, 2l - 2$) assuming the symplectic Weyl curvature tensor field vanishes. In this way, we will obtain two complexes. Unfortunately, one can not expect $T_l T_{l-1} = 0$ in general. This will influence the way, how we construct one complex of the two complexes introduced above. Let us notice that similar complex was investigated in Severa [16] in the case of spheres equipped with the conformal structure of their round metrics.

The reader interested in applications of the symplectic spinor fields in theoretical physics is referred to Green, Hull [5], where the symplectic spinors are used in the context of 10 dimensional super string theory. In Reuter [15], symplectic spinors are used in the theory of the so called Dirac-Kähler fields.

In the second section, some basic facts on the metaplectic representation and higher symplectic spinors are recalled. In this section, we also intro-

duce several mappings acting on the graded space $\bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S}$, derive some (super -) commutation relations between them and determine a superset of the image of two of them, which are components of the "infinitesimal version" of the symplectic Ricci curvature tensor field. In the section 3, basic properties of torsion-free symplectic connections and their curvature tensor field are mentioned and the metaplectic structure is introduced. In the subsection 3.1., the theorem on the complex consisting of the symplectic twistor operators is presented and proved.

2 Metaplectic representation, higher symplectic spinors and basic notation

To fix a notation, let us recall some notions from symplectic linear algebra. Let us consider a real symplectic vector space (\mathbb{V}, ω) of dimension $2l$, i.e., \mathbb{V} is a $2l$ dimensional real vector space and ω is a non-degenerate antisymmetric bilinear form on \mathbb{V} . Let us choose two Lagrangian subspaces² $\mathbb{L}, \mathbb{L}' \subseteq \mathbb{V}$ such that $\mathbb{L} \oplus \mathbb{L}' = \mathbb{V}$. It follows that $\dim(\mathbb{L}) = \dim(\mathbb{L}') = l$. Throughout this article, we shall use a symplectic basis $\{e_i\}_{i=1}^{2l}$ of \mathbb{V} chosen in such a way that $\{e_i\}_{i=1}^l$ and $\{e_i\}_{i=l+1}^{2l}$ are respective bases of \mathbb{L} and \mathbb{L}' . Because the definition of a symplectic basis is not unique, let us fix one which shall be used in this text. A basis $\{e_i\}_{i=1}^{2l}$ of \mathbb{V} is called symplectic basis of (\mathbb{V}, ω) if $\omega_{ij} := \omega(e_i, e_j)$ satisfies $\omega_{ij} = 1$ if and only if $i \leq l$ and $j = i + l$; $\omega_{ij} = -1$ if and only if $i > l$ and $j = i - l$ and finally, $\omega_{ij} = 0$ in other cases. Let $\{\epsilon^i\}_{i=1}^{2l}$ be the basis of \mathbb{V}^* dual to the basis $\{e_i\}_{i=1}^{2l}$. For $i, j = 1, \dots, 2l$, we define ω^{ij} by $\sum_{k=1}^{2l} \omega_{ik} \omega^{jk} = \delta_i^j$, for $i, j = 1, \dots, 2l$. Notice that not only $\omega_{ij} = -\omega_{ji}$, but also $\omega^{ij} = -\omega^{ji}$, $i, j = 1, \dots, 2l$.

As in the orthogonal case, we would like to rise and lower indices. Because the symplectic form ω is antisymmetric, we should be more careful in this case. For coordinates $K_{ab\dots c\dots d}{}^{rs\dots t\dots u}$ of a tensor K over \mathbb{V} , we denote the expression $\omega^{ic} K_{ab\dots c\dots d}{}^{rs\dots t}$ by $K_{ab\dots}{}^i{}_{\dots d}{}^{rs\dots t}$ and $K_{ab\dots c}{}^{rs\dots t\dots u} \omega_{ti}$ by $K_{ab\dots c}{}^{rs\dots}{}_{i\dots}{}^u$ and similarly for other types of tensors and also in the geometric setting when we will be considering tensor fields over a symplectic manifold (M, ω) .

Let us denote the symplectic group of (\mathbb{V}, ω) by G , i.e., $G := Sp(\mathbb{V}, \omega) \simeq Sp(2l, \mathbb{R})$. Because the maximal compact subgroup K of G is isomorphic to the unitary group $K \simeq U(l)$ which is of homotopy type \mathbb{Z} , there exists a nontrivial two-fold covering \tilde{G} of G . See, e.g., Habermann, Habermann [8] for details. This two-fold covering is called metaplectic group of (\mathbb{V}, ω) and it is denoted by $Mp(\mathbb{V}, \omega)$. Let us remark that $Mp(\mathbb{V}, \omega)$ is reductive in the sense of Vogan [19]. In the considered case, we have $\tilde{G} \simeq Mp(2l, \mathbb{R})$. For a later use, let us reserve the symbol λ for the mentioned covering. Thus $\lambda : \tilde{G} \rightarrow G$ is a fixed member of the isomorphism class of all nontrivial $2 : 1$ covering homomorphisms of G . Because $\lambda : \tilde{G} \rightarrow G$ is a homomorphism of Lie groups and G is a subgroup of the general linear group $GL(\mathbb{V})$ of \mathbb{V} , the mapping λ is also a representation of the metaplectic group \tilde{G} on the vector space \mathbb{V} . Let us define $\tilde{K} := \lambda^{-1}(K)$.

²maximal isotropic wr. to ω

Obviously, \tilde{K} is a maximal compact subgroup of \tilde{G} . Further, one can easily see that $\tilde{K} \simeq \widetilde{U(l)} := \{(g, z) \in U(l) \times \mathbb{C}^\times \mid \det(g) = z^2\}$ and thus in particular, \tilde{K} is connected. The Lie algebra $\tilde{\mathfrak{g}}$ of \tilde{G} is isomorphic to the Lie algebra \mathfrak{g} of G and we will identify them. One has $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}, \omega) \simeq \mathfrak{sp}(2l, \mathbb{R})$.

Now let us recall some notions from representation theory which we shall need in this paper. From the point of view of this article, these notions are rather of a technical character. Let $\mathcal{R}(\tilde{G})$ be the category the object of which are complete, locally convex, Hausdorff topological spaces with a continuous linear \tilde{G} -action, such that the resulting representation is admissible and of finite length; the morphisms are continuous \tilde{G} -equivariant linear maps between the objects. Let $\mathcal{HC}(\mathfrak{g}, \tilde{K})$ be the category of Harish-Chandra $(\mathfrak{g}, \tilde{K})$ -modules and let us consider the forgetful Harish-Chandra functor $HC : \mathcal{R}(\tilde{G}) \rightarrow \mathcal{HC}(\mathfrak{g}, \tilde{K})$. It is well known that there exists an adjoint functor $mg : \mathcal{HC}(\mathfrak{g}, \tilde{K}) \rightarrow \mathcal{R}(\tilde{G})$ to the Harish-Chandra functor HC . This functor is usually called the minimal globalization functor and its existence is a deep result in representation theory. For details and for the existence of the minimal globalization functor mg , see Kashiwara, Schmid [9] or Vogan [19].

