(1a) $\int_0^T (y')^2 \, dx$ $\qquad y(0)=0 \qquad y(T)=B.$ $\qquad a=0 \quad b=T$

(1) $f(x,y,y')=(y')^2$ $\qquad$ (1') f nez na $x, y$

$$\frac{\partial f}{\partial y}=0 \qquad \frac{\partial f}{\partial y'}=2y' \qquad\qquad \frac{\partial^2 f}{(\partial y')^2}\, y''=0$$

$$2\cdot y''=0$$

$$\frac{d(2y')}{dx}=2y''$$

celkom $\quad 2y''=0$

(1'') $$\frac{\partial^2 f}{\partial y'^2}\, y'' + \frac{\partial^2 f}{\partial y'\partial y}\, y' + \left(\frac{\partial^2 f}{\partial y'\partial x} - \frac{\partial f}{\partial y}\right)=0$$

$$2\cdot 1\cdot y'' + 0 + 0 - 0 = 0$$

každopádne: $\quad 2y''=0 \to y''=0$

$$y'=k$$

$$y=kx+c$$

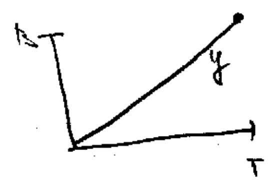
podmienky $\quad y(0)=c \qquad c=0$

$$y=kx \qquad k=\frac{B}{T}$$

$$y(T)=kT$$

celkom $\quad \boxed{y(x)=\frac{B}{T}x} \qquad x\in[0,T]$

obr.



(2b) $\int_0^T c_1 y'^2 + c_2 y \, dx$ $\quad a=0 \quad b=T$

$y(0)=0, \quad y(T)=B \qquad y' \geq 0$

$f(x,y,y') = c_1 y'^2 + c_2 y$

$\frac{\partial f}{\partial y'} = 2c_1 y' \qquad \frac{\partial f}{\partial y} = c_2$

celkem ... $= 0$

$y'(y'' 2c_1 - c_2) = 0$

$y' = 0 \quad \vee \quad y'' = \frac{c_2}{2c_1}$

(a) $y'=0 \quad y = k \quad \Rightarrow B = 0$ jinak nelze

(b) $y'' = \frac{c_2}{2c_1} \Rightarrow y' = \frac{c_2}{2c_1} x + k \Rightarrow$

$y = \frac{c_2}{4c_1} x^2 + kx + m$

$0 = y(0) = m \qquad B = y(T) = \frac{c_2}{4c_1} T^2 + kT$

$\Rightarrow k = \frac{B - \frac{c_2}{4c_1} T^2}{T}$

celkem $y = \frac{c_2}{4c_1} x^2 + \frac{B - \frac{c_2}{4c_1} T^2}{T} x \qquad x \in (0,T)$

(2c) $\int_0^1 y'^2 + 10xy \, dx \qquad y(0) = 1, \quad y(1) = 2$

$$f(x, y, y') = y'^2 + 10xy$$

$$\frac{\partial f}{\partial y} = 10x \qquad \frac{\partial f}{\partial y'} = 2y'$$

$$2y'' - 10x = 0$$

$$\underline{y'' = 5x}$$

$$y' = \frac{5}{2}x^2 + c$$

$$y = \frac{5}{6}x^3 + cx + d$$

$$1 = y(0) = d \qquad 2 = \frac{5}{6} + c + 1$$

$$\frac{1}{6} = c$$

celkem $\quad \underline{y(x) = \frac{5}{6}x^3 + \frac{1}{6}x + 1}$

(2d) $\int_a^b 3y' - xy'^2 \, dx \qquad y(a) = x_0, \quad y(b) = x_1$

$$f = 3y' - xy'^2 \qquad \frac{\partial f}{\partial y'} = 3 - 2xy'$$

$$(3 - 2xy') = c$$

$$3 - c = 2xy'$$

$$\left(\frac{3-c}{2}\right) \frac{1}{x} = y' \qquad \left(\frac{dy}{dx}\right)$$

$$\boxed{\left(\frac{3-c}{2}\right) \ln x + k = y(x)} \qquad \text{(nedopočítávat)}$$

