Decomposition of a tensor product of a higher symplectic spinor module and the defining representation of $\mathfrak{sp}(2n, \mathbb{C})$

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September 20, 2005

Abstract

Let $L(\lambda)$ be the irreducible highest weight module over the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$ with a highest weight λ, such that $L(\lambda)$ is with bounded multiplicities, and let $F(\varpi_1)$ be the defining representation of $\mathfrak{sp}(2n, \mathbb{C})$. In this article, the tensor product $L(\lambda) \otimes F(\varpi_1)$ is decomposed into irreducible summands explicitely.

Key words: decomposition of tensor products, symplectic modules with bounded multiplicities.

1 Introduction

Let $L(\lambda)$ stays for the irreducible highest weight module with a highest weight λ. We denote $L(\lambda)$ by $F(\lambda)$, if λ is dominant and integral weight, i.e., if and only if the corresponding module $L(\lambda)$ is finite dimensional.

The study of the tensor product $L(\lambda) \otimes F(\varpi_1)$ over complex symplectic Lie algebras, where λ is some nonintegral weight from a suitable set, which will be denoted by $\mathbb{A}$, and ϖ_1 is the highest weight of the defining representation, was motivated by an interest in some first order invariant differential operators.

These operators are acting between sections of vector bundles over projective contact geometries. The mentioned vector bundles are associated to these geometries via the so called higher symplectic spinor representations $L(\lambda)$, $(\lambda \in \mathbb{A})$. To explain the motivation, let us mention that the projective contact geometries belong to Cartan geometries defined by a contact grading

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of the tangent bundle and a projective class of partial affine connections. In physics, these geometries play a role of a phase-space of the time dependent Hamiltonian mechanics.

The process of classifying of the invariant differential operators involves (on the infinitesimal level at least) the decomposition of the mentioned tensor product $L(\lambda)\otimes F(\varpi_1)$ as its first step. The higher symplectic spinor modules, to which the bundles are associated are "symplectic" analogues of spinor modules over orthogonal Lie algebras as it is mentioned in Kostant [13]. In [1], Britten, Hooper and Lemire and in [2] Britten and Hooper, described the decomposition of $L(\lambda_i) \otimes F(\nu)$ for $i = 0, 1$, where ν is a dominant integral weight, $\lambda_0 = -\frac{1}{2}\varpi_n$ and $\lambda_1 = \varpi_{n-1} - \frac{3}{2}\varpi_n$, i.e., λ_i are the highest weights of the so called basic symplectic spinor modules $L(\lambda_i)$. In [1] and [2], a characterization of all infinite dimensional symplectic modules with bounded multiplicities is given.

In this article, we study a problem, which is complementary to that one of Britten, Hooper and Lemire in a sense. Namely, we describe the decomposition of a tensor product $L(\lambda) \otimes F(\varpi_1)$ of an arbitrary infinite dimensional module with bounded multiplicities $L(\lambda)$, $\in \mathbb{A}$, and the defining representation $F(\varpi_1)$ of the symplectic Lie algebra. The techniques, which are used to decompose the mentioned tensor product, are based on a result of Humphreys in [6] on a formula for formal character of tensor products of an irreducible highest weight module and an irreducible finite dimensional module over simple complex Lie algebras. The assumption, under which this formula is valid is the same as that one of Kostant, see [12], to which it goes back. The second ingredient we have used is the famous Kac-Wakimoto formula in Kac, Wakimoto [9], which was published for complex simple Lie algebras in Jantzen, see [7], earlier, but which is valid for slightly different set of weights.

One of the invariant differential operators serving the motivation of our paper appeared already in Kostant [13] and is known as the Kostant's Dirac operator. Analytical and geometrical aspects of the Kostant's Dirac operator were studied by many authors, see, e.g., Habermann [4] and Klein [11]. Other operators from the set related to our tensor product decomposition, involving, e.g., the so called symplectic twistor and symplectic Rarita-Schwinger (each having its corresponding orthogonal counterparts) were also studied, e.g., by Kadlčáková [10]. Let us also mention that the study of the corresponding first order differential operators has its application in theoretical physics, namely in the 10 dimensional super string theory, see Green, Hull [3], and the theory of Dirac-Kähler fields, see, e.g., Reuter [14].