From now on, we shall restrict ourselves to the case $l \geq 2$ not always mentioning it explicitly. The case $l = 1$ should be handled separately (though analogously) because the shape of the root system of $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R})$ is different from that one of the root system of $\mathfrak{sp}(2l, \mathbb{R})$ for $l \geq 2$. As usual, we shall denote the complexification of \mathfrak{g} by $\mathfrak{g}^{\mathbb{C}}$. Obviously, $\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sp}(2l, \mathbb{C})$.

Further, for any Lie group G and a principal G -bundle $(p : \mathcal{P} \rightarrow M, G)$ over a manifold M , we shall denote the vector bundle associated to this principal bundle via a representation $\sigma : G \rightarrow \text{Aut}(\mathbf{W})$ of G on \mathbf{W} by \mathcal{W} , i.e., $\mathcal{W} = \mathcal{G} \times_{\sigma} \mathbf{W}$. Let us also mention that we shall often use the Einstein summation convention for repeated indices (lower and upper) without mentioning it explicitly.

2.1 Metaplectic representation and symplectic spinors

There exists a distinguished infinite dimensional unitary representation of the metaplectic group \tilde{G} which does not descend to a representation of the symplectic group G . This representation, called *Segal-Shale-Weil*,³ plays an important role in geometric quantization of Hamiltonian mechanics, see, e.g., Woodhouse [21]. We shall not give a definition of this representation here and refer the interested reader to Weil [20] or Habermann, Habermann [8].

The Segal-Shale-Weil representation, which we shall denote by U , is a complex infinite dimensional unitary representation of \tilde{G} on the space of complex valued square Lebesgue integrable functions defined on the Lagrangian subspace \mathbb{L} , i.e.,

$$U : \tilde{G} \rightarrow \mathcal{U}(\mathbf{L}^2(\mathbb{L})),$$

where $\mathcal{U}(\mathbf{W})$ denotes the group of unitary operators on a Hilbert space \mathbf{W} . In

³The names oscillator or metaplectic representation are also used in the literature. We shall use the name Segal-Shale-Weil in this text, and reserve the name metaplectic for certain representation arising from the Segal-Shale-Weil one.

order to be precise, let us refer to the space $\mathbf{L}^2(\mathbb{L})$ as to the Segal-Shale-Weil module. It is known that the Segal-Shale-Weil module belongs to the category $\mathcal{R}(\tilde{G})$. (See Kashiwara, Vergne [10] for details and Segal-Shale-Weil representation in general.) It is easy to see that the Segal-Shale-Weil representation splits into two irreducible $Mp(2l, \mathbb{R})$ -submodules $\mathbf{L}^2(\mathbb{L}) \simeq \mathbf{L}^2(\mathbb{L})_+ \oplus \mathbf{L}^2(\mathbb{L})_-$. The first module consists of even and the second one of odd complex valued square Lebesgue integrable functions on the Lagrangian subspace \mathbb{L} . Let us remark that a typical construction of the Segal-Shale-Weil representation is based on the so called Schrödinger representation of the Heisenberg group of $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega)$ and a use of the Stone-von Neumann theorem.

For technical reasons, we shall need the minimal globalization of the underlying Harish-Chandra $(\mathfrak{g}, \tilde{K})$ -module $HC(\mathbf{L}^2(\mathbb{L}))$ of the introduced Segal-Shale-Weil module. We shall call this minimal globalization *metaplectic representation* and denote it by *meta*, i.e.,

$$meta : \tilde{G} \rightarrow \text{Aut}(mg(HC(\mathbf{L}^2(\mathbb{L})))),$$

where mg is the minimal globalization functor (see this section and the references therein). For our convenience, let us denote the module $mg(HC(\mathbf{L}^2(\mathbb{L})))$ by \mathbf{S} . Similarly we define \mathbf{S}_+ and \mathbf{S}_- to be the minimal globalizations of the underlying Harish-Chandra $(\mathfrak{g}, \tilde{K})$ -modules of the modules $\mathbf{L}^2(\mathbb{L})_+$ and $\mathbf{L}^2(\mathbb{L})_-$. Accordingly to $\mathbf{L}^2(\mathbb{L}) \simeq \mathbf{L}^2(\mathbb{L})_+ \oplus \mathbf{L}^2(\mathbb{L})_-$, we have $\mathbf{S} \simeq \mathbf{S}_+ \oplus \mathbf{S}_-$. We shall call the $Mp(\mathbb{V}, \omega)$ -module \mathbf{S} the symplectic spinor module and its elements *symplectic spinors*. For the name "spinor", see Kostant [11] or the Introduction.

Further notion related to the symplectic vector space $(\mathbb{V} = \mathbb{L} \oplus \mathbb{L}', \omega)$ is the so called symplectic Clifford multiplication of elements of \mathbf{S} by vectors from \mathbb{V} . For $i = 1, \dots, l$ and a symplectic spinor $f \in \mathbf{S}$, we define

$$\begin{aligned} (e_i \cdot f)(x) &:= ix^i f(x) \text{ and} \\ (e_{i+1} \cdot f)(x) &:= \frac{\partial f}{\partial x^i}(x), \end{aligned}$$

where $x = \sum_{i=1}^l x^i e_i \in \mathbb{L}$ and $i = \sqrt{-1}$ denotes the imaginary unit. Extending this multiplication \mathbb{R} -linearly, we get the mentioned symplectic Clifford multiplication. Let us mention that the multiplication and the differentiation make sense for any $f \in \mathbf{S}$ because of the "analytic" interpretation of the minimal globalization. (See Vogan [19] for details.) Let us remark that in the physical literature, the symplectic Clifford multiplication is usually called Schrödinger quantization prescription.

The following lemma is an easy consequence of the definition of the symplectic Clifford multiplication.

Lemma 1: For $v, w \in \mathbb{V}$ and $s \in \mathbf{S}$, we have

$$v \cdot (w \cdot s) - w \cdot (v \cdot s) = -\omega(v, w)s.$$

Proof. See Habermann, Habermann [8], pp. 11. \square

Sometimes, we shall write $v \cdot w \cdot s$ instead of $v \cdot (w \cdot s)$ for $v, w \in \mathbb{V}$ and a symplectic spinor $s \in \mathbf{S}$ and similarly for higher number of multiplying elements.