$$x_0 = \frac{3-c}{2} \ln a + k$$

$$x_1 = \frac{3-c}{2} \ln b + k$$

(2a) $\partial$ hat

$$\frac{\partial^2 f}{\partial y'^2} y'' + \frac{\partial^2 f}{\partial y' \partial x} = 0$$

$$-2x\, y'' + -2y' = 0$$

$$y'' = \frac{-y'}{x} \qquad z = y'$$

$$z' = -\frac{z}{x}$$

$$\frac{z'}{z} = -\frac{1}{x}$$

(2e) $\int_a^b y' \, dx$ $\qquad y(a) = x_0 \quad y(b) = x_2$

$$f = y' \qquad \frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = 1 \qquad \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad 0 = 0$$

$\forall y$ to vyjde, navíc

$$\int_a^b y' \, dx = y(b) - y(a) = \underline{x_2 - x_0}$$

(2f) $\int_a^b xy' + y'^2 \, dx$ $\qquad x(a) = x_0, \qquad y(b) = x_1$

$$f = xy' + y'^2 \qquad \frac{\partial f}{\partial y'} = x + 2y'$$

tedy

$$x + 2y' = c$$

$$2y' = c - x$$

$$\underline{\underline{y = \frac{cx}{2} - \frac{x^2}{4} + d}}$$

navíc

$$\frac{ca}{2} - \frac{a^2}{4} + d = x_0$$

$$\frac{cb}{2} - \frac{b^2}{4} + d = x_1$$

(2g) $\int_0^1 y' \cdot x + y'^2 \qquad y(0)=0 \quad y(1)=2$

$f = y'x + y'^2 \qquad \frac{\partial f}{\partial y'} = x + 2y'$

pak $\qquad x + 2y' = k$

$$2y' = k - x$$

$$y' = \frac{k-x}{2}$$

$$y = \frac{k}{2}x - \frac{x^2}{4} + c$$

$0 = y(0) = c \qquad 2 = \frac{k}{2} - \frac{1}{4} \qquad \frac{k}{2} = \frac{9}{4}$

celkem $\qquad y = \frac{9}{4}x - \frac{x^2}{4}$

$\boxed{\text{jinak}}$ $\qquad \frac{\partial^2 f}{\partial y'^2} = 2 \qquad \frac{\partial^2 f}{\partial y' \partial x} = 1$

pak $\qquad 2y'' + 1 = 0$

(2h) $\int_0^{\pi/2} y'(x)^2 - y(x)^2 \, dx$ $\qquad y(0)=0$

$y(\frac{\pi}{2}) = 1$

$f = y'^2 - y^2$

$\frac{\partial f}{\partial y'} = 2y' \qquad \frac{\partial f}{\partial y} = -2y$

EL: $y'(2y'' + 2y) = 0$

$\to \quad y'' = -y$

$y = c_1 \sin x + c_2 \cos x$

$0 = y(0) = c_2 \cdot 1$

$1 = y(\frac{\pi}{2}) = c_1 \cdot 1$

$y(x) = \sin x$

(2i) $\int_a^b \sqrt{\frac{1+y'^2}{y}} \, dx \qquad f = \sqrt{\frac{1+y'^2}{y}} \qquad \frac{\partial f}{\partial y} = \sqrt{1+y'^2}\; y^{-3/2}\left(-\frac{1}{2}\right)$

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{y}} \cancel{\frac{1}{2}} \frac{1}{\sqrt{1+y'^2}} \cdot (+\cancel{2}y') = \frac{+y'}{\sqrt{y}\sqrt{1+y'^2}}$$

$$\cancel{y''\left(\frac{-\sqrt{1-y'^2} + y'\frac{1}{2}\frac{1}{\sqrt{1-y'^2}}(-2y')}{\sqrt{y}\,(1+y'^2)}\right) + \frac{-y'}{\sqrt{1-y'^2}}\left(-\frac{1}{2}\right)y^{-3/2} + \frac{1}{2}y^{-3/2}\sqrt{1-y'^2}}$$

$$\frac{+y'^2}{\sqrt{y}\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{\sqrt{y}} = c$$