In the second section of this article, some known results on formal characters of irreducible highest weight modules (theorem 1), decomposition of tensor products (theorem 2, 3) and formal character of a tensor product (theorem 4) are presented. The second part contains also a lemma 1, in which the theorem 1 is adapted to the situation we are studying. The third part of this paper is devoted to the formulation of the decomposition of $L(\lambda) \otimes F(\varpi_1)$ for $\lambda \in \mathbb{A}$ and to its proof.

2 Tensor products and higher symplectic modules

2.1 Tensor products decompositions

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n and let $(\,,)$ denote the Killing form of $\mathfrak{g}$. For the choice of a Cartan subalegbra $\mathfrak{h} \subseteq \mathfrak{g}$ and of a system of positive roots Φ^+ of $\mathfrak{g}$, there is a set of fundamental weights, which will be denoted by $\{\varpi_i\}_{i=1}^n$. The set of roots determines its $\mathbb{R}$-linear span, denoted by $\mathfrak{h}_0^*$. Having chosen the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we can fix an orthonormal basis $\{\epsilon_i\}_{i=1}^n$ of $\mathfrak{h}_0^*$ wr. respect of the restriction of the Killing form $(\,,)$ to the subspace $\mathfrak{h}_0^* \times \mathfrak{h}_0^*$. Let $\{e_i\}_{i=1}^n \subseteq \mathfrak{h}_0$ be the dual basis to the orthonormal basis $\{\epsilon\}_{i=1}^n$. With help of the Killing form on $\mathfrak{g}$, we can introduce a mapping $\langle\,,\rangle : \mathfrak{h}_0^* \times (\mathfrak{h}_0^* - \{0\}) \to \mathbb{R}$ by the following equation

$$\langle v, w \rangle := 2\frac{(v,w)}{(w,w)},$$

for $v \in \mathfrak{h}_0^*$, $w \in \mathfrak{h}_0^* - \{0\}$. Further, let us denote the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$ by $\mathcal{W}$. The determinant of an element $\sigma \in \mathcal{W}$ is denoted by $\epsilon(\sigma)$. If $\lambda \in \mathfrak{h}^*$ then by $\mathcal{W}^\lambda$ we mean a subgroup of the Weyl group $\mathcal{W}$ generated by reflections in planes perpendicular to such simple roots γ, for which $\langle \lambda, \gamma \rangle \in \mathbb{Z}$. Further, let us denote the affine action of a Weyl group element by a dot, thus $\sigma.\lambda := \sigma(\lambda + \delta) - \delta$ is an affine action of an element $\sigma \in \mathcal{W}$ on $\lambda \in \mathfrak{h}^*$. For $\lambda, \mu \in \mathfrak{h}^*$, let us denote $\lambda \sim \mu$, if there is an element $\sigma \in \mathcal{W}$ such that $\sigma.\lambda = \mu$. We will call such weights linked to each other. The sum of the fundamental weights is denoted by $\delta := \sum_{i=1}^n \varpi_i$. Let us denote by R_+ the set of positive coroots and by R_+^λ the set $\{X \in R_+; \lambda(X) \in \mathbb{Z}\}$, for some $\lambda \in \mathfrak{h}^*$. Further, denote the basis of the root system R^λ by $B^\lambda (\subseteq R_+^\lambda)$.

For a complex simple Lie algebra $\mathfrak{g}$, let $L(\lambda)$ be the irreducible highest weight module over $\mathfrak{g}$ with a highest weight λ and $M(\lambda)$ be the Verma module of a highest weight λ. To stress, that λ is integral and dominant for a choice of $(\mathfrak{h}, \Phi^+)$, i.e., the corresponding module $L(\lambda)$ is finite dimensional, we will denote it by $F(\lambda)$ or simply by F, if the highest weight is not important or clear from the context. Let $\Pi(\lambda)$ be the set of all weights of the module $L(\lambda)$ and $n(\nu)$ be the multiplicity of a weight $\nu \in \Pi(\varpi_1)$. For $\lambda \in \mathfrak{h}^*$, the symbol

L_λ stays for the weight space of the weight λ in a highest weight module L. Further, let us denote the formal character of a highest weight module L by $ch\ L$. The central character corresponding to a weight λ is denoted by χ_λ, i.e., for an element $Z \in \mathfrak{Z} := \mathfrak{cen}(\mathfrak{U}(\mathfrak{g}))$ of the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ one has $z.v = \chi_\lambda(z)v$, for an element v of a highest weight module with a highest weight λ.

