SYMPLECTIC SPINOR VALUED FORMS AND INVARIANT OPERATORS ACTING BETWEEN THEM

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Abstract. Exterior differential forms with values in the (Kostant's) symplectic spinor bundle on a manifold with a given metaplectic structure are decomposed into invariant subspaces. Projections to these invariant subspaces of a covariant derivative associated to a torsion-free symplectic connection are described.

1. Introduction

While the spinor twisted de Rham sequence for orthogonal spin structures is well understood from the point of view of representation theory (see, e.g., Delanghe, Sommen, Souček [4]), its symplectic analogue seems to be untouched till present days. In Riemannian geometry, a decomposition of spinor twisted de Rham sequence (i.e., exterior differential forms with values in basic spinor bundles) into invariant parts is well known. Suppose a principal connection on the frame bundle of orthogonal repers (of the tangent bundle) is given. It induces in a canonical way a covariant derivative on differential forms with values in the basic spinor bundles. In this case, it is known, which parts of the covariant derivatives acting between the spinor bundle valued forms are zero if we restrict it to an invariant part of the sequence. Namely, the covariant derivative maps each invariant part only in at most three invariant parts sitting in the next gradation (some degeneracies on the ends of the sequence could be systematically described). In symplectic geometry, the first question which naturally arises is, what are the spinors for a symplectic Lie algebra. This question was successfully answered by Bertram Kostant in [14]. He offered a candidate for symplectic spinors. We will call these spinors *basic symplectic spinors* and denote their underlying vector spaces $\mathbb{S}_+$ and $\mathbb{S}_-$. They are analogous to the ordinary orthogonal spinors in at least two following ways.

> First, they could be found in a *symmetric algebra* of an isotropic subspace, while the orthogonal spinors could be found in an *exterior algebra* of certain isotropic subspace.

The paper is in final form and no version of it will be submitted elsewhere.

Second, the highest weights of the basic symplectic spinors are also half integral like the highest weights of orthogonal spinors (both with respect to the usual basis of the dual of an appropriate Cartan subalgebra).

Unlike the orthogonal spinors, the symplectic ones are of infinite dimension, thus not so easy to handle.

The main results of this article are the decomposition of the spinor twisted de Rham sequence in the symplectic case and a theorem, which says that the image of each covariant derivative (associated to a symplectic torsion-free connection and restricted to an invariant subspace) lies in at most three invariant subspaces, i.e., a similar theorem to that one, which is valid in Riemannian geometry. To derive the first mentioned result, we need to decompose the ordinary exterior forms into irreducible summands over the symplectic Lie algebra – a procedure, which is well known. We also need to know, how to decompose a tensor product of such irreducible summand and the basic symplectic spinor module $\mathbb{S}_+$. This was done by Britten, Lemire, Hooper in [2] and by Britten, Lemire in [3] even in a more general setting. To derive the second result (the description of the image of the covariant derivative), we need one more ingredient. In particular, we should know, how to decompose a tensor product of the defining representation $\mathbb{V}$ of the symplectic Lie algebra and each infinite-dimensional representation of the symplectic Lie algebra, which is an irreducible summand in the space $\bigwedge^i \mathbb{V} \otimes \mathbb{S}$. These irreducible summands belong to a broader class of infinite dimensional modules over a symplectic algebra, so called *higher symplectic spinor modules*, which are also known as harmonic spinor representations in the literature. To describe the decomposition of the tensor product of the defining representation and the mentioned irreducible summand, we shall use a theorem which was derived by the author in [15].

Investigation of the decomposition of the twisted de Rham complex for metaplectic structures has been motivated by a search for symplectic analogues of an (orthogonal) Dirac operator and its generalizations, which naturally appear in the twisted de Rham sequence in the orthogonal setting. Namely, it is known that the Dirac, Rarita-Schwinger and twistor operators could be found in the twisted de Rham sequence for an orthogonal spin structure. The symplectic Dirac operator has been found by B. Kostant, see [14], and and has been studied intensively by many authors, see, e.g., Habermann [7], Klein [12] and Kadlčáková [10]. We will recover all of these operators (Dirac, Rarita-Schwinger and twistor) in a more systematic way by an investigation of invariant differential operators appearing in the symplectic spinor twisted de Rham sequence for metaplectic structures. No definition of the symplectic Rarita-Schwinger operator within mathematics is known to the author. In physical literature, there are some references to symplectic Majorana fields or symplectic Rarita-Schwinger fields, see Reuter [17] and Green, Hull [6], in the context of super-gravity of strings.

In the algebraic part of this article (part 2), some basic and known facts on higher symplectic spinor modules and decomposition of the mentioned tensor products are written (Lemma 1, Theorem 1, Theorem 2). Besides these theorems, the first main result (the decomposition of the symplectic spinor twisted de Rham sequence) is described (Lemma 2) together with the theorem on the decomposition of the tensor product of the defining representation and a higher symplectic spinor module (Theorem 3). In this part, an information on intersection of $\mathfrak{g}$-modules is written (Lemma

3). Third part of this article is the geometrical one. It contains a general lemma on an image of a covariant derivative (Lemma 4) and the second main result (Theorem 4), namely the characterization of the image of a covariant derivative associated to a torsion-free symplectic connection.

