Limita posloupnosti - komplexní úloha XII

Určete limitu posloupnosti

$$\lim_{n\to\infty} \frac{2^n\sqrt{n^2+n}-n\sqrt{4^n+1}}{\sqrt[n]{2^{n^2}+1}}.$$

Řešení

$$= \lim_{n\to\infty} \frac{\cancel{2^n}\left(\sqrt{n^2+n}-\frac{\sqrt{n^2 4^n+n^2}}{2^n}\right)}{\cancel{2^n}\sqrt[n]{1+\frac{1}{2^{n^2}}}} = \lim_{n\to\infty} \frac{\sqrt{n^2+n}-\sqrt{\frac{n^2 4^n+n^2}{4^n}}}{\sqrt[n]{1+\frac{1}{2^{n^2}}}} = \lim_{n\to\infty} \frac{\sqrt{n^2+n}-\sqrt{n^2+\frac{n^2}{4^n}}}{\sqrt[n]{1+\frac{1}{2^{n^2}}}} =$$

$$= \lim_{n\to\infty} \frac{\left(\sqrt{n^2+n}-\sqrt{n^2+\frac{n^2}{4^n}}\right)\left(\sqrt{n^2+n}+\sqrt{n^2+\frac{n^2}{4^n}}\right)}{\sqrt[n]{1+\frac{1}{2^{n^2}}}\left(\sqrt{n^2+n}+\sqrt{n^2+\frac{n^2}{4^n}}\right)} = \lim_{n\to\infty} \frac{n^2+n-n^2-\frac{n^2}{4^n}}{\sqrt[n]{1+\frac{1}{2^{n^2}}}\left(n\sqrt{1+\frac{1}{n}}+n\sqrt{1+\frac{1}{4^n}}\right)} =$$

$$= \lim_{n\to\infty} \frac{n\left(1-\frac{n}{4^n}\right)}{\sqrt[n]{1+\frac{1}{2^{n^2}}}\cdot n\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{1}{4^n}}\right)} \overset{\text{VOAL}}{=} \frac{1}{1(1+1)} = \frac{1}{2}$$

$\lim_{n\to\infty} \frac{n^{\cancel{k}}}{b^n} = 0 \quad b>0$ .... $\frac{n}{4^n} \to 0$

Věta o dvou policajtech

$\sqrt[n]{1} \le \sqrt[n]{1+\frac{1}{2^{n^2}}} \le \sqrt[n]{2}$

$\frac{1}{2^{n^2}} \to 0$; $\lim_{n\to\infty} \sqrt[n]{a} = 1 \quad a>0$ .... $\lim_{n\to\infty} \sqrt[n]{1+\frac{1}{2^{n^2}}} = 1$

$\frac{1}{n} \to 0$; $\lim_{n\to\infty} a_n^q = \left(\lim_{n\to\infty} a_n\right)^q \quad q\in\mathbb{Q} \quad a_n>0$ .... $\lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = 1$

podobně i pro $\sqrt{1+\frac{1}{4^n}}$ $\left(\frac{1}{4^n} \to 0\right)$