Let L be an $\mathfrak{h}$-diagonalizable module over the complex semisimple algebra $\mathfrak{g}$. We call L a module with bounded multiplicies, if there is $k \in \mathbb{N}$ such that $\dim V_\lambda \leq k$ for all weights λ of the module L. The minimal such k is called degree of L. We call a module with bounded multiplicities completely pointed provided its degree is 1. Let us mention, that the basic symplectic spinor modules $L(\lambda_i)$, $i = 0, 1$ are completely pointed and these are the only ones among infinite dimensional irreducible highest weight modules over the symplectic Lie algebra, see Britten, Hooper, Lemire [1].

There is a result on a formal character of an irreducible highest weight module over a complex semisimple algebra. In this theorem, the formal characters of Verma modules $M(\sigma.\lambda)$ for some Weyl group elements σ are related to the formal character of the irreducible module $L(\lambda)$.

Theorem 1. Let $\lambda \in \mathfrak{h}^*$ be such that $(\lambda + \delta)\check{\alpha} > 0$ for all $\check{\alpha} \in B^\lambda$. Then we have

$$ch\ L(\lambda) = \sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.\lambda).$$

Proof. See Kac, Wakimoto [9], Theorem 1, pp. 4957. □

A version of the previous theorem appeared in Jantzen [7], Theorem 2.23, pp. 70 but for a slightly different set of weights. We will refer to the formula in the preceding theorem as Kac-Wakimoto formal character formula.

In the next theorem a decomposition of a tensor product of an irreducible highest weight module possibly of infinite dimension and a finite dimensional irreducible module into irreducible summands is described, for further comments see Humphreyes [6], pp. 1 - 64.

Theorem 2. Let F be a finite dimensional module over a complex semisimple Lie algebra $\mathfrak{g}$, $L(\lambda)$ an irreducible highest weight module with a highest weight λ, then one has a canonical decomposition $F \otimes L(\lambda) = M^{(1)} \oplus \ldots \oplus M^{(t)}$, where $M^{(i)}$ is the generalized eigenspace corresponding to $\chi_{\lambda+\mu_i}$, where μ_i runs over a subset of the weights of F, so that the indicated central characters are distinct.

Proof. See Humphreys [6], sect 4.4. and pp. 39. □

Let us recall the famous Harish-Chandra theorem, which says that $\chi_\lambda = \chi_\mu$, if and only if $\lambda \sim \mu$. In the next theorem, the generalized eigenspaces are specified more precisely.

Theorem 3. Keep the above notation. Suppose $\mu := \mu_i$ is a weight of F such that for all weights $\nu \neq \mu$ of F, $\lambda + \nu$ and $\lambda + \mu$ are not linked to each other. Then $M := M^{(i)}$ is a direct sum of n copies of $L(\lambda + \mu)$, where $n = \dim\, M_{\lambda+\mu}$.

Proof. See Humphreys [6], sect. 6.3., pp. 40. $\square$

In the next theorem, the formal character of the generalized eigenspace is related to formal characters of some Verma modules and to multiplicities of corresponding weights of the finite dimensional module F.

Theorem 4. Keep the above notation and denote by $n(\mu)$ the multiplicity of the weight μ in the irreducible finite dimensional module F. Then

$$n(\mu) \sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.(\lambda + \mu)) = n\, ch\, L(\lambda + \mu).$$

Proof. See Humphreys, [6] sect. 6.4., pp. 42 and use the Kac-Wakimoto formal character formula in the substitution for $a(w, \lambda)$ from Humphreys. $\square$

2.2 The case of $\mathfrak{sp}(2n, \mathbb{C})$ and higher symplectic spinor modules

In this subsection, we focus our attention to a specific Lie algebra $\mathfrak{g}$, namely to the symplectic Lie algebra, i.e., $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})(= C_n)$, and to a distinguished class of infinite dimensional irreducible highest weight modules. First, let us describe the modules we shall be dealing with.

Definition 1: Let us denote the set of weights

$$\{\lambda = \sum_{i=1}^{n} \lambda_i \varpi_i; \lambda_i \geq 0, \lambda_n \in \mathbb{Z} + \frac{1}{2}, \lambda_{n-1} + 2\lambda_n + 3 > 0\}$$

by $\mathbb{A}$. We will call the modules $L(\lambda)$ for $\lambda \in \mathbb{A}$ *higher symplectic spinor modules.*

Theorem 5. The following are equivalent

1.) $L(\lambda)$ is a higher symplectic spinor module, i.e., $\lambda \in \mathbb{A}$,

2.) $L(\lambda)$ has bounded multiplicities,

3.) $L(\lambda)$ is equivalent to a direct summand of the tensor product $L(-\frac{1}{2}\varpi_n) \otimes F(\varpi_1)$ for some choice of dominant integral weight ν.