2. Higher symplectic spinor modules

Let $(\mathbb{V}, \omega)$ be a complex symplectic space of complex dimension $2l$, $l \in \mathbb{N}$. Let $G = Sp(\mathbb{V}, \omega) \simeq Sp(2l, \mathbb{C})$ be a complex symplectic group of $(\mathbb{V}, \omega)$ and $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}, \omega) \simeq \mathfrak{sp}(2l, \mathbb{C})$ its Lie algebra.[1] Consider a Cartan subalgebra $\mathfrak{h}$ of the symplectic Lie algebra is given together with a choice of positive roots Φ^+ of the system of all roots Φ. The set of fundamental weights $\{\varpi_i\}_{i=1}^l$ is then uniquely determined. For later use, we shall need an orthogonal basis (with respect to the Killing form on $\mathfrak{g}$), $\{\epsilon_i\}_{i=1}^l$, for which $\varpi_i = \sum_{j=1}^i \epsilon_j$ for $i = 1, \ldots, l$.

For $\lambda \in \mathfrak{h}^*$, let $L(\lambda)$ be the (up to a $\mathfrak{g}$-isomorphism uniquely defined) irreducible module with the highest weight λ. If λ happens to be integral and dominant (wr. to the choice $(\mathfrak{h}, \Phi^+)$), i.e., $L(\lambda)$ is finite dimensional, we shall write $F(\lambda)$ instead of $L(\lambda)$. Let L be an arbitrary (finite or infinite dimensional) weight module over a complex simple Lie algebra. We call L *module with bounded multiplicities*, if there is a $k \in \mathbb{N}_0$, such that for each $\mu \in \mathfrak{h}^*$, $\dim L_\mu \leq k$, where L_μ is the weight space of weight μ.

Let us introduce the following set of weights

$$A := \left\{\lambda = \sum_{i=1}^{l} \lambda_i \varpi_i \mid \lambda_{l-1} + 2\lambda_l + 3 > 0, \lambda_i \in \mathbb{N}_0,\ i = 1, \ldots, l-1, \lambda_l \in \mathbb{Z} + \frac{1}{2}\right\}$$

Definition 1. For a weight $\lambda \in A$, we call the module $L(\lambda)$ *higher symplectic spinor module*. We shall denote the module $L(-\frac{1}{2}\varpi_l)$ by $\mathbb{S}_+$ or simply by $\mathbb{S}$ and the module $L(\varpi_{l-1} - \frac{3}{2}\varpi_l)$ by $\mathbb{S}_-$. We shall call these two representations *basic symplectic spinor modules*.

The next theorem says that the class of higher symplectic spinor modules is quite natural and broad in a sense.

Theorem 1. *Let $\mathfrak{g} \simeq \mathfrak{sp}(2l, \mathbb{C})$ and $\lambda \in \mathfrak{h}^*$. Then the following are equivalent*

1) *$L(\lambda)$ is a module with bounded multiplicities;*
2) *$L(\lambda)$ is a direct summand in the completely reducible tensor product $\mathbb{S} \otimes F(\nu)$ for some integral dominant $\nu \in \mathfrak{h}^*$;*
3) *$\lambda \in A$.*

Proof. See Britten, Hooper, Lemire [2] and Britten, Lemire [3]. □

In this paper, we shall first study the irreducible decomposition of spaces $\bigwedge^i \mathbb{V}^* \otimes \mathbb{S}$ for $i = 0, \ldots, 2l$. To do it, we need to decompose the wedge powers $\bigwedge^i \mathbb{V}$ into irreducible modules. In the symplectic case (contrary to the orthogonal one), the wedge powers are not irreducible generically. This decomposition is well known and we state it as Lemma 1.

[1] Different choices of the symplectic form lead to isomorphic symplectic groups and algebras.

Lemma 1. *Let $\mathbb{V}$ be the $2l$ dimensional defining representation of the symplectic Lie algebra $\mathfrak{sp}(\mathbb{V},\omega)$, then*

$$\bigwedge^i \mathbb{V} \simeq \bigoplus_{p=0}^{[i/2]} F(\varpi_{i-2p})$$

for $i=0,\ldots,l$, where $[q]$ is the lower integral part of an element $q\in\mathbb{R}$.

Proof. See Goodman, Wallach [5], pp. 237. □

In the next theorem, the decomposition of the tensor product of an irreducible finite dimensional $\mathfrak{sp}(\mathbb{V},\omega)$-module and the basic symplectic spinor module $\mathbb{S}$ is described.

Theorem 2. *Let $\mathfrak{g}\simeq\mathfrak{sp}(2l,\mathbb{C})$ and $\nu=\sum_{i=1}^l \nu_i\varpi_i\in\mathfrak{h}^*$ be an integral dominant weight for $(\mathfrak{h},\Phi^+)$. Let us define a set $T_\nu := \big\{\nu-\sum_{i=1}^l d_i\epsilon_i \mid d_i\in\mathbb{N}_0, \sum_{i=1}^l d_i\in 2\mathbb{Z}, 0\le d_i\le \nu_i, i=1,\ldots,l-1, 0\le d_l\le 2\nu_l+1\big\}$. Then*

$$F(\nu)\otimes\mathbb{S} = \bigoplus_{\kappa\in T_\nu} L(\kappa-\frac{1}{2}\varpi_l)\,.$$

Proof. See Britten, Lemire [3], Theorem 1.2. □

We use the two last written claims to decompose the tensor products $\bigwedge^i\mathbb{V}^*\otimes\mathbb{S}$, for $i=0,\ldots,l$. We shall introduce the following convention. For a weight $\sum_{i=1}^l\lambda_i\varpi_i$, we shall write $(\lambda_1\lambda_2\ldots\lambda_l)$ briefly instead of $L(\sum_{i=1}^l\lambda_i\varpi_i)$.