$$\frac{y'^2 - 1 - y'^2}{\sqrt{y}\sqrt{1+y'^2}} = c$$

$$y(1+y'^2) = k$$

$$y' = \sqrt{\frac{k-y}{y}}$$

pak $\quad x = -(ky - y^2)^{1/2} + k \arccos(1 - y/k_{/2}) + c.$

(2j) $\int_a^b y'^2 e^{-y'}\, dx$

$$f = y'^2 e^{-y'}$$

$$\frac{\partial f}{\partial y'} = -e^{-y}\left(y(y+2)+2\right)$$

$$\frac{\partial^2 f}{\partial y'^2} = e^{-y}\left(y^2+4y+6\right)$$

Ece: $\quad y''\left(e^{-y}\left(y^2+4y+6\right)\right) = 0$

$$y'' = 0 \rightarrow \quad \underline{\underline{y(x) = cx + d}}$$

stačí pro zkladnoi

$$y' \quad - \quad 1$$

(2k) $\int_a^b 2\pi y \sqrt{1+y'^2}\, dx$

$$f = 2\pi\, y \sqrt{1+y'^2} \qquad \frac{\partial f}{\partial y} = 2\pi \sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y'} = 2\pi y \cancel{\frac{1}{2}} \frac{1}{\sqrt{1+y'^2}} \cdot \cancel{2}y'$$

$$\cancel{2\pi} y \left( \frac{\sqrt{1+y'^2} - y' \cdot \cancel{\frac{1}{2}} \frac{1}{\sqrt{1+y'^2}} \cdot \cancel{2}y'}{1+y'^2} \right) \cdot y''$$

$$+ \frac{\cancel{2\pi}\, y' \cdot y'}{\sqrt{1+y^2}} - \cancel{2\pi} \sqrt{1+y'^2} = 0$$

Zkratka $\quad y' \frac{\partial f}{\partial y'} - f = c$

$$y'\, 2\pi y \frac{1}{\sqrt{1+y'^2}} \cdot y' - 2\pi y \sqrt{1+y'^2} = c$$

pak $\quad y \left( \frac{y'^2 - (1+y'^2)}{\sqrt{1+y'^2}} \right) = c$

$$y = c_1 \sqrt{1+y'^2} \qquad |^2$$

$$y^2 = c_2 + c_2 y'^2$$

$$y'^2 = \frac{y^2 - c_2}{c_2} \quad \to \quad y' = \pm \frac{\sqrt{y^2 - c_2}}{c_2}$$

(2m)

$$\int_0^1 y'^2 - 2yy' + 10yx \, dx \qquad y(0) = 1 \qquad y(1) = 2$$

$$f = y'^2 - 2yy' + 10yx$$

$$\frac{\partial f}{\partial y} = -2y' + 10x \qquad \frac{\partial f}{\partial y'} = 2y' - 2y$$

$$+2y'' \quad -\cancel{2y'} + \cancel{2y'} - 10x = 0$$

$$y'' = 5x$$

tedy $\quad y' = \frac{5}{2}x^2 + c$

$$y = \frac{5}{6}x^3 + cx + d$$

$d = 1 \qquad 2 = \frac{5}{6} + c + 1$

$$\frac{1}{6} = c$$

celkem $\quad \underline{y = \frac{5}{6}x^3 + \frac{x}{6} + 1}$

Differentiating (2),

$$Y'(x) = y'(x) + \epsilon\eta'(x) \tag{4}$$

Replacing $y$ and $y'$ in (1) $Y$ and $Y'$ from (2) and (4), we obtain the integral

$$I(\epsilon) = \int_{x_1}^{x_2} f(x, Y, Y')dx \tag{5}$$

which is a function of the parameter $\epsilon$. Thus the problem of determining $y(x)$ reduces to finding the extremum of $I(\epsilon)$ at $\epsilon = 0$ which is obtained by solving $I'(\epsilon = 0) = 0$. For this, differentiate (5) w.r.t. $\epsilon$, we get

$$\frac{dI}{d\epsilon} = I'(\epsilon) = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial Y}\frac{\partial Y}{\partial \epsilon} + \frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial \epsilon}\right)dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial Y}\eta + \frac{\partial f}{\partial Y'}\eta'\right)dx$$

putting $\epsilon = 0$,

$$I'(0) = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right)dx \tag{6}$$

because for $\epsilon = 0$, we have from (2) $Y = y$ and $Y' = y'$. Integrating the second integral in R.H.S. of (6) by parts, we have

$$I'(0) = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta + \left[\frac{\partial f}{\partial y'}\eta\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx\right]$$