Proof. See Britten, Lemire [2], theorem 2.1 pp. 3417 and theorem 1.2. pp. 3415. $\square$

Let us stress, that in the case of $\mathfrak{g} = C_n$ one has the formula $\varpi_i = \sum_{j=1}^{i} \epsilon_j$, for $i = 1, \ldots, n$, and that one can easily compute that the set $\Pi(\varpi_1)$ of weights of $F(\varpi_1)$ equals $\{\pm\epsilon_i; i = 1, \ldots, n\}$.

In the next lemma, the theorem 1 is adapted to the situation we are studying.

Lemma 1. Let $\nu \in \Pi(\varpi_1)$ and $\lambda, \lambda + \nu \in \mathbb{A}$, then

$$ch\ L(\lambda + \nu) = \sum_{\sigma \in \mathcal{W}^\lambda} \epsilon(\sigma) ch\ M(\sigma.(\lambda + \nu))$$

Proof. We must check whether the assumption of theorem 1 is satisfied. At first, we determine the set $R_+^{\lambda+\nu}$ for $\nu \in \Pi(\varpi_1)$ and $\lambda, \lambda + \nu \in \mathbb{A}$. Looking at the definition of the set $R_+^{\lambda+\nu}$, we easily obtain that

$$R_+^{\lambda+\nu} = \{e_i + e_j, 1 \leq i \leq j < n\} \cup \{e_i - e_j, 1 \leq i < j < n\} \cup \{e_k, 1 \leq k < n\}.$$

The basis $B^{\lambda+\nu}$ of $R_+^{\lambda+\nu}$ is

$$B^{\lambda+\nu} = \{e_i - e_{i+1}, 1 \leq i \leq n-2\} \cup \{e_{n-1}\}.$$

Secondly, we need to compute $(\lambda+\nu)\check{\alpha}$ for $\check{\alpha} \in B^{\lambda+\nu}$. Suppose that $\nu = t\epsilon_p$ for some $p = 1, \ldots, n$ and $t \in \{-1, 1\}$.

1.) $A := (\lambda + \nu + \delta)(e_i - e_{i+1}) = [\sum_{r=1}^{n}(\sum_{s=r}^{n} \lambda_s + n - r + 1 + t\delta_{rp})\epsilon_r](e_i - e_{i+1}) = \lambda_i + 1 + t(\delta_{ip} - \delta_{i+1,p})$, $i = 1, \ldots, n-2$. We know that $\lambda + \nu \in \mathbb{A}$, from which it follows that $\lambda_i + t(\delta_{ip} - \delta_{i,p-1}) \geq 0$ for $i = 1, \ldots, n-1$,, because $\epsilon_p = \varpi_p - \varpi_{p-1}, p = 1, \ldots, n$, where $\varpi_0 = 0$ and $\delta_{i,-1} := 0$ for $i = 1, \ldots, n$. are to be understood. Thus the condition $A > 0$, we have had to check, is satisfied.

2.) $B := (\lambda + \nu + \delta)(e_{n-1}) = [\sum_{r=1}^{n}(\sum_{s=r}^{n} \lambda_s + n - r + 1 + t\delta_{rp})\epsilon_r](e_{n-1}) = \lambda_{n-1} + \lambda_n + 2 + t\delta_{n-1,p}$. If $\lambda_n > 0$, then the inequality $B > 0$ is evidently satisfied. Now suppose, that $\lambda_n \leq -\frac{1}{2}$. If $p = n-1$, then using the inequality $\lambda_{n-1} + 2\lambda_n + 3 + t \geq 1$ and $\lambda_n \leq -\frac{1}{2}$, one obtains, that $\lambda_{n-1} + \lambda_n + \frac{1}{2} + t \geq 0$, from which $B > 0$ easily follows. Suppose that $p = n$ and $t = -1$, then using the inequality $\lambda_{n-1} + 2\lambda_n + 3 - 2 \geq 1$ and $\lambda_n \leq -\frac{1}{2}$, one obtains, that $\lambda_{n-1} + \lambda_n + \frac{1}{2} \geq 0$, from which $B > 0$ follows. Now, suppose that $p = n$ and $t = 1$, then using the inequality $\lambda_{n-1} + 2\lambda_n + 3 \geq 1$ and $\lambda_n \leq -\frac{1}{2}$, one gets that $\lambda_{n-1} + \lambda_n + \frac{3}{2} \geq 0$ from which $B > 0$ follows, too.