Further instead of $e_i.e_j.s$, we shall write $e_{ij}.s$ simply and similarly for expressions with higher number of multiplying elements, e.g., $e_{ijk}.s$ abbreviates $e_i.e_j.e_k.s$.

2.2 Higher symplectic spinors

In this subsection, we shall present a result on a decomposition of the tensor product of the metaplectic representation $meta : \tilde{G} \rightarrow \text{Aut}(\mathbf{S})$ with the wedge power of the representation $\lambda^* : \tilde{G} \rightarrow GL(\mathbb{V}^*)$ of \tilde{G} (dual to the representation λ) into irreducible summands. Let us reserve the symbol ρ for the mentioned tensor product representation of \tilde{G} , i.e.,

$$\begin{aligned} \rho : \tilde{G} &\rightarrow \text{Aut}\left(\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}\right) \\ \rho(g)(\alpha \otimes s) &:= \lambda(g)^{* \wedge r} \alpha \otimes meta(g)s \end{aligned}$$

for $r = 0, \dots, 2l$, $g \in \tilde{G}$, $\alpha \in \bigwedge^r \mathbb{V}^*$, $s \in \mathbf{S}$ and extend it linearly. For definiteness, let us equip the tensor product $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ with the so called Grothendieck tensor product topology. See Vogan [19] and Treves [17] for details on this topological structure. In a parallel to the Riemannian case, we shall call the elements of $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ higher symplectic spinors.

Let us introduce the following subsets of the set of pairs of non-negative integers. We define

$$\begin{aligned} \Xi &:= \{(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid i = 0, \dots, l; j = 0, \dots, i\} \cup \\ &\quad \cup \{(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid i = l + 1, \dots, 2l, j = 0, \dots, 2l - i\}, \\ \Xi_+ &:= \Xi - \{(i, i) \mid i = 0, \dots, l\} \text{ and} \\ \Xi_- &:= \Xi - \{(i, 2l - i) \mid i = l, \dots, 2l\}. \end{aligned}$$

For each $(i, j) \in \Xi$, a $\mathfrak{g}^{\mathbb{C}}$ -module \mathbb{E}_{\pm}^{ij} was introduced in Krýsl [14]. These modules are irreducible infinite dimensional highest modules over $\mathfrak{sp}(\mathbb{V}, \omega)^{\mathbb{C}}$ and they are described via their highest weights in the mentioned article. In the next theorem, the module of symplectic spinor valued exterior forms $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ is decomposed into irreducible submodules.

Theorem 2: For $l \geq 2$, the following decomposition into irreducible $Mp(\mathbb{V}, \omega)$ -submodules

$$\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}_{\pm} \simeq \bigoplus_{j, (i, j) \in \Xi} \mathbf{E}_{\pm}^{ij}, \quad i = 0, \dots, 2l, \text{ holds.}$$

The modules \mathbf{E}_{\pm}^{ij} are determined, as objects in the category $\mathcal{R}(\tilde{G})$, by the fact that first they are submodules of the corresponding tensor product and second the $\mathfrak{g}^{\mathbb{C}}$ -structure of $HC(\mathbf{E}_{\pm}^{ij})$ is isomorphic to \mathbb{E}_{\pm}^{ij} .

Proof. See Krýsl [14] or Krýsl [12]. \square

In the Figure 1, the decomposition in the case $l = 3$ is displayed. In the i^{th} column of the Figure 1, when counted from zero, the summands of $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}$,

$i = 0, \dots, 6$, are written. The meaning of the arrows at the figure will be explained later.

Remark: Let us mention that for any $(i, j), (i, k) \in \Xi$, $j \neq k$, we have $\mathbb{E}_{\pm}^{ij} \not\cong \mathbb{E}_{\pm}^{ik}$ (as $\mathfrak{g}^{\mathbb{C}}$ -modules) for all combinations of \pm on the left hand as well as on the right hand side. Using this fact, we have that for $i = 0, \dots, 2l$ the \tilde{G} -modules $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}_{\pm}$ are multiplicity free. Moreover for $(i, j), (k, j) \in \Xi$, we have $\mathbb{E}_{\pm}^{ij} \simeq \mathbb{E}_{\mp}^{kj}$. These facts will be crucial in the paper.

For our convenience, let us set $\mathbf{E}_{\pm}^{ij} := \{0\}$ for $(i, j) \in \mathbb{Z} \times \mathbb{Z} - \Xi$ and $\mathbf{E}^{ij} := \mathbf{E}_{+}^{ij} \oplus \mathbf{E}_{-}^{ij}$.

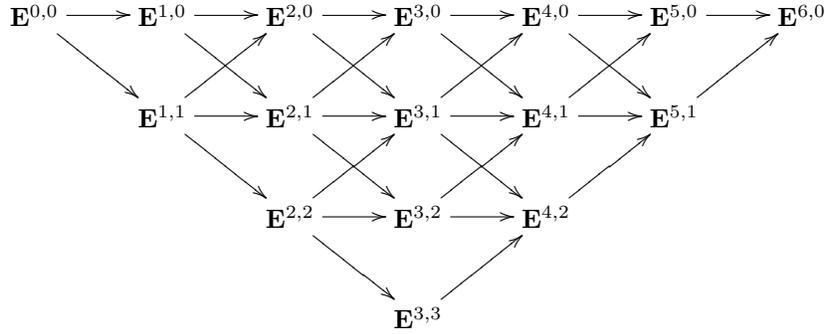


Figure 1.

Now, we shall introduce four operators which help us to describe the action of the symplectic Ricci curvature tensor field acting on symplectic spinor valued exterior differential forms. For $r = 0, \dots, 2l$, $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$ and $\sigma \in \odot^2 \mathbb{V}^*$, we set

$$\begin{aligned}
X & : \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \rightarrow \bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S}, \quad X(\alpha \otimes s) := \sum_{i=1}^{2l} \epsilon^i \wedge \alpha \otimes e_{i.s}, \\
Y & : \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \rightarrow \bigwedge^{r-1} \mathbb{V}^* \otimes \mathbf{S}, \quad Y(\alpha \otimes s) := \sum_{i,j=1}^{2l} \omega^{ij} \iota_{e_i} \alpha \otimes e_{j.s}, \\
\Sigma^\sigma & : \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \rightarrow \bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S}, \quad \Sigma^\sigma(\alpha \otimes s) := \sum_{i,j=1}^{2l} \sigma^{ij} \epsilon^j \wedge \alpha \otimes e_{i.s} \text{ and} \\
\Theta^\sigma & : \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \rightarrow \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}, \quad \Theta^\sigma(\alpha \otimes s) := \sum_{i,j=1}^{2l} \alpha \otimes \sigma^{ij} e_{ij.s}
\end{aligned}$$

and extend it linearly. Here $\sigma_{ij} := \sigma(e_i, e_j)$, $i, j = 1, \dots, 2l$, and the contraction of an exterior form $\alpha \in \bigwedge^r \mathbb{V}^*$ by a vector $v \in \mathbb{V}$ is denoted by $\iota_v \alpha$.