Lemma 2. *The following decompositions hold:*

For $2i+1\le l-1$,

$$\bigwedge^{2i+1}\mathbb{V}^*\otimes\mathbb{S}\simeq\bigwedge^{2l-2i-1}\mathbb{V}^*\otimes\mathbb{S}\simeq\Big(10\ldots0-\frac{1}{2}\Big)\oplus\Big(0010\ldots0-\frac{1}{2}\Big)\oplus\ldots\oplus\Big(0\ldots01_{2i+1}0\ldots0-\frac{1}{2}\Big)$$
$$\oplus\Big(0\ldots01-\frac{3}{2}\Big)\oplus\Big(010\ldots01-\frac{3}{2}\Big)\oplus\ldots\oplus\Big(0\ldots01_{2i}0\ldots01-\frac{3}{2}\Big)\,.$$

For $2i\le l-1$,

$$\bigwedge^{2i}\mathbb{V}^*\otimes\mathbb{S}\simeq\bigwedge^{2l-2i}\mathbb{V}^*\otimes\mathbb{S}\simeq\Big(0\ldots0-\frac{1}{2}\Big)\oplus\Big(010\ldots0-\frac{1}{2}\Big)\oplus\ldots\oplus\Big(0\ldots01_{2i}0\ldots0-\frac{1}{2}\Big)$$
$$\oplus\Big(10\ldots01-\frac{3}{2}\Big)\oplus\Big(0010\ldots01-\frac{3}{2}\Big)\oplus\ldots\oplus\Big(0\ldots01_{2i-1}0\ldots01-\frac{3}{2}\Big)\,.$$

For l even,

$$\bigwedge^{l}\mathbb{V}^*\otimes\mathbb{S}\simeq\Big(0\ldots0-\frac{1}{2}\Big)\oplus\Big(010\ldots0-\frac{1}{2}\Big)\oplus\ldots\oplus\Big(0\ldots010-\frac{1}{2}\Big)\oplus\Big(0\ldots\ldots0\frac{1}{2}\Big)$$
$$\oplus\Big(10\ldots01-\frac{3}{2}\Big)\oplus\Big(0010\ldots01-\frac{3}{2}\Big)\oplus\ldots\oplus\Big(0\ldots0101-\frac{3}{2}\Big)\oplus\Big(0\ldots02-\frac{3}{2}\Big)\,.$$

For l odd,

$$\bigwedge^{l}\mathbb{V}^*\otimes\mathbb{S}\simeq\Big(10\ldots0-\frac{1}{2}\Big)\oplus\Big(0010\ldots0-\frac{1}{2}\Big)\oplus\ldots\oplus\Big(0\ldots010-\frac{1}{2}\Big)\oplus(0\ldots0\frac{1}{2}\Big)$$
$$\oplus\Big(0\ldots001-\frac{3}{2}\Big)\oplus\Big(010\ldots01-\frac{3}{2}\Big)\oplus\ldots\oplus\Big(0\ldots0101-\frac{3}{2}\Big)\oplus\ldots\Big(0\ldots02-\frac{3}{2}\Big)\,.$$

Proof. Since $\omega : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ is a non degenerate $\mathfrak{g}$-invariant bilinear form, it gives a $\mathfrak{g}$-module isomorphism $\mathbb{V} \simeq \mathbb{V}^*$. Thus the decomposition of the product $\bigwedge^i \mathbb{V}^* \otimes \mathbb{S}$ is equivalent to the decomposition of $\bigwedge^i \mathbb{V} \otimes \mathbb{S}$. For obtaining a further isomorphism, choose a symplectic basis $\{e_j\}_{j=1}^{2l}$ of $(\mathbb{V}, \omega)$ and define a mapping

$$\phi : \bigwedge^i \mathbb{V} \times \bigwedge^{2l-i} \mathbb{V} \to \mathbb{C}$$

for $i = 0, \dots, 2l$ on homogeneous elements by a formula $\phi(v_1 \wedge \dots \wedge v_i, w_1 \wedge \dots \wedge w_{2l-i}) =: q \in \mathbb{C}$, if and only if $qe_1 \wedge \dots \wedge e_{2l} = v_1 \wedge \dots \wedge v_i \wedge w_1 \wedge \dots \wedge w_{2l-i}$, where $v_1, \dots, v_i, w_1, \dots, w_{2l-i} \in \mathbb{V}$. Obviously, one extends the definition by linearity. Since the symplectic group $G = Sp(\mathbb{V}, \omega)$ is a subgroup of the special linear group $SL(\mathbb{V})$, we have $\phi(gv, gw) = gv \wedge gw = \det(g) v \wedge w = v \wedge w = \phi(v, w)$ for each $v \in \bigwedge^i \mathbb{V}$ and $w \in \bigwedge^{2l-i} \mathbb{V}$ and $g \in Sp(\mathbb{V}, \omega)$, i.e., the mapping ϕ is $Sp(\mathbb{V}, \omega)$- and also $\mathfrak{sp}(\mathbb{V}, \omega)$-invariant in the appropriate manners. Thus $\bigwedge^i \mathbb{V} \simeq (\bigwedge^{2l-i} \mathbb{V})^*$, which is naturally isomorphic to $\bigwedge^{2l-i} \mathbb{V}^*$, which is in turn isomorphic to $\bigwedge^{2l-i} \mathbb{V}$. Thus we need to decompose the spaces $\bigwedge^i \mathbb{V} \otimes \mathbb{S}$ for $i = 0, \dots, l$ only. After a straightforward but tedious application of Lemma 1 and Theorem 2 we would get the decompositions written in the statement of this lemma. $\square$