Thus, we have proved that the assumption of the theorem 1 is satisfied and therefor the conclusion of this lemma follows. $\square$

3 Decomposition of $L(\lambda)\otimes F(\varpi_1)$ for $\lambda\in\mathbb{A}$

Theorem 6: Let $L(\lambda)$ be a higher symplectic spinor module, i.e., $\lambda\in\mathbb{A}$. Then

$$L(\lambda)\otimes F(\varpi_1)=\bigoplus_{\mu\in\mathbb{A}_\lambda}L(\mu),$$

where $\mathbb{A}_\lambda=\{\lambda+\nu;\nu\in\Pi(\varpi_1)\}\cap\mathbb{A}$.

Proof. **Part I.** We would like to use theorem 3. We shall actually prove that $\lambda+\mu$ and $\lambda+\nu$, for $\mu,\nu\in\Pi(\varpi_1)$ are not conjugated by the affine action of an element of the Weyl group $\mathcal{W}$ of the algebra C_n if $\nu\neq\mu$. Two elements $\phi,\psi\in\mathfrak{h}^*$ are conjugated by the affine action of an element of the Weyl group if and only if $\phi+\delta$ and $\psi+\delta$ are conjugated by an element of the Weyl group, i.e., if and only if $\sigma(\phi+\delta)=\psi+\delta$, for some $\sigma\in\mathcal{W}$. We prove that $\{\lambda+\nu+\delta,\lambda+\mu+\delta\}\subseteq\overline{W_1}\cup\overline{W_2}$, where

$$W_1:=\{\sum_{i=1}^n\beta_i\epsilon_i;\beta_1>\ldots>\beta_n>0\},$$

$$W_2:=\{\sum_{i=1}^n\beta_i\epsilon_i;\beta_1>\ldots>\beta_{n-1}>-\beta_n>0\}$$

are two open neighbor Weyl chambers of C_n. Let us denote $\mu=s\epsilon_p$ for $s\in\{-1,1\}$ and some $p=1,\ldots,n$. In the case of C_n the element $\delta=n\epsilon_1+(n-1)\epsilon_2+\ldots+\epsilon_n$. Using the relation $\varpi_j=\sum_{i=1}^j\epsilon_i$ for $j=1,\ldots,n$, one easily computes that

$$\lambda+\mu+\delta=:\sum_{i=1}^n\beta_i\epsilon_i=\sum_{i=1}^n[(\sum_{j=i}^n\lambda_j)+n-i+1+s\delta_{ip}]\epsilon_i$$

for $\lambda=\sum_{i=1}^n\lambda_i\varpi_i$. Thus the requirement $\lambda+\mu+\delta\in\overline{W_1}$ reduces to $\lambda_i+1\geq s(\delta_{i+1,p}-\delta_{ip})$ which is evidently satisfied for all $i=1,\ldots,n-1$, see the definition 1. For $i=n$, the condition we need to check is $\beta_n\geq 0$ or $\beta_{n-1}\geq-\beta_n\geq 0$. If $\beta_n\geq 0$, we are ready. Suppose $\beta_n<0$, then the remaining condition we need to check is

$$\lambda_{n-1}+2\lambda_n+3+s(\delta_{n-1,p}+\delta_{np})\geq 0,\qquad (1)$$

because $-\beta_n\geq 0$ under our assumption. Condition (1) is satisfied because of the last inequality in the definition 1 of higher symplectic spinor modules.

Suppose that $\lambda+\mu+\delta$ and $\lambda+\nu+\delta$ are conjugated by an element σ of the Weyl group of C_n, i.e., $\sigma(\lambda+\mu+\delta)=\lambda+\nu+\delta$.