Remark:

- 1) One easily finds out that the operators are independent of the choice of a symplectic basis $\{e_i\}_{i=1}^{2l}$. The operators X and Y are used to prove the

Howe correspondence for $Mp(\mathbb{V}, \omega)$ acting on $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ via the representation ρ . See Krýsl [12] for details.

- 2) The symmetric tensor σ is an infinitesimal version of a part of the curvature of a Fedosov connection. This part is called symplectic Ricci curvature tensor field and will be introduced bellow. The operators Σ^σ and Θ^σ will help us to describe the action of the symplectic Ricci curvature tensor field acting on symplectic spinor valued exterior differential forms.

In what follows, we shall write $\iota_{e_{ij}}\alpha$ instead of $\iota_{e_i}\iota_{e_j}\alpha$, $i, j = 1, \dots, 2l$, and similarly for higher number of contracting elements.

Using the Lemma 1, it is easy to compute that

$$X^2(\alpha \otimes s) = -\frac{l}{2}\omega_{ij}\epsilon^i \wedge \epsilon^j \wedge \alpha \otimes s \quad \text{and} \quad Y^2(\alpha \otimes s) = \frac{l}{2}\omega^{ij}\iota_{e_{ij}}\alpha \otimes s \quad (1)$$

for any element $\alpha \otimes s \in \bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$.

In order to be able to use the operators X and Y in a geometric setting and some further reasons, we shall need the following

Lemma 3:

- 1) The operators X, Y are \tilde{G} -equivariant wr. to the representation ρ of \tilde{G} .
- 2) For $(i, j) \in \Xi_-$, the operator X is an isomorphism if restricted to \mathbf{E}^{ij} .
For $(i, j) \in \Xi_+$, the operator Y is an isomorphism if restricted to \mathbf{E}^{ij} .

Proof. For the \tilde{G} -equivariance of X and Y , see Krýsl [13]. The fact that the mentioned restrictions are isomorphisms is proved in Krýsl [12]. \square

In the next lemma, four relations are proved which will be used later in order to determine a superset of the image of a restriction of the symplectic Ricci curvature tensor field acting on symplectic spinor valued exterior differential forms. Often, we shall write Σ and Θ simply instead of the more explicit Σ^σ and Θ^σ . The symmetric tensor σ is assumed to be chosen. The symbol $\{, \}$ denotes the anticommutator on $\text{End}(\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S})$.

Lemma 4: The following relations

$$\{\Sigma, X\} = 0, \quad (2)$$

$$[\{\Sigma, Y\}, Y^2] = 0, \quad (3)$$

$$[X, \Theta] = 2i\Sigma \quad \text{and} \quad (4)$$

$$[\Theta, Y^2] = 0 \quad (5)$$

hold on $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$.

Proof. We shall prove these identities for $\alpha \otimes s \in \bigwedge^i \mathbb{V}^* \otimes \mathbf{S}$, $i = 0, \dots, 2l$ only. The statement then follows by linearity of the considered operators.

1) Let us compute

$$\begin{aligned}
(X\Sigma + \Sigma X)(\alpha \otimes s) &= X(\sigma^i_j \epsilon^j \wedge \alpha \otimes e_i \phi) + \Sigma(\epsilon^i \wedge \alpha \otimes e_i \cdot s) \\
&= \sigma^i_j \epsilon^k \wedge \epsilon^j \wedge \alpha \otimes e_{ki} \cdot s + \sigma^j_k \epsilon^k \wedge \epsilon^i \wedge \alpha \otimes e_{ji} \cdot s \\
&= \sigma^i_k \epsilon^j \wedge \epsilon^k \wedge \alpha \otimes e_{ji} \cdot s + \sigma^i_k \epsilon^k \wedge \epsilon^j \wedge \alpha \otimes e_{ij} \cdot s \\
&= \sigma^i_k \epsilon^j \wedge \epsilon^k \wedge \alpha \otimes (e_{ji} - e_{ij}) \cdot s \\
&= -\iota \sigma^i_k \omega_{ji} \epsilon^j \wedge \epsilon^k \wedge \alpha \otimes s \\
&= \iota \sigma_{jk} \epsilon^j \wedge \epsilon^k \wedge \alpha \otimes s \\
&= 0,
\end{aligned}$$

where we have renumbered indices, used the Lemma 1 and the fact that σ is symmetric. In what follows, we shall use similar procedures without mentioning it explicitly.

2) Let us compute

$$\begin{aligned}
P(\alpha \otimes s) &:= \{\Sigma, Y\}(\alpha \otimes s) \\
&= Y(\sigma^i_j \epsilon^j \wedge \alpha \otimes e_i \cdot s) + \Sigma(\omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot s) \\
&= \sigma^i_j \omega^{kl} \iota_{e_k} (\epsilon^j \wedge \alpha) \otimes e_{li} \cdot s + \omega^{ij} \sigma^k_l \epsilon^l \wedge \iota_{e_i} \alpha \otimes e_{kj} \cdot s \\
&= \sigma^i_j \omega^{kl} (\delta_k^j \alpha - \epsilon^j \wedge \iota_{e_k} \alpha) \otimes e_{li} \cdot s + \omega^{ij} \sigma^k_l \epsilon^l \wedge \iota_{e_i} \alpha \otimes e_{kj} \cdot s \\
&= \sigma^{il} \alpha \otimes e_{li} \cdot s - \sigma^i_j \omega^{kl} \epsilon^j \wedge \iota_{e_k} \alpha \otimes e_{li} \cdot s + \omega^{ij} \sigma^k_l \epsilon^l \wedge \iota_{e_i} \alpha \otimes e_{kj} \cdot s \\
&= \sigma^{il} \alpha \otimes e_{li} \cdot s - \sigma^k_l \omega^{ij} \epsilon^l \wedge \iota_{e_i} \alpha \otimes e_{jk} \cdot s + \omega^{ij} \sigma^k_l \epsilon^l \wedge \iota_{e_i} \alpha \otimes e_{kj} \cdot s \\
&= \sigma^{il} \alpha \otimes e_{li} \cdot s - \omega^{ij} \omega_{kj} \sigma^k_l \epsilon^l \wedge \iota_{e_i} \alpha \otimes s \\
&= \sigma^{il} \alpha \otimes e_{li} \cdot s - \iota \sigma^i_j \epsilon^j \wedge \iota_{e_i} \alpha \otimes s.
\end{aligned}$$