For sake of brevity, let us introduce the following notation. First, let us define a finite subset Ξ of pairs of non-negative integers.

$$\Xi := \{(i,j) | i = 0, \dots, l; j = 0, \dots, i\} \cup \{(i,j) | i = l+1, \dots, 2l, j = 0, \dots, 2l - i\}.$$

Further, let us define

$$\mathbb{E}_{0,2j} := \left(0 \dots 0 - \tfrac{1}{2}\right),\ (0, 2j) \in \Xi - \{(l,l), (l, l-1)\},$$
$$\mathbb{E}_{0,2j+1} := \left(0 \dots 01 - \tfrac{3}{2}\right),\ (0, 2j+1) \in \Xi - \{(l,l), (l, l-1)\},$$
$$\mathbb{E}_{2i,2j} := \left(0 \dots 01_{2j} 0 \dots 0 - \tfrac{1}{2}\right),\ (2i, 2j) \in \Xi - \{(l,l), (l, l-1)\},$$
$$\mathbb{E}_{2i+1,2j} := \left(0 \dots 01_{2j} 0 \dots 01 - \tfrac{3}{2}\right),\ (2i+1, 2j) \in \Xi - \{(l,l), (l, l-1)\},$$
$$\mathbb{E}_{2i,2j+1} := \left(0 \dots 01_{2j+1} 0 \dots 01 - \tfrac{3}{2}\right),\ (2i, 2j+1) \in \Xi - \{(l,l), (l, l-1)\},$$
$$\mathbb{E}_{2i+1,2j+1} := (0 \dots 01_{2j+1} 0 \dots 0 - \tfrac{1}{2}),\ (2i+1, 2j+1) \in \Xi - \{(l,l), (l, l-1)\},$$
$$\mathbb{E}_{l,l-1} := \left(0 \dots 02 - \tfrac{3}{2}\right), \quad \mathbb{E}_{l,l} = (0 \dots 0 \tfrac{1}{2}).$$

Let us remark, that a little bit more systematic way of defining the modules $\mathbb{E}_{i,j}$ for $(i,j) \in \Xi$ would be that one, in which the basis $\{\epsilon_i\}_{i=1}^l$ is used.

Using this notation, we can reformulate the lemma 2 in the following way

$$\bigwedge^{2l-i} \mathbb{V}^* \otimes \mathbb{S} \simeq \bigwedge^{i} \mathbb{V}^* \otimes \mathbb{S} \simeq \bigoplus_{j=0}^{i} \mathbb{E}_{i,j}$$

for $i = 0, \dots, l$ or

$$\bigwedge^{i} \mathbb{V}^* \otimes \mathbb{S} \simeq \bigoplus_{(i,j) \in \Xi} \mathbb{E}_{i,j}$$

for $i = 0, \dots, 2l$.

To visualize the system described by Lemma 2, we display a picture for rank $l = 3$. The i^{th} column corresponds to the space $\bigwedge^i \mathbb{V}^* \otimes \mathbb{S}$ and each member of a column corresponds to an irreducible representation in $\bigwedge^i \mathbb{V}^* \otimes \mathbb{S}$ with a displayed highest weight.

$$\begin{array}{ccccccc}
\left(00-\frac{1}{2}\right) & \left(01-\frac{3}{2}\right) & \left(00-\frac{1}{2}\right) & \left(01-\frac{3}{2}\right) & \left(00-\frac{1}{2}\right) & \left(01-\frac{3}{2}\right) & \left(00-\frac{1}{2}\right) \\
 & \left(10-\frac{1}{2}\right) & \left(11-\frac{3}{2}\right) & \left(10-\frac{1}{2}\right) & \left(11-\frac{3}{2}\right) & \left(10-\frac{1}{2}\right) & \\
 & & \left(01-\frac{1}{2}\right) & \left(02-\frac{3}{2}\right) & \left(01-\frac{1}{2}\right) & & \\
 & & & \left(00-\frac{1}{2}\right) & & &
\end{array}$$

In the next theorem, a decomposition of a tensor product of a higher symplectic spinor module and the defining representation $\mathbb{V} \simeq F(\varpi_1)$ over $\mathfrak{sp}(\mathbb{V}, \omega) \simeq \mathfrak{sp}(2l, \mathbb{C})$ is described.

Theorem 3. *Let* $\mathfrak{g} \simeq \mathfrak{sp}(2l, \mathbb{C})$ *and* $\lambda \in A$. *Then*

$$L(\lambda) \otimes F(\varpi_1) = \bigoplus_{\mu \in A_\lambda} L(\mu)\,,$$

where $A_\lambda := A \cap \{\lambda + \nu | \nu \in \Pi(\varpi_1)\}$ *and* $\Pi(\varpi_1)$ *is the saturated set of weights of the defining representation.*[2]

Proof. See Krýsl, [15] or [16]. $\square$

Let us remark, that the proof of this theorem is based on the so called Kac-Wakimoto formal character formula of Kac and Wakimoto published in [9] and on some results of Humphreys, see [8], who specified results of Kostant from [13] on tensor products of finite and infinite dimensional modules admitting a central character.

In the next lemma, a property is formulated, which is valid for an arbitrary simple Lie algebra $\mathfrak{g}$.