(1) Suppose that $\lambda+\mu+\delta \in W_1$ and $\lambda+\nu+\delta \in W_2$ (or $\lambda+\mu+\delta \in W_2$ and $\lambda+\nu+\delta \in W_1$, which is analogous). The condition $\sigma(\lambda+\mu+\delta) = \lambda+\nu+\delta$ implies $\sigma W_1 = W_2$. It is evident that $\sigma_{\epsilon_n} W_1 = W_2$. The Weyl group acts simply transitively on the set of open (or closed) Weyl chambers. Hence $\sigma = \sigma_{\epsilon_n}$. Although ϵ_n does not belong to the system of simple roots, it is evident that we could have written $\sigma_{2\epsilon_n}$ instead of σ_{ϵ_n}. Now, $\sigma_{\epsilon_n}(\lambda+\mu+\delta) = \lambda+\mu+\delta-2(\epsilon_n, \lambda+\mu+\delta)\epsilon_n = \lambda+\mu+\delta-2(\lambda_n+s\delta_{np}+1)\epsilon_n$. This element equals $\lambda+\nu+\delta$ if and only if $\mu-\nu = 2(\lambda_n+s\delta_{np}+1)\epsilon_n$ which is impossible due to the structure of the set $\Pi(\varpi_1)$ and the condition $\lambda_n \in \mathbb{Z}+\frac{1}{2}$.

(2) The case $\lambda+\mu+\delta, \lambda+\nu+\delta \in W_i$ and $\sigma(\lambda+\mu+\delta) = \lambda+\nu+\delta$ for $i = 1, 2$ leads to the condition $\sigma = id$, i.e., $\nu = \mu$ - a contradiction.

(3) The remaining case is $\lambda+\mu+\delta, \lambda+\nu+\delta \in \overline{W_1} \cup \overline{W_2} - (W_1 \cup W_2)$, i.e., the considered elements lie on the walls of the two Weyl chambers. (The other cases are impossible: if there is an element lying on a wall of a closed Weyl chamber and an another one is lying in the open Weyl chamber, then they cannot be conjugated.) The inspection of the fact $\lambda+\mu+\delta, \lambda+\nu+\delta \in \overline{W_1} \cup \overline{W_2}$ showed that if these elements lie on the walls of $\overline{W_1}$ and $\overline{W_2}$, then they lie in their interior (i.e., they do not lie on the walls of codimension 2): the inequations $\beta_j \geq \beta_{j+1}$, for $j = 1, \ldots, n-1$ happen equations exactly for one $j \in \{1, \ldots, n-1\}$. Let us define two families of open Weyl chambers

$$Y_r := \{\sum_{i=1}^{n} \beta_i\epsilon_i; \beta_1 > \ldots > \beta_{r-1} > -\beta_r > \beta_{r+1} > \ldots > \beta_n > 0\},$$

$r = 1, \ldots, n-1$ and

$$Y'_t := \{\sum_{i=1}^{n} \beta_i\epsilon_i; \beta_1 > \ldots > \beta_{t-1} > -\beta_t > \beta_{t+1} > \ldots > -\beta_n > 0\},$$

$t = 1, \ldots, n-1$.

(3.1) Suppose that $\lambda+\mu+\delta \in \overline{W_1} \cap \overline{Y_r}$ and $\lambda+\nu+\delta \in \overline{W_2} \cap \overline{Y'_t}$ for some $r, t = 1, \ldots, n-1$. If we suppose that $\sigma(\lambda+\mu+\delta) = \lambda+\nu+\delta$, then the fact that these elements lie in the interior of the walls implies that $\sigma W_1 = W_2$ or $\sigma W_1 = Y'_t$. The first case leads to a contradiction as we have shown. Using the fact that the Weyl group acts simply transitively, we easily find that $\sigma = \sigma_{\epsilon_t}\sigma_{\epsilon_n}$ in the second case. Let us compute $\sigma_{\epsilon_t}\sigma_{\epsilon_n}(\lambda+\mu+\delta) = \lambda+\mu+\delta -$