Since by (3), $\eta(x_1) = \eta(x_2) = 0$, the second term vanishes and using $I'(0) = 0$, we get

$$I'(0) = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta\, dx = 0 \tag{7}$$

Since $\eta(x)$ is arbitrary, equation (7) holds good only when the integrand is zero

i.e.,
$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \tag{2}$$

*Note:* Equation (2) is not sufficient condition. Solution of (2) may be maximum or minimum or a horizontal inflexion. Thus $y(x)$ is known as extremizing function or extremal and the term extremum includes maximum or minimum or stationary value.

## EQUIVALENT FORMS OF EULER'S EQUATION:

(I) Differentiating $f$, which is a function of $x, y, y'$, w.r.t. $x$, we get

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} \tag{8}$$

Consider

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y'}y'' \tag{9}$$

Subtracting (9) from (8), we have

$$\frac{df}{dx} - \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

Rewriting this

$$\frac{d}{dx}\left\{f - y'\frac{\partial f}{\partial y'}\right\} - \frac{\partial f}{\partial x} = y'\left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right\} \tag{10}$$

Since by Euler's Equation (2), the R.H.S. of (10) is zero, we get another form of Euler's equtaion

$$\boxed{\frac{d}{dx}\left\{f - y'\frac{\partial f}{\partial y'}\right\} - \frac{\partial f}{\partial x} = 0} \tag{11}$$

(II) Since $\frac{\partial f}{\partial y'}$ is also function $\phi$ of $x, y, y'$ say $\frac{\partial f}{\partial y'} = \phi(x, y, y')$. Differentiating w.r.t. $x$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} + \frac{\partial \phi}{\partial y'}\frac{dy'}{dx}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) + y'\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right) + y''\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial^2 f}{\partial x \partial y'} + y'\frac{\partial^2 f}{\partial y \partial y'} + y''\frac{\partial^2 f}{\partial y'^2} \tag{12}$$

Substituting (12) in the Euler's equation (2), we have

$$\boxed{\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y'\frac{\partial^2 f}{\partial y \partial y'} - y''\frac{\partial^2 f}{\partial y'^2} = 0} \tag{13}$$

**General case:** the necessary condition for the occurrence of extremum of the general integral

$$\int_{x_1}^{x_2} f(x, y_1, y_2, \ldots, y_\eta, y_1', y_2', \ldots, y_\eta')dx$$

involving $\eta$ functions $y_1, y_2, \ldots, y_\eta$, is given by the





set of $\eta$ Euler's equations

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0$$

for $i = 1, 2, 3, \ldots, \eta$.

**First integrals of the Euler-Lagrang's equation:** Degenerate cases: Euler's equation is readily integrable in the following cases:

Case (a): If $f$ is independent of $x$, then $\frac{\partial f}{\partial x} = 0$ and equivalent form of Euler's Equation (11) reduces to

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

Integrating, we get the first integral of Euler's equation

$$\boxed{f - y'\frac{\partial f}{\partial y'} = \text{constant}} \tag{14}$$

Thus the extremizing function $y$ is obtained as the solution of a first-order differential equation (14) involving $y$ and $y'$ only.

Case (b): If $f$ is independent of $y$, then $\frac{\partial f}{\partial y} = 0$, and the Euler's Equation (2) reduces to

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Integrating, we get the first integral of the Euler's equation as,

$$\boxed{\frac{\partial f}{\partial y'} = \text{constant}} \tag{15}$$

which is a first order differential equation involving $y'$ and $x$ only.