Lemma 3. *Let X be a $\mathfrak{g}$-module and $\mathbb{V}, \mathbb{W} \subseteq X$ its two $\mathfrak{g}$-submodules* (*of finite or infinite dimension*). *Suppose $\mathbb{V} = \mathbb{V}_1 \oplus \ldots \oplus \mathbb{V}_a$ and $\mathbb{W} = \mathbb{W}_1 \oplus \ldots \oplus \mathbb{W}_b$ are decompositions into irreducible $\mathfrak{g}$-submodules. Define a subset I of the set $\{1, \ldots, a\}$ by the prescription $I := \big\{i \in \{1, \ldots, a\} \mid \exists j \in \{1, \ldots, b\} : \mathbb{V}_i \simeq \mathbb{W}_j\big\}$. Then*

$$\mathbb{V} \cap \mathbb{W} \subseteq \bigoplus_{i \in I} \mathbb{V}_i\,.$$

Proof. Let us consider the projections $p_i : \mathbb{V} \to \mathbb{V}_i$, $i = 1, \ldots, a$ and $q_j : \mathbb{W} \to \mathbb{W}_j$, $j = 1, \ldots, b$. Suppose, that we have defined the projections p_i also on the space $\mathbb{W}$ in the following way. One can easily show, that a finite direct sum of $\mathfrak{g}$-modules is actually completely reducible (see, e.g., Krýsl [15]). Thus there is a (generally non-unique) $\mathfrak{g}$-submodule $\mathbb{U}$, such that $(\mathbb{V} \cap \mathbb{W}) \oplus \mathbb{U} = \mathbb{W}$ is a direct sum of $\mathfrak{g}$-modules. Therefore given any $x \in \mathbb{W}$, we can write it as a sum $x = v + u$, where $v \in \mathbb{V} \cap \mathbb{W}$ and $u \in \mathbb{U}$, in a unique way and define $p_i(x) := p_i(v)$. Now, take an element $x \in$

[2]One can easily compute,that $\Pi(\varpi_1) = \{\pm\epsilon_i | i = 1, \ldots, l\}$.

$\mathbb{V} \cap \mathbb{W}$. We have $x = \sum_{i=1}^{a} \sum_{j=1}^{b} p_i\big(q_j(x)\big)$. To get a contradiction, suppose there are elements $i \notin I$ and $j \in \{1, \ldots, b\}$ such that $p_i\big(q_j(x)\big) \neq 0$. Thus we have a $\mathfrak{g}$-module homomorphism $R := p_i \circ q_j|_{\mathbb{W}_j} : \mathbb{W}_j \to \mathbb{V}_i$, which is nonzero. The classical Schur lemma type argument shows that R is an isomorphism of $\mathbb{W}_j$ and $\mathbb{V}_i$, which contradicts the condition $i \notin I$. $\square$

Remark. The proof of the above written lemma does not use any information about the Lie algebra over which we took the module X, thus it could be generalized to each module over a general algebraic structure (group, commutative, associative, super-Lie algebra e.t.c.), which admits modules over itself. Let us note, that the statement of the lemma can be improved in an easy way. To see the weakness of the lemma, consider two nonzero equivalent representations $\mathbb{V}_1$ and $\mathbb{V}_2$, form their direct sum $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$ and suppose a submodule $\mathbb{W} \simeq \mathbb{V}_1 \simeq \mathbb{V}_2$, for which $\mathbb{V}_1 \neq \mathbb{W} \neq \mathbb{V}_2$, is given. Then clearly $\mathbb{V} \cap \mathbb{W} = \mathbb{W} \subsetneq \mathbb{V}_1 \oplus \mathbb{V}_2$, but the lemma gives only $\mathbb{V} \cap \mathbb{W} \subseteq \mathbb{V}_1 \oplus \mathbb{V}_2$. We will not try to improve this lemma, because first we shall need it only in the above written form and second the reformulation would be a bit inefficient because of its increased length.

3. Spin symplectic geometry

We shall begin with a short observation about covariant derivatives on vector bundle valued forms and then we are going to consider basic aspects of metaplectic structures.

Lemma 4. *Let $p : F \to M$ be a smooth vector bundle equipped by a vector bundle connection ∇^F. Consider a subbundle $E \subseteq \bigwedge^i T^*M \otimes F \to M$ for some $i = 0, \ldots, dimM$ and a section $s \in \Gamma(M, E)$. Since s could be viewed as an exterior differential form with values in F, the covariant derivative d^{∇^F} could be applied. Then*

$$d^{\nabla^F} s \in \Gamma\big(M, (T^*M \otimes E) \cap \big(\bigwedge^{i+1} T^*M \otimes F\big)\big)\,.$$

Proof. This is an easy observation. We know that $s \in \Gamma(M, \bigwedge^i T^*M \otimes F)$ and therefore $d^{\nabla^F} s \in \Gamma\big(M, \bigwedge^{i+1} T^*M \otimes F\big)$. The assumption $s \in \Gamma(M, E)$ implies $d^{\nabla^F} s \in \Gamma(M, T^*M \otimes E)$. Summing up, we obtain $d^{\nabla^F} s \in \Gamma\big(M, (T^*M \otimes E) \cap (\bigwedge^{i+1} T^*M \otimes F)\big)$. $\square$

To define a metaplectic structure, we will use a definition of Katharina Habermann from [7], which is quite analogous to the Riemannian case. Now, let $(\mathbb{V}_0, \omega)$ be a real symplectic space of dimension $2l$. Let $\tilde{G}_0$ be a nontrivial 2-fold covering of the group $G_0 = Sp(\mathbb{V}_0, \omega) \simeq Sp(2l, \mathbb{R})$, thus $\tilde{G}_0 \simeq Mp(2l, \mathbb{R})$ (the metaplectic group) and fix a 2:1 covering $\lambda : \tilde{G}_0 \to G_0$.