Now, we use the derived prescription for P and the equation (1) to compute

$$\begin{aligned}
[P, 2\iota Y^2](\alpha \otimes s) &= 2\iota P Y^2(\alpha \otimes s) - 2\iota Y^2 P(\alpha \otimes s) \\
&= -P(\omega^{ij} \iota_{e_{ij}} \alpha \otimes s) - 2\iota Y^2(\sigma^{ij} \alpha \otimes e_{ji} \cdot s - \iota \sigma^i_j \epsilon^j \wedge \iota_{e_i} \alpha \otimes s) \\
&= -\omega^{ij} \sigma^{kl} \iota_{e_{ij}} \alpha \otimes e_{lk} \cdot s + \omega^{ij} \sigma^k_l \epsilon^l \wedge \iota_{e_{kij}} \alpha \otimes s \\
&\quad + \sigma^{ij} \omega^{kl} \iota_{e_{kl}} \alpha \otimes e_{ij} \cdot s - \iota \sigma^i_j \omega^{kl} \iota_{e_{kl}} (\epsilon^j \wedge \iota_{e_i} \alpha) \otimes s \\
&= -\omega^{ij} \sigma^{kl} \iota_{e_{ij}} \alpha \otimes e_{kl} \cdot s + \omega^{ij} \sigma^k_l \epsilon^l \wedge \iota_{e_{kij}} \alpha \otimes s \\
&\quad + \sigma^{ij} \omega^{kl} \iota_{e_{kl}} \alpha \otimes e_{ji} \cdot s - \omega^{kl} \sigma^i_j (\delta_l^j \iota_{e_{ki}} \alpha - \delta_k^j \iota_{e_{li}} \alpha + \epsilon^j \wedge \iota_{e_{kli}} \alpha) \otimes s \\
&= -\omega^{ij} \sigma^{kl} \iota_{e_{ij}} \alpha \otimes e_{kl} \cdot s + \omega^{ij} \sigma^k_l \epsilon^l \wedge \iota_{e_{kij}} \alpha \otimes s \\
&\quad + \sigma^{kl} \omega^{ij} \iota_{e_{ij}} \alpha \otimes e_{kl} \cdot s - \omega^{ij} \sigma^k_l \epsilon^l \wedge \iota_{e_{ijk}} \alpha \otimes s \\
&= 0.
\end{aligned}$$

3) Due to the definition of Θ , we have

$$\begin{aligned}
[X, \Theta](\alpha \otimes s) &= \epsilon^k \wedge \alpha \otimes \sigma^{ij} e_{kij}.s - \epsilon^i \wedge \alpha \otimes \sigma^{jk} e_{jki}.s \\
&= \epsilon^k \wedge \alpha \otimes \sigma^{ij} e_{kij}.s - \epsilon^k \wedge \alpha \otimes \sigma^{ij} e_{ijk}.s \\
&= \sigma^{ij} \epsilon^k \wedge \alpha \otimes (e_{ikj}.s - \omega_{kj} e_{j}.s - e_{ijk}.s) \\
&= \sigma^{ij} \epsilon^k \wedge \alpha \otimes (e_{ijk}.s - \omega_{kj} e_{i}.s - \omega_{ki} e_{j}.s - e_{ijk}.s) \\
&= 2i\Sigma(\alpha \otimes s).
\end{aligned}$$

4) This relation follows easily from the definition of Θ and the relation (1).
□

In the next proposition, a superset of the image of Σ and Θ restricted to \mathbf{E}^{ij} , for $(i, j) \in \Xi$, is determined.

Proposition 5: For $(i, j) \in \Xi$, we have

$$\begin{aligned}
\Sigma|_{\mathbf{E}^{ij}} &: \mathbf{E}^{ij} \rightarrow \mathbf{E}^{i+1, j-1} \oplus \mathbf{E}^{i+1, j} \oplus \mathbf{E}^{i+1, j+1} \text{ and} \\
\Theta|_{\mathbf{E}^{ij}} &: \mathbf{E}^{ij} \rightarrow \mathbf{E}^{i+1, j-1} \oplus \mathbf{E}^{i+1, j} \oplus \mathbf{E}^{i+1, j+1}.
\end{aligned}$$

Proof.

1) For $i = 0, \dots, l$, let us choose an element $\psi = \alpha \otimes s \in \mathbf{E}^{ii}$. Using the relation (3), we have $0 = [P, Y^2]\psi = (PY^2 - Y^2P)\psi = (\Sigma Y^3 + Y\Sigma Y^2 - Y^2\Sigma Y + Y^3\Sigma)\psi$. Because Y is \tilde{G} -equivariant (Lemma 3 item 1), decreasing the form degree of ψ by one and there is no summand isomorphic to \mathbf{E}_+^{ii} or \mathbf{E}_-^{ii} in $\bigwedge^{i-1} \mathbb{V}^* \otimes \mathbf{S}$ (Remark below the Theorem 2), $Y\psi = 0$. Using this equation, we see that the first three summands in the above expression for $[P, Y^2]$ are zero. Therefore we have $0 = Y^3\Sigma\psi$. Because Y is injective on \mathbf{E}^{ij} for $(i, j) \in \Xi_+$ (Lemma 3 item 2), we see that $\Sigma\psi \in \mathbf{E}^{i+1, i-1} \oplus \mathbf{E}^{i+1, i} \oplus \mathbf{E}^{i+1, i+1}$.

Now, let us consider a general $(i, j) \in \Xi$ and $\psi \in \mathbf{E}^{ij}$. Let us take an element $\psi' \in \mathbf{E}^{jj}$ such that $\psi = X^{(i-j)}\psi'$. This element exists because according to Lemma 3 item 2, the operator X is an isomorphism when restricted to \mathbf{E}^{ij} for $(i, j) \in \Xi_-$. Because of the relation (2), we have $\Sigma\psi = \Sigma X^{(i-j)}\psi' = \pm X^{(i-j)}\Sigma\psi'$. From the previous item, we know that $\Sigma\psi' \in \mathbf{E}^{j+1, j-1} \oplus \mathbf{E}^{j+1, j} \oplus \mathbf{E}^{j+1, j+1}$. Because X is \tilde{G} -equivariant (Lemma 3 item 1) and the only summands in $\bigwedge^{i+1} \mathbb{V}^* \otimes \mathbf{S}$ isomorphic to $\mathbf{E}^{j+1, j-1} \oplus \mathbf{E}^{j+1, j} \oplus \mathbf{E}^{j+1, j+1}$ are those described in the formulation of this proposition (see the Remark below the Theorem 2), the statement follows.