Case (c): If $f$ is independent of $x$ and $y$ then the partial derivative $\frac{\partial f}{\partial y'}$ is independent of $x$ and $y$ and is therefore function of $y'$ alone. Now (15) of case (b) $\frac{\partial f}{\partial y'} =$ constant has the solution.

$$y' = \text{constant} = c_1$$

Integrating, the extremizing function is a linear function of $x$ given by

$$y = c_1 x + c_2$$

Case (d): If $f$ is independent of $y'$, then $\frac{\partial f}{\partial y'} = 0$ and the Euler's Equation (2) reduces to

$$\frac{\partial f}{\partial y} = 0$$

Integrating, we get $\boxed{f = f(x)}$, i.e., function of $x$ alone.

**Geodesics:** A geodesic on a surface is a curve on the surface along which the distance between any two points of the surface is a minimum.

## 4.4 STANDARD VARIATIONAL PROBLEMS

### *Shortest distance*

**Example 1:** Find the shortest smooth plane curve joining two distinct points in the plane. (3)

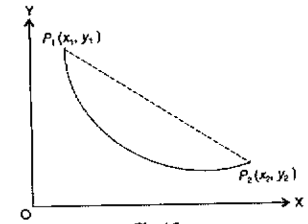



Fig. 4.2

*Solution:* Assume that the two distinct points be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie in the $XY$-Plane. If $y = f(x)$ is the equation of any plane curve $c$ in $XY$-Plane and passing through the points $P_1$ and $P_2$, then the length $L$ of curve $c$ is given by

$$L[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}dx \tag{1}$$

The variational problem is to find the plane curve whose length is shortest i.e., to determine the function $y(x)$ which minimizes the functional (1). The condition for extrema is the Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Here $f = \sqrt{1 + y'^2}$ so $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial y'} = \frac{1}{2}\frac{2y'}{\sqrt{1+y'^2}}$

Then

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

or $\quad y' = k\sqrt{1 + y'^2} \quad$ where $k$ = constant

Squaring $\quad y'^2 = k^2(1 + y'^2)$

i.e., $y' = \sqrt{\frac{k^2}{1-k^2}} = m = \text{constant}.$

Integrating, $y = mx + c$, where $c$ is the constant of integration. Thus the straight line joining the two points $P_1$ and $P_2$ is the curve with shortest length (distance).

### *Brachistochrone (shortest time) problem*

**Example 2:** Determine the plane curve down which a particle will slide without friction from the point $A(x_1, y_1)$ to $B(x_2, y_2)$ in the shortest time.

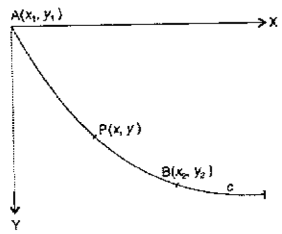



Fig. 4.3

*Solution:* Assume the positive direction of the y-axis is vertically downward and let $x_1 < x_2$. Let $P(x, y)$ be the position of the particle at any time $t$, on the curve $c$. Since energy is conserved, the speed $v$ of the particle sliding along any curve is given by

$$v = \sqrt{2g(y - y^*)}$$

where $y^* = y_1 - \left(\frac{v_1^2}{2g}\right)$. Here $g$ is acceleration due to gravity, $v_1$ is the initial speed. Choose the origin at $A$ so that $x_1 = 0$, $y_1 = 0$ and assume that $v_1 = 0$. Then

$$\frac{ds}{dt} = v = \sqrt{2gy}$$

Integrating this, we get the time taken by the particle moving under gravity (and neglecting friction along the curve and neglecting resistance of the medium) from $A(0, 0)$ to $B(x_2, y_2)$ is

$$t[y(x)] = \int \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int_{x=0}^{x=x_2} \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx \quad (1)$$

subject to the boundary conditions $y(0) = 0$ and $y(x_2) = y_2$. Integral (1) is convergent although it is improper. Here

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{y}}$$

which is independent of $x$. Now

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{y}} \frac{1}{\sqrt{1+y'^2}} \cdot \frac{1}{2} \cdot 2y'$$

The Euler's equation

$$\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0$$

reduces to

$$\frac{d}{dx}\left[\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}}\right] = 0$$