Definition 2. Let (M, ω) be a symplectic manifold of dimension $2l$. Let $\big(p_1 : \mathcal{P} \to M, Sp(2l, \mathbb{R})\big)$ be a principal fiber bundle of symplectic repers (in TM) and $\big(p_2 : \mathcal{Q} \to M, Mp(2l, \mathbb{R})\big)$ be a principal fiber bundle with a structure group $Mp(2l, \mathbb{R})$. We call a surjective bundle homomorphism $\Lambda : \mathcal{Q} \to \mathcal{P}$ (over the identity on M) *metaplectic* *structure*, if the following diagram commutes.

$$\begin{tikzcd}
Mp(2l,\mathbb{R}) \times \mathcal{Q} \arrow[r] \arrow[dd, "\lambda\times\Lambda"] & \mathcal{Q} \arrow[dd, "\Lambda"] \arrow[dr, "p_2"] & \\
& & M \\
Sp(2l,\mathbb{R}) \times \mathcal{P} \arrow[r] & \mathcal{P} \arrow[ur, "p_1"'] &
\end{tikzcd}$$

Let (M, ω) be a symplectic manifold of dimension $2l$. It is well known that there is no unique *symplectic connection.* Symplectic connection is a torsion-free connection ∇ on the tangent bundle which preserves the symplectic structure ω, i.e.,

$$\omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z) = X\omega(Y, Z)$$

for each $X, Y, Z \in \Gamma(M, TM)$, see, e.g., Habermann [7]. Nevertheless, we may take any symplectic connection ∇ and associate to it a principal bundle connection $Z^{\mathcal{P}} : T\mathcal{P} \to \mathfrak{g}_0$. For this connection, there is a lifted connection $Z^{\mathcal{Q}} : T\mathcal{Q} \to \tilde{\mathfrak{g}_0} \simeq \mathfrak{g}_0$ on the metaplectic structure. To this connection, $Z^{\mathcal{Q}}$ we can associate a linear connection ∇^S on the spinor bundle $p : S \to M$. By the spinor bundle $p : S \to M$ we mean the associated vector bundle $S := \mathcal{Q} \times_{\tilde{G}_0} \mathbb{S}$ to the principle $\tilde{G}_0$-bundle via the basic symplectic spinor representation $\mathbb{S}$. In this way we may construct the covariant derivative d^{∇^S}. (For the correctness of this definition, see Kashiwara, Vergne [11], where certain globalization of the basic spinor modules to the group $\tilde{G}_0$ is described.)

For the sake of brevity, let us introduce the notation for the following associated vector bundles $E_{i,j} := \mathcal{Q} \times_{\tilde{G}_0} \mathbb{E}_{i,j}$ for all $(i, j) \in \Xi$. For technical reasons, we define $E_{i,j} := 0$ if $(i, j) \notin \Xi$.

Theorem 4. *Let ∇ be a torsion-free symplectic connection on the tangent bundle of the (symplectic) base manifold M of a metaplectic structure. Let us denote the induced covariant derivative for the associated symplectic spinor bundle $p : S \to M$ by d^{∇^S}. Then with the notation introduced above,*

$$d^{\nabla^S} : \Gamma(M, E_{i,j}) \to \Gamma(M, E_{i+1,j-1} \oplus E_{i+1,j} \oplus E_{i+1,j+1})$$

for all $(i, j) \in \Xi$.

Proof. We must prove that the image is of the form described by the statement of the theorem. Lemma 4 (for $F := S, E := E_{i,j} \subseteq \bigwedge^i T^*M \otimes S$ and a connection ∇^S induced by a symplectic connection ∇ on TM) implies, that $d^{\nabla^S} s \in \Gamma(M, (E_{i,j} \otimes T^*M) \cap \bigwedge^{i+1} T^*M \otimes S)$ for a section $s \in \Gamma(M, E_{i,j})$. The vector bundle $T^*M \otimes E_{i,j} \cap \bigwedge^{i+1} T^*M \otimes S$ is isomorphic to the associated vector bundle $\mathcal{Q} \times_{\tilde{G}} (\mathbb{V}^* \otimes \mathbb{E}_{i,j} \cap \bigwedge^{i+1} \mathbb{V}^* \otimes \mathbb{S})$. Our strategy is to use lemma 3 to compute the intersection. Therefore we need to find the decomposition of the modules $\mathbb{V}^* \otimes \mathbb{E}_{i,j}$ and $\bigwedge^{i+1} \mathbb{V}^* \otimes \mathbb{S}$ into irreducible summands. The latter was done in lemma 2. We shall use theorem 3 to decompose $\mathbb{V}^* \otimes \mathbb{E}_{i,j} \simeq \mathbb{V} \otimes \mathbb{E}_{i,j}$. There are in principle 6 forms of $\mathbb{E}_{i,j}$: $L(0 \ldots 0 - \frac{1}{2})$, $L(0 \ldots 01 - \frac{3}{2})$, $L(0 \ldots 01_k 0 \ldots 0 - \frac{1}{2})$, $L(0 \ldots 01_k 0 \ldots 01 - \frac{3}{2})$, $L(0 \ldots 02 - \frac{3}{2})$ and $L(0 \ldots 0\frac{1}{2})$. Because we will be more careful and will distinguish between odd and even subscripts i, j for the space $\mathbb{E}_{i,j}$ at the beginning of our analysis, we will be investigating eleven cases actually.