2) For $i = 0, \dots, l$, let us consider an element $\psi = \alpha \otimes s \in \mathbf{E}^{ii}$. Using the relation (5), we have $0 = [\Theta, Y^2]\psi = \Theta Y^2\psi + Y^2\Theta\psi$. Using similar reasoning to that one in the first item, we get $Y\psi = 0$. Using the expression for $[\Theta, Y^2]$ above, we get $Y^2\Theta\psi = 0$ and consequently, $\Theta\psi \in \mathbf{E}^{ii} \oplus \mathbf{E}^{i, i-1}$. Now, let us suppose $\psi \in \mathbf{E}^{ij}$ for $(i, j) \in \Xi$. There exists an element $\psi' \in \mathbf{E}^{jj}$ such that $\psi = X^{(i-j)}\psi'$ (Lemma 3 item 2). Using the relations (4) and (2), we have $\Theta\psi = \Theta X^{(i-j)}\psi' = X^{(i-j)}\Theta\psi'$ if $i - j$ is even and

$(X^{(i-j)}\Theta - 2iX^{(i-j-1)}\Sigma)\psi'$ if $i - j$ is odd. Using the fact $\Sigma|_{\mathbf{E}^{ij}} : \mathbf{E}^{ij} \rightarrow \mathbf{E}^{i+1,j-1} \oplus \mathbf{E}^{i+1,j} \oplus \mathbf{E}^{i+1,j+1}$, the statement follows by similar lines of reasoning as in the first item. \square

3 Metaplectic structures and symplectic curvature tensors

After we have finished the algebraic part of the paper, let us start describing the geometric structure we shall be investigating. We begin with a recollection of results of Vaisman in [18] and of Gelfand, Retakh and Shubin in [4]. Let (M, ω) be a symplectic manifold and ∇ be a symplectic torsion-free affine connection. By symplectic and torsion-free, we mean $\nabla\omega = 0$ and $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$ for all $X, Y \in \mathfrak{X}(M)$, respectively. Such connections are usually called Fedosov connections. In what follows, we shall call the triple (M, ω, ∇) Fedosov manifolds.

To fix our notation, let us recall the classical definition of the curvature tensor R^∇ of the connection ∇ , we shall be using here. Let

$$R^\nabla(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Let us choose a local symplectic frame $\{e_i\}_{i=1}^{2l}$ over an open subset $U \subseteq M$. We shall often write expressions in which indices i, j, k, l e.t.c. occur. We will implicitly mean i, j, k, l are running from 1 to $2l$ without mentioning it explicitly. We set

$$R_{ijkl} := \omega(R(e_k, e_l)e_j, e_i).$$

Let us mention that we are using the convention of Vaisman [18] which is different from that one used in Habermann, Habermann [8].

From the symplectic curvature tensor field R^∇ , we can build the symplectic Ricci curvature tensor field σ^∇ defined by the classical formula

$$\sigma^\nabla(X, Y) := \text{Tr}(V \mapsto R^\nabla(V, X)Y)$$

for each $X, Y \in \mathfrak{X}(M)$ (the variable V denotes a vector field on M). For the chosen frame and $i, j = 1, \dots, 2l$, we set

$$\sigma_{ij} := \sigma^\nabla(e_i, e_j).$$

Further, let us define

$$2(l+1)\tilde{\sigma}_{ijkl}^\nabla := \omega_{il}\sigma_{jk} - \omega_{ik}\sigma_{jl} + \omega_{jl}\sigma_{ik} - \omega_{jk}\sigma_{il} + 2\sigma_{ij}\omega_{kl}, \quad (6)$$

$$\begin{aligned} \tilde{\sigma}^\nabla(X, Y, Z, V) &:= \tilde{\sigma}_{ijkl} X^i Y^j Z^k V^l \text{ and} \\ W^\nabla &:= R^\nabla - \tilde{\sigma}^\nabla \end{aligned} \quad (7)$$

for local vector fields $X = X^i e_i$, $Y = Y^j e_j$, $Z = Z^k e_k$ and $V = V^l e_l$. We will call the tensor field $\tilde{\sigma}$ the extended symplectic Ricci curvature tensor field and

W^∇ the symplectic Weyl curvature tensor field. These tensor fields were already introduced in Vaisman [18]. We shall often drop the index ∇ in the previous expressions. Thus, we shall often write W , σ and $\tilde{\sigma}$ instead of W^∇ , σ^∇ and $\tilde{\sigma}^\nabla$, respectively.

In the next lemma, the symmetry of σ is stated.

Lemma 6: The symplectic Ricci curvature tensor field σ is symmetric.

Proof. See Vaisman [18]. \square

Let us start describing the geometric structure with help of which the action of the symplectic twistor operators are defined. This structure, called metaplectic, is a precise symplectic analogue of the notion of a spin structure in the Riemannian geometry. For a symplectic manifold (M^{2l}, ω) of dimension $2l$, let us denote the bundle of symplectic repers in TM by \mathcal{P} and the foot-point projection of \mathcal{P} onto M by p . Thus $(p : \mathcal{P} \rightarrow M, G)$, where $G \simeq Sp(2l, \mathbb{R})$, is a principal G -bundle over M . As in the subsection 2, let $\lambda : \tilde{G} \rightarrow G$ be a member of the isomorphism class of the non-trivial two-fold coverings of the symplectic group G . In particular, $\tilde{G} \simeq Mp(2l, \mathbb{R})$. Further, let us consider a principal \tilde{G} -bundle $(q : \mathcal{Q} \rightarrow M, \tilde{G})$ over the symplectic manifold (M, ω) . We call a pair (\mathcal{Q}, Λ) metaplectic structure if $\Lambda : \mathcal{Q} \rightarrow \mathcal{P}$ is a surjective bundle homomorphism over the identity on M and if the following diagram,

$$\begin{array}{ccc}
 \mathcal{Q} \times \tilde{G} & \longrightarrow & \mathcal{Q} \\
 \downarrow \Lambda \times \lambda & & \downarrow \Lambda \\
 \mathcal{P} \times G & \longrightarrow & \mathcal{P}
 \end{array}
 \begin{array}{c}
 \nearrow q \\
 \\
 \searrow p
 \end{array}
 \begin{array}{c}
 \\
 \\
 M
 \end{array}$$

with the horizontal arrows being respective actions of the displayed groups, commutes. See, e.g., Habermann, Habermann [8] and Kostant [11] for details on metaplectic structures. Let us only remark, that typical examples of symplectic manifolds admitting a metaplectic structure are cotangent bundles of orientable manifolds (phase spaces), Calabi-Yau manifolds and complex projective spaces $\mathbb{C}P^{2k+1}$, $k \in \mathbb{N}_0$.