Integrating

$$\frac{\sqrt{1+y'^2}\sqrt{1+y'^2} - y'^2}{\sqrt{y}\sqrt{1+y'^2}} = k_1 = \text{constant}$$

or $$y(1 + y'^2) = k_2 \quad (1)$$

where $k_2 = \left(\frac{1}{k_1}\right)^2$, put $y' = \cot\theta$ where $\theta$ is a parameter. Then from (1)

$$y = \frac{k_2}{1+y'^2} = \frac{k_2}{1+\cot^2\theta} = k_2 \sin^2\theta = \frac{k_2}{2}(1 - \cos 2\theta) \quad (2)$$

Now

$$dx = \frac{dy}{y'} = \frac{\frac{k_2}{2}(+2 \cdot \sin 2\theta)d\theta}{\cot\theta}$$

$$= \frac{k_2 2 \cdot \sin\theta \cdot \cos\theta d\theta}{\cot\theta} = 2k_2 \sin^2\theta d\theta$$

$$dx = k_2 \cdot (1 - \cos 2\theta)d\theta.$$

Integrating, $x = k_2\left(\theta - \frac{\sin 2\theta}{2}\right) + k_3$, where $k_3$ is constant of integration. So

$$x - k_3 = \frac{k_2}{2}(2\theta - \sin 2\theta) \quad (2)$$

Since $y = 0$ at $x = 0$, we have $k_3 = 0$. Put $2\theta = \phi$ in (1) and (2), then

$$x = \frac{k_2}{2}(\phi - \sin\phi), \; y = \frac{k_2}{2}(1 - \cos 2\phi) \quad (3)$$





## WORKED OUT EXAMPLES

### *Variational problems.*
### *f is dependent on x, y, y'*

1e

**Example 1:** Find a complete solution of the Euler-Lagrange equation for

$$\int_{x_1}^{x_2}\left[y^2 - (y')^2 - 2y\cosh x\right]dx \quad (1)$$

*Solution:* Here $f(x, y, y') = y^2 - (y')^2 - 2y\cosh x$, which is a function of $x, y, y'$. The Euler-Lagrange equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad (2)$$

Differentiating (1) partially w.r.t. $y$ and $y'$, we get

$$\frac{\partial f}{\partial y} = 2y - 2\cosh x \quad (3)$$

$$\frac{\partial f}{\partial y'} = -2y' \quad (4)$$

Substituting (3) and (4) in (2), we have

$$2y - 2\cosh x - \frac{d}{dx}(-2y') = 0$$

$$y'' + y = \cosh x \quad (5)$$

The complimentary function of (5) is

$$y_c = c_1\cos x + c_2 \sin x$$

and particular integral of (5) is

$$y = \frac{1}{2}\cosh x.$$

Thus the complete solution Euler-Lagrange Equation (5) is

$$y(x) = c_1 \cos x + c_2 \sin x + \frac{1}{2}\cosh x.$$

### *f is independent of x*

**Example 1:** Find the extremals of the functional

$$I[y(x)] = \int_{x_1}^{x_2} \frac{(1+y^2)}{y'^2} dx$$

*Solution:* Here $f = \frac{1+y^2}{y'^2}$ which is independent of $x$. So the Euler's equation becomes

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0 \quad (1)$$

Here $$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left(\frac{1+y^2}{y'^2}\right) = -\frac{2(1+y^2)}{y'^3} \quad (2)$$

Substituting (2) in (1), we have

$$\frac{d}{dx}\left(\frac{1+y^2}{y'^2} - y'\frac{(-2)(1+y^2)}{y'^3}\right) = 3\frac{d}{dx}\left(\frac{1+y^2}{y'^2}\right) = 0$$

$$\frac{y'^2(2yy') - (1+y^2)2y'y''}{y'^4} = 0$$

or $$(1+y^2)y'' - yy'^2 = 0 \quad (3)$$

Put $y' = p$, then $y'' = \frac{d}{dx}y' = \frac{d}{dx}p = \frac{dp}{dy}\frac{dy}{dx} = y'\frac{dp}{dy} = p\frac{dp}{dy}$. Putting these values in (3),

$$(1+y^2)p\frac{dp}{dy} - yp^2 = 0 \quad \text{or} \quad \frac{dp}{dy} = \frac{py}{1+y^2}$$

Integrating $$\frac{dp}{p} = \frac{y\,dy}{1+y^2} = \frac{1}{2}\frac{d(1+y^2)}{(1+y^2)}$$

$$p^2 = c_1^2(1+y^2).$$

so $p = c_1\sqrt{(1+y^2)}$ or $\frac{dy}{dx} = c\sqrt{1+y^2}$

Integrating $$\frac{dy}{\sqrt{1+y^2}} = c_1\,dx \quad \text{we get}$$

$$\sinh^{-1} y = c_1 x + c_2$$

Thus the required extremal function is

$$y(x) = \sinh(c_1 x + c_2)$$

where $c_1$ and $c_2$ are two arbitrary constant.