$2(\epsilon_t, \lambda+\mu+\delta)\epsilon_t - 2(\epsilon_n, \lambda+\mu+\delta)\epsilon_n = \lambda+\mu+\delta-2(\lambda_t+s\delta_{pt}+n-t+1)\epsilon_t - 2(\lambda_n+s\delta_{pn}+1)\epsilon_n$. This element equals $\lambda+\nu+\delta$ if and only if $\mu-\nu = 2(\lambda_t+s\delta_{pt}+n-t+1)\epsilon_t + 2(\lambda_n+\delta_{pt}+1)\epsilon_n$. Because of the structure of $\Pi(\varpi_1)$, we obtain: $\mu-\nu \in \{\pm 2\epsilon_t, \pm 2\epsilon_n, \pm\epsilon_t \pm \epsilon_t, \pm\epsilon_t \mp \epsilon_n\}$. The first possibility leads to $0 = \lambda_n + s\delta_{np}+1$, which is impossible because λ_n is half-integral. The second possibility implies $0 = \lambda_t + s\delta_{tp} + n - t + 1 \geq \lambda_t + n - t > 0$ - a contradiction. The third and fourth possibilities force $\pm 1 = 2(\lambda_t + s\delta_{tp} + n - t + 1)$ - an odd number equals an even one, i.e., a contradiction.

(3.2) Suppose that $\lambda+\mu+\delta \in \overline{W_1} \cap \overline{Y_r}$ and $\lambda+\nu+\delta \in \overline{W_1} \cap \overline{Y_t}$. In this case, $\sigma W_1 = W_1$ or $\sigma W_1 = Y_t$. The first case leads to a contradiction as we already know. In the second case, one easily finds that $\sigma_{\epsilon_t} W_1 = Y_t$, i.e., using the simplicity of the Weyl group action, this implies $\sigma = \sigma_{\epsilon_t}$. Let us compute $\sigma_{\epsilon_t}(\lambda+\mu+\delta) = \lambda+\mu+\delta-2(\lambda_t+s\delta_{pt}+n-t+1)\epsilon_t$. This element equals $\lambda+\nu+\delta$ if and only if $\{\mu,\nu\} = \{\epsilon_t, -\epsilon_t\}$, i.e., $\mu-\nu = \pm 2\epsilon_t$. That means that $1 = \lambda_t + 1 + n - t + 1$ or $-1 = \lambda_t - 1 + n - t + 1$ which are impossible because $\lambda_t \geq 0$ and $t < n$ for $t = 1, \ldots, n-1$.

(3.3) The remaining cases are analogous to the previous ones and actually have been done.

Part II. Summarizing the part 1 of the proof, we have proved that the condition of the theorem 3 is satisfied, and therefor in the canonical decomposition $L(\lambda) \otimes F(\varpi_1) = M^{(1)} \oplus \ldots \oplus M^{(t)}$, we have $M^{(i)} = n_i L(\lambda+\nu_i)$, for all weights $\nu_i \in \Pi(\varpi_1)$, for which $\nu_i + \lambda \in \mathbb{A}$, and $\nu_i \neq \nu_j$, if $i \neq j$, further n_i is some nonnegative integer. Using the theorem 4, we get $n(\nu_i)\sum_{\sigma\in\mathcal{W}^\lambda}\epsilon(\sigma)ch\ M(\sigma.(\lambda+\nu_i)) = n_i chL(\lambda+\nu_i)$. Because we know, that $n(\nu_i) = 1$ for all weights $\nu_i \in \Pi(\varpi_1)$, we get

$$\sum_{\sigma\in\mathcal{W}^\lambda}\epsilon(\sigma)ch\, M(\sigma.(\lambda+\nu_i)) = n_i ch\, L(\lambda+\nu_i).$$

Using the formal character formula of Kac and Wakimoto from lemma 1, we get $ch\, L(\lambda+\nu_i) = n_i ch\, L(\lambda+\nu_i)$, which implies $n_i = 1$ for such ν_i for which $\lambda+\nu_i \in \mathbb{A}$. From theorem 3, we know that if $L(\mu)$ appears in the decomposition of $L(\lambda)\otimes F(\nu)$, then $\mu = \lambda+\eta$, where $\eta \in \Pi(\varpi_1)$. Till yet, we have shown that $\mu \in (\lambda+\Pi(\varpi_1)) \cap \mathbb{A} = \mathbb{A}_\lambda$ occurs in the decomposition, we are interested in, with multiplicity 1. The remaining question is, whether a weight from $\lambda+\Pi(\varpi_1) - \mathbb{A}$ may occur in the decomposition. But this is not possible, because the highest weight μ of an irreducible summand $L(\mu)$ of the decomposition lies in the set $\mathbb{A}$, see theorem 5 (3. $\Rightarrow$ 1.). $\square$

I am very grateful to Vladimír Souček who helped me orientating in this topic and also to James Humphreys, who recommended me his text [6] for reading. The author acknowledges the support by grant of

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