Let us denote the vector bundle associated to the introduced principal \tilde{G} -bundle $(q : \mathcal{Q} \rightarrow M, \tilde{G})$ via the representation $meta$ on \mathbf{S} by \mathcal{S} . We shall call this associated vector bundle symplectic spinor bundle. Thus, we have $\mathcal{S} = \mathcal{Q} \times_{meta} \mathbf{S}$. The sections $\phi \in \Gamma(M, \mathcal{S})$, will be called symplectic spinor fields. Let us denote the space of symplectic valued exterior differential forms $\Gamma(M, \mathcal{Q} \times_\rho (\wedge^\bullet \mathbb{V}^* \otimes \mathbf{S}))$ by $\Omega^\bullet(M, \mathcal{S})$ and call it the space of symplectic spinor valued forms simply. Further for $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, we define the associated vector bundles \mathcal{E}^{ij} by the prescription $\mathcal{E}^{ij} := \mathcal{Q} \times_\rho \mathbf{E}^{ij}$.

Because the operators X, Y are \tilde{G} -equivariant (Lemma 3 item 1), they lift to operators acting on sections of the corresponding associated vector bundles. We shall use the same symbols as for the defined operators as for their "lifts" to the associated vector bundle structure. Because for each $i = 0, \dots, 2l$, the

decomposition $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S} \simeq \bigoplus_{j, (i,j) \in \Xi} \mathbf{E}^{ij}$ is multiplicity free (see the Remark below the Theorem 2), there exist uniquely defined projections $p^{ij} : \Omega^i(M, \mathcal{S}) \rightarrow \Gamma(M, \mathcal{E}^{ij})$, $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.

Now, let us suppose that (M, ω) is equipped with a Fedosov connection ∇ . The connection ∇ determines the associated principal bundle connection Z on the principal bundle $(p : \mathcal{P} \rightarrow M, G)$. This connection lifts to a principal bundle connection on the principal bundle $(q : \mathcal{Q} \rightarrow M, \tilde{G})$ and defines the associated covariant derivative on the symplectic bundle \mathcal{S} , which we shall denote by $\nabla^{\mathcal{S}}$ and call it the symplectic spinor covariant derivative. See Habermann, Habermann [8] for details. The symplectic spinor covariant derivative induces the exterior symplectic spinor derivative $d^{\nabla^{\mathcal{S}}}$ acting on $\Omega^\bullet(M, \mathcal{S})$. The curvature tensor field $R^{\Omega^\bullet(M, \mathcal{S})}$ acting on the symplectic spinor valued forms is given by the classical formula

$$R^{\Omega^\bullet(M, \mathcal{S})} := d^{\nabla^{\mathcal{S}}} d^{\nabla^{\mathcal{S}}}.$$

In the next theorem, a superset of the image of $d^{\nabla^{\mathcal{S}}}$ restricted to $\Gamma(M, \mathcal{E}^{ij})$, $(i, j) \in \Xi$, is determined.

Theorem 7: Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Then for the exterior symplectic spinor derivative $d^{\nabla^{\mathcal{S}}}$, we have

$$d^{\nabla^{\mathcal{S}}}_{|\Gamma(M, \mathcal{E}^{ij})} : \Gamma(M, \mathcal{E}^{ij}) \rightarrow \Gamma(M, \mathcal{E}^{i+1, j-1} \oplus \mathcal{E}^{i+1, j} \oplus \mathcal{E}^{i+1, j+1}),$$

where $(i, j) \in \Xi$.

Proof. See Krýsl [14]. \square

Remark: From the proof of the theorem, it is easy to see that it can be extended to the case (M, ω) is presymplectic and the symplectic connection ∇ has a non-zero torsion. For $l = 3$ and any $(i, j) \in \Xi$, the mappings $d^{\nabla^{\mathcal{S}}}$ restricted to $\Gamma(M, \mathcal{E}^{ij})$ are displayed as arrows at the Figure 1 above. (The exterior covariant derivative $d^{\nabla^{\mathcal{S}}}$ maps $\Gamma(M, \mathcal{E}^{ij})$ into the three "neighbor" subspaces.)

3.1 Curvature tensor on symplectic spinor valued forms and the complex of symplectic twistor operators

Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure (\mathcal{Q}, Λ) . In the next lemma, the action of $R^{\mathcal{S}} := d^{\nabla^{\mathcal{S}}} \circ \nabla^{\mathcal{S}}$ on the space of symplectic spinors fields is described using just the symplectic curvature tensor field R of ∇ .

Lemma 8: Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. Then for a symplectic spinor field $\phi \in \Gamma(M, \mathcal{S})$, we have

$$R^{\mathcal{S}} \phi = \frac{l}{2} R^{ij}_{kl} \epsilon^k \wedge \epsilon^l \otimes e_i \cdot e_j \cdot \phi.$$

Proof. See Habermann, Habermann [8] pp. 42. \square

For our convenience, let us set $m_i := i$ for $i = 0, \dots, l$ and $m_i := 2l - i$ for $i = l + 1, \dots, 2l$. Now, we can define the symplectic twistor operators, which we

shall need to introduce the mentioned complex. For $i = 0, \dots, 2l - 1$, we set

$$T_i : \Gamma(M, \mathcal{E}^{im_i}) \rightarrow \Gamma(M, \mathcal{E}^{i+1, m_{i+1}}), \quad T_i := p^{i+1, m_{i+1}} d_{|\Gamma(M, \mathcal{E}^{im_i})}^{\nabla^S}$$

and call these operators *symplectic twistor operators*. Informally, one can say that the operators are going on the edge of the triangle at the Figure 1. Let us notice that $Y(\nabla^S - T_0)$ is, up to a nonzero scalar multiple, the so called symplectic Dirac operator introduced by K. Habermann in [7].

Theorem 9: Let (M^{2l}, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. If $l \geq 2$ and the symplectic Weyl tensor field $W^\nabla = 0$, then

$$0 \longrightarrow \Gamma(M, \mathcal{E}^{00}) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^{11}) \xrightarrow{T_1} \dots \xrightarrow{T_{l-1}} \Gamma(M, \mathcal{E}^{ll}) \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow \Gamma(M, \mathcal{E}^{ll}) \xrightarrow{T_l} \Gamma(M, \mathcal{E}^{l+1, l+1}) \xrightarrow{T_{l+1}} \dots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l, 2l}) \longrightarrow 0$$

are complexes.