### *f is independent of y*

**Example 3:** If the rate of motion $v = \frac{ds}{dt}$ is equal to $x$ then the time $t$ spent on translation along the curve $y = y(x)$ from one point $P_1(x_1, y_1)$ to another point $P_2(x_2, y_2)$ is a functional. Find the extremal of this functional, when $P(1, 0)$ and $P_2(2, 1)$.

*Solution:* Given $\frac{ds}{dt} = x$ or $\frac{ds}{x} = dt.$

But $ds = \sqrt{1+y'^2}dx$ so $\sqrt{1+y'^2}\frac{dx}{x} = dt.$

## Section 4

# Examples and Interpretations

1a

**Example 1.** Find a function $x(t)$ to solve the problem of Section 2 above:

$$\min \int_0^T x'^2(t)\,dt$$

$$\text{subject to} \quad x(0) = 0, \quad x(T) = B.$$

The integrand is $F(t, x, x') = x'^2$. Thus, $F_{x'} = 2x'$. Since $F$ is independent of $x$ in this case, $F_x = 0$. Hence Euler's equation is $0 = 2x''(t)$, that is equivalent to $x'' = 0$, integrating once yields $x'(t) = c_1$, where $c_1$ is a constant of integration. Integrating again gives $x(t) = c_1 t + c_2$, where $c_2$ is another constant of integration. The two constants are determined from the boundary conditions of the problem; evaluating the function $x$ at $t = 0$ and $t = T$ gives conditions that $c_1$ and $c_2$ must satisfy, namely

$$x(0) = 0 = c_2, \qquad x(T) = B = c_1 T + c_2.$$

Solving, one obtains $c_1 = B/T$, $c_2 = 0$; therefore the solution to the Euler equation with the given boundary conditions is

$$x(t) = Bt/T, \qquad 0 \le t \le T.$$

If there is a solution to the problem posed, this must be it. Of course, this is the solution found in Section 2.

1c

**Example 2.** Find extremals for

$$\int_0^1 \{[x'(t)]^2 + 10tx(t)\}\,dt \qquad \text{subject to} \quad x(0) = 1, \quad x(1) = 2.$$



Since $F(t, x, x') = x'^2 + 10tx$, we have $F_x = 10t$, $F_{x'} = 2x'$ and $dF_{x'}/dt = 2x''$; thus the Euler equation is $10t = 2x''$ or, equivalently,

$$x''(t) = 5t.$$

The variables $x$ and $t$ are separated. Integrate, introducing constants of integration $c_1$ and $c_2$:

$$x'(t) = 5t^2/2 + c_1$$

$$x(t) = 5t^3/6 + c_1 t + c_2.$$

The constants are found by using the boundary conditions; they obey

$$x(0) = 1 = c_2, \qquad x(1) = 2 = 5/6 + c_1 + c_2.$$

Solving, $c_1 = 1/6$, $c_2 = 1$, and the extremal sought is

$$x(t) = 5t^3/6 + t/6 + 1.$$

1f

**Example 3.** Find extremals for

$$\int_{t_0}^{t_1} \left[tx'(t) + (x'(t))^2\right] dt \qquad \text{subject to} \quad x(t_0) = x_0, \quad x(t_1) = x_1,$$

where $t_0$, $t_1$, $x_0$ and $x_1$ are given parameters. Write $F(t, x, x') = tx' + x'^2$ and compute $F_x = 0$ and $F_{x'} = t + 2x'$. Therefore, the Euler equation is

$$dF_{x'}/dt = d(t + 2x')/dt = 0.$$

Since the right side is zero, there is no need to carry out the differentiation; a function whose derivative is zero must itself be constant. Hence