1) For $\mathbb{E}_{2i,0} = (0\dots 0 - \frac{1}{2})$, we obtain that $\mathbb{E}_{2i,0} \otimes \mathbb{V} = (10\dots 0 - \frac{1}{2}) \oplus (0\dots 01 - \frac{3}{2})$
   - For $i = 0,\dots,l-1$, we obtain that $((10\dots 0 - \frac{1}{2}) \oplus (0\dots 01 - \frac{3}{2})) \cap \bigwedge^{2i+1} \mathbb{V} \otimes \mathbb{S} \subseteq (10\dots 0 - \frac{1}{2}) \oplus (0\dots 01 - \frac{3}{2}) = \mathbb{E}_{2i+1,-1} \oplus \mathbb{E}_{2i+1,0} \oplus \mathbb{E}_{2i+1,1}$ (the first summand is zero by definition).
   - For $i = l$, we obtain that the intersection is zero, because $\bigwedge^{2i+1} \mathbb{V} \otimes \mathbb{S}$ is zero. But in this case the vector space $\mathbb{E}_{2l,-1} \oplus \mathbb{E}_{2l+1,0} \oplus \mathbb{E}_{2l+1,1}$ is zero, too (by definition).

   In each cases, we have obtain that $\mathbb{E}_{2i,0} \otimes \mathbb{V}^* \subseteq \mathbb{E}_{2i,-1} \oplus \mathbb{E}_{2i,0} \oplus \mathbb{E}_{2i,1}$ according to the statement.
2) For $\mathbb{E}_{2i+1,0} = (0\dots 01 - \frac{3}{2})$, we obtain that $\mathbb{E}_{2i+1,0} \otimes \mathbb{V} = (10\dots 01 - \frac{3}{2}) \oplus (0\dots 0 - \frac{1}{2}) \oplus (0\dots 010 - \frac{3}{2})$.
   - For $i = 0,\dots,l-2$, we get $((10\dots 01 - \frac{3}{2}) \oplus (0\dots 0 - \frac{1}{2}) \oplus (0\dots 010 - \frac{3}{2})) \cap \bigwedge^{2i+1} \mathbb{V} \otimes \mathbb{S} = (0\dots 0 - \frac{3}{2}) \oplus (10\dots 01 - \frac{3}{2}) = \mathbb{E}_{2i+2,-1} \oplus \mathbb{E}_{2i+2,0} \oplus \mathbb{E}_{2i+2,1}$, because the first summand is zero (by definition).
   - For $i = l-1$, we obtain $((10\dots 01 - \frac{3}{2}) \oplus (0\dots 0 - \frac{1}{2}) \oplus (0\dots 010 - \frac{3}{2})) \cap \bigwedge^{2l} \mathbb{V} \otimes \mathbb{S} = (0\dots 0 - \frac{1}{2}) = \mathbb{E}_{2l,-1} \oplus \mathbb{E}_{2l,0} \oplus \mathbb{E}_{2l,1}$, because the first and last summands are zero (by definition).
3) For $\mathbb{E}_{2i,2j} = (0\dots 01_{2j}0\dots 0 - \frac{1}{2})$ ($l-2 \geq 2j > 0$ is to be supposed, $j = 0$ has been already handled), we obtain $\mathbb{E}_{2i,2j} \otimes \mathbb{V} = (0\dots 01_{2j-1}0\dots 0 - \frac{1}{2}) \oplus (0\dots 01_{2j+1}0\dots 0 - \frac{1}{2}) \oplus (0\dots 01_{2j}0\dots 1 - \frac{3}{2}) \oplus (10\dots 01_{2j}0\dots 0 - \frac{1}{2})$.
   - For $2i + 2j < 2l$, the intersection $((0\dots 01_{2j-1}0\dots 0 - \frac{1}{2}) \oplus (0\dots 01_{2j+1}0\dots 0 - \frac{1}{2}) \oplus (0\dots 01_{2j}0\dots 1 - \frac{3}{2}) \oplus (10\dots 01_{2j}0\dots 0 - \frac{1}{2})) \cap \bigwedge^{2i+1} \mathbb{V} \otimes \mathbb{S} = (0\dots 01_{2j-1}0\dots 0 - \frac{1}{2}) \oplus (0\dots 01_{2j+1}0\dots 0 - \frac{1}{2}) \oplus (0\dots 01_{2j}0 \dots 1 - \frac{3}{2}) = \mathbb{E}_{2i+1,2j-1} \oplus \mathbb{E}_{2i+1,2j} \oplus \mathbb{E}_{2i+1,2j+1}$.
   - For $2i + 2l = 2l$, we get that the intersection equals $(0\dots 01_{2j-1}0\dots 0 - \frac{1}{2})$.

   In each case, we have obtained that the intersection is in $\mathbb{E}_{2i+1,2j-1} \oplus E_{2i+1,2j} \oplus \mathbb{E}_{2i+1,2j+1}$.

The remaining cases will not be handled so carefully. We will write only the result of the appropriate decomposition and not the result of the intersection. In each cases, one can compute the intersection like in the previous ones and check that the condition for the intersection to be a vector subspace of $\mathbb{E}_{i+1,j-1} \oplus \mathbb{E}_{i+1,j} \oplus \mathbb{E}_{i+1,j+1}$ for appropriate i, j is fulfilled. We will also not distinguish between the parities of i, j.