Proof.

1) In this item, we prove that for an element $\psi \in \Omega^\bullet(M, \mathcal{S})$,

$$R^{\Omega^\bullet(M, \mathcal{S})} \psi = \frac{l}{l+1} (lX^2\Theta^\sigma - X\Sigma^\sigma) \psi.$$

For $\psi = \alpha \otimes \phi \in \Omega^\bullet(M, \mathcal{S})$, we can write

$$\begin{aligned} R^{\Omega^\bullet(M, \mathcal{S})}(\alpha \otimes \phi) &= d^{\nabla^S} d^{\nabla^S}(\alpha \otimes \phi) = d^{\nabla^S}(d\alpha \otimes \phi + (-1)^{\deg(\alpha)} \alpha \wedge \nabla^S \phi) \\ &= d^2 \alpha \otimes \phi + (-1)^{\deg(\alpha)+1} d\alpha \wedge \nabla^S \phi + (-1)^{\deg(\alpha)} d\alpha \wedge \nabla^S \psi + \\ &\quad (-1)^{\deg(\alpha)} (-1)^{\deg(\alpha)} \alpha \wedge d^{\nabla^S} \nabla^S \phi = \alpha \wedge \frac{l}{2} R^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \otimes e_{ij} \cdot \phi \\ &= \frac{l}{2} R^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij} \cdot \phi, \end{aligned}$$

where we have used the Lemma 8. Using this computation, the definition of the symplectic Weyl curvature tensor field W^∇ (Eqn. (7)), the definition of the extended symplectic Ricci curvature tensor field $\tilde{\sigma}$ (Eqn. (6)) and the assumption $W^\nabla = 0$, we get

$$\begin{aligned} -4(l+1)lR^{\Omega^\bullet(M, \mathcal{S})}(\alpha \otimes \phi) &= 2(l+1)R^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij} \cdot \phi \\ &= 2(l+1)(W^{ij}{}_{kl} + \tilde{\sigma}^{ij}{}_{kl}) \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij} \cdot \phi \\ &= 2(l+1)\tilde{\sigma}^{ij}{}_{kl} \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij} \cdot \phi \\ &= (\omega^i{}_l \sigma^j{}_k - \omega^i{}_k \sigma^j{}_l + \omega^j{}_l \sigma^i{}_k - \omega^j{}_k \sigma^i{}_l + 2\sigma^{ij} \omega_{kl}) \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij} \cdot \phi \\ &= (4\omega^i{}_l \sigma^j{}_k \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij} \cdot \phi + 2\sigma^{ij} \omega_{kl}) \epsilon^k \wedge \epsilon^l \wedge \alpha \otimes e_{ij} \cdot \phi \\ &= 4lX^2(\alpha \otimes \sigma^{ij} e_{ij} \cdot \phi) - 4X(\sigma^j{}_k \epsilon^k \wedge \alpha \otimes e_j \cdot \phi) = (4lX^2\Theta^\sigma - 4X\Sigma^\sigma) \psi, \end{aligned}$$

where we have used the relation (1) in the second last step. Extending the result by linearity, we get the statement of this item for arbitrary $\psi \in \Omega^\bullet(M, \mathcal{S})$.

- 2) Using the derived formula for $R^{\Omega^\bullet(M,S)}$, the Proposition 5, the \tilde{G} -equivariance of X (Lemma 3 item 1) and the decomposition structure of $\bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ (see the Remark below the Theorem 2), we see that for $(i, j) \in \Xi$ and an element $\psi \in \Gamma(M, \mathcal{E}^{ij})$, the section $R^{\Omega^\bullet(M,S)}\psi \in \Gamma(M, \mathcal{E}^{i+2, j-1} \oplus \mathcal{E}^{i+2, j} \oplus \mathcal{E}^{i+2, j+1})$. Thus especially, $p^{i+2, m_i+2} R^{\Omega^\bullet(M,S)}\psi = 0$ for $i = 0, \dots, l-2, l, \dots, 2l-2$ and $\psi \in \Gamma(M, \mathcal{E}^{im_i})$. For $i = 0, \dots, l-2$, we get

$$\begin{aligned}
0 &= p^{i+2, i+2} R^{\Omega^\bullet(M,S)} = p^{i+2, i+2} d^{\nabla^S} d^{\nabla^S} \\
&= p^{i+2, i+2} d^{\nabla^S} (p^{i+1, 0} + \dots + p^{i+1, i+1}) d^{\nabla^S} \\
&= p^{i+2, i+2} d^{\nabla^S} p^{i+1, 0} d^{\nabla^S} + \dots + p^{i+2, i+2} d^{\nabla^S} p^{i+1, i+1} d^{\nabla^S} \\
&= T_{i+1} T_i,
\end{aligned}$$

where we have used the Theorem 7 in the last step. Similarly, one proceeds in the case $i = l, \dots, 2l-2$.

□

Corollary 10. Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. If $l \geq 2$ and the symplectic Weyl tensor field $W^\nabla = 0$, then

$$\begin{aligned}
0 &\longrightarrow \Gamma(M, \mathcal{E}^{00}) \xrightarrow{T_0} \dots \xrightarrow{T_{l-2}} \Gamma(M, \mathcal{E}^{l-1, l-1}) \xrightarrow{T_l T_{l-1}} \\
&\xrightarrow{T_i T_{i-1}} \Gamma(M, \mathcal{E}^{l+1, l+1}) \xrightarrow{T_{l+1}} \dots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l, 2l}) \longrightarrow 0
\end{aligned}$$

is a complex.

Proof. Follows easily from the Theorem 9. □

The question of the existence of a symplectic connection with vanishing symplectic Weyl curvature tensor field was treated, e.g., in Cahen, Gutt, Rawnsley [2]. These connections are called connections of Ricci type. For instance it is known that if a compact simply connected symplectic manifold (M, ω) admits a connection of Ricci type, then (M, ω) is affinely symplectomorphic to a $\mathbb{P}^n \mathbb{C}$ with the symplectic form, given by the standard complex structure and the Fubini-Study metric, and the Levi-Civita connection of this metric. Let us refer an interested reader to the paper of Cahen, Gutt, Schwachhöfer [3], where also a relation of symplectic connections to contact projective geometries is treated.

Further research could be devoted to the investigation and the interpretation of the cohomology of the introduced complex and to the investigation of analytic properties of the introduced symplectic twistor operators.

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