4) For $\mathbb{E} = (0\dots 01_k0\dots 01 - \frac{3}{2})$ ($k > 1$), we get $\mathbb{E} \otimes \mathbb{V} = (0\dots 01_{k-1}0\dots 01 - \frac{3}{2}) \oplus (0\dots 01_{k+1}0\dots 01 - \frac{3}{2}) \oplus (10\dots 01_k0\dots 01 - \frac{3}{2}) \oplus (0\dots 01_k0\dots 0 - \frac{1}{2})$.
5) For $\mathbb{E} = (0\dots 01_k0\dots 0 - \frac{1}{2})$ ($k > 1$), we get $\mathbb{E} \otimes \mathbb{V} = (0\dots 01_{k-1}0\dots 0 - \frac{1}{2}) \oplus (0\dots 01_{k+1}0\dots 0 - \frac{1}{2}) \oplus (10\dots 01_k0\dots 0 - \frac{1}{2}) \oplus (0\dots 01_k0\dots 01 - \frac{3}{2})$.
6) For $\mathbb{E} = (10\dots 0 - \frac{1}{2})$, we get $\mathbb{E} \otimes \mathbb{V} = (0\dots 0 - \frac{1}{2}) \oplus (010\dots 0 - \frac{1}{2}) \oplus (1\dots 01 - \frac{3}{2})$.
7) For $\mathbb{E} = (10\dots 01 - \frac{3}{2})$, we get $\mathbb{E} \otimes \mathbb{V} = (0\dots 01 - \frac{3}{2}) \oplus (010\dots 01 - \frac{3}{2}) \oplus (1\dots 0 - \frac{1}{2})$.
8) For $\mathbb{E} = (0\dots 01 - \frac{1}{2})$, we get $\mathbb{E} \otimes \mathbb{V} = (10\dots 01 - \frac{1}{2}) \oplus (0\dots 02 - \frac{3}{2}) \oplus (0\dots 0\frac{1}{2})$.

9) For $\mathbb{E} = (0\dots011-\frac{3}{2})$, we get $\mathbb{E}\otimes\mathbb{V} = (10\dots011-\frac{3}{2}) \oplus (0\dots02-\frac{3}{2}) \oplus (0\dots010-\frac{1}{2})$.

10) For $\mathbb{E} = (0\dots02-\frac{3}{2})$, we get $\mathbb{E}\otimes\mathbb{V} = (10\dots02-\frac{3}{2})\oplus(0\dots011-\frac{3}{2})\oplus(0\dots03-\frac{5}{2}) \oplus (0\dots01-\frac{1}{2})$.

11) For $\mathbb{E} = (0\dots0\frac{1}{2})$, we get $\mathbb{E}\otimes\mathbb{V} = (10\dots0\frac{1}{2}) \oplus (0\dots0-\frac{1}{2})$.

The irreducible representation of $\mathfrak{g}$ could be define also for the split real form $\mathfrak{g}_0 = \mathfrak{sp}(2l,\mathbb{R})$. The irreducibility does not change and also the decompositions remain the same. Let us remark, that according to the result of Kashiwara, Vergne [11], there are some L^2-globalizations of our representation to the metaplectic group $\tilde{G}_0$ and that there are also the canonically defined ones. The decompositions do not change if we turn our attention to the $(\mathfrak{g}_0,\tilde{K})$-structure,[3] because $\tilde{K}$ is connected (see Baldoni [1]), and do not change even if we take the appropriate globalization (e.g., the Casselman-Wallach, i.e., the minimal one). □

In the next picture, the system described by theorem 4 is displayed for $l = 3$, i.e., for metaplectic structures over a six dimensional symplectic manifold (M^6,ω).

$$\begin{tikzcd}
\mathbb{E}_{0,0} \arrow[r] \arrow[dr] & \mathbb{E}_{1,0} \arrow[r] \arrow[dr] & \mathbb{E}_{2,0} \arrow[r] \arrow[dr] & \mathbb{E}_{3,0} \arrow[r] \arrow[dr] & \mathbb{E}_{4,0} \arrow[r] \arrow[dr] & \mathbb{E}_{5,0} \arrow[r] & \mathbb{E}_{6,0} \\
 & \mathbb{E}_{1,1} \arrow[r] \arrow[ur] \arrow[dr] & \mathbb{E}_{2,1} \arrow[r] \arrow[ur] \arrow[dr] & \mathbb{E}_{3,1} \arrow[r] \arrow[ur] \arrow[dr] & \mathbb{E}_{4,1} \arrow[r] \arrow[ur] & \mathbb{E}_{5,1} \arrow[ur] & \\
 & & \mathbb{E}_{2,2} \arrow[r] \arrow[ur] \arrow[dr] & \mathbb{E}_{3,2} \arrow[r] \arrow[ur] & \mathbb{E}_{4,2} \arrow[ur] & & \\
 & & & \mathbb{E}_{3,3} \arrow[ur] & & &
\end{tikzcd}$$

Having the analogous result in the Riemannian case in mind, we are entitled to call the horizontally going operators in the first arrow *symplectic Dirac operators*, that ones going from the first arrow down-right *symplectic twistor* and in the second arrow the horizontally going ones *symplectic Rarita-Schwinger operators*. The horizontally going operators on the remaining arrows could be eventually called *symplectic generalized Rarita-Schwinger operators*.

The further research could be devoted to other real symplectic groups, various types of globalizations of the modules in question, to a coordinate-way description of the operators, we have obtained, and to their analytic properties.

Acknowledgement. I am very grateful to Vladimír Souček for motivations coming from Riemannian geometry and orthogonal spin structures. The author is also grateful to the Grant Agency of Czech Republic for the support from the grant for young researchers GAČR 201/06/P223.

[3] $\tilde{K}$ is the maximal compact subgroup of $\tilde{G}_0 \simeq Mp(2l,\mathbb{R})$, i.e., the λ-preimage of the maximal compact subgroup K of $Sp(2l,\mathbb{R})$, which is isomorphic to the unitary group, $K \simeq U(2l)$.

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