$$t + 2x'(t) = c_1$$

for some constant $c_1$. Separate the variables, integrate again, and rearrange the result slightly to obtain

$$x(t) = c_2 + c_1 t/2 - t^2/4.$$

The constants of integration must satisfy the pair of equations

$$x(t_0) = x_0 = c_2 + c_1 t_0/2 - t_0^2/4,$$

$$x(t_1) = x_1 = c_2 + c_1 t_1/2 - t_1^2/4.$$

1b

**Example 4.** Returning to Example 1.1, we seek a production and inventory accumulation plan to minimize the sum of production and storage costs:

$$\min \int_0^T \{c_1[x'(t)]^2 + c_2 x(t)\}\,dt$$

$$\text{subject to} \quad x(0) = 0, \quad x(T) = B, \quad x'(t) \ge 0.$$

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where $c_1$ and $c_2$ are given nonnegative constants. Suppose the optimal solution satisfies the nonnegativity condition $x'(t) \geq 0$ with strict inequality, so that this constraint is never binding. Since $F_x = c_2$, $F_{x'} = 2c_1 x'$, the Euler equation is $2c_1 x'' = c_2$ or

$$x''(t) = c_2/2c_1. \tag{1}$$

Integration twice yields

$$x(t) = c_2 t^2/4c_1 + k_1 t + k_2,$$

where $k_1$ and $k_2$ are constants of integration determined by the boundary conditions:

$$x(0) = 0 = k_2, \qquad x(T) = B = c_2 T^2/4c_1 + k_1 T + k_2.$$

Thus

$$k_1 = B/T - c_2 T/4c_1, \qquad k_2 = 0$$

so

$$\underline{x(t) = c_2 t(t - T)/4c_1 + Bt/T,} \qquad 0 \leq t \leq T \tag{2}$$

is the extremal sought.

We check whether (2) obeys $x' \geq 0$. From the Euler equation (1), it follows immediately that $x'' > 0$ so that $x'$ is an increasing function of $t$. Therefore $x'(t) \geq 0$ for all $t$ if and only if it holds initially; $x'(0) = k_1 \geq 0$. This means the constraint $x'(t) \geq 0$ will be satisfied by (2) provided that

$$B \geq c_2 T^2/4c_1. \tag{3}$$

Therefore (2) is the solution to the problem if required total production $B$ is sufficiently large relative to the time period $T$ available, and the storage cost $c_2$ is sufficiently small relative to the unit production cost $c_1$. If (3) does not hold, then start of production is postponed in the optimal plan. This will be demonstrated later (Section II.10) when we study the case where $x' \geq 0$ is a tight constraint.

The Euler equation $2c_1 x'' = c_2$ has an interpretation. Recall that $c_2$ is the cost of holding one additional unit of inventory for one time period. Also $c_1[x'(t)]^2$ is the total production cost at $t$, so $2c_1 x'$ is the instantaneous marginal cost of production and $2c_1 x''$ is its time rate of change. Therefore, the Euler equation calls for balancing the rate of change of the marginal production cost against the marginal inventory holding cost to minimize the cost of delivering $B$ units of product at time $T$.

This interpretation may be clearer after integrating the Euler equation over a very small segment of time, say $\Delta$. Since the equation must hold for all $t$ along the path, we have



$$\int_t^{t+\Delta} 2c_1 x''(t)\, ds = \int_t^{t+\Delta} c_2\, ds,$$

that is, using (A1.1)

$$2c_1[x'(t+\Delta) - x'(t)] = c_2 \Delta$$

or, rearranging,

$$2c_1 x'(t) + c_2 \Delta = 2c_1 x'(t+\Delta).$$

Thus, the marginal cost of producing a unit at $t$ and holding it for an increment of time $\Delta$ must be the same as the marginal cost of producing it at $t + \Delta$. That is, we are indifferent between producing a marginal unit at $t$ or postponing it a very little while. Indeed, all along the optimal path, no shift in the production schedule can reduce cost.

**Example 5.** In Example 4, suppose expenditures are discounted at a continuous rate $r$ (review the appendix to Section 1).

$$\min \int_0^T e^{-rt}[c_1 x'^2 + c_2 x]\, dt$$

$$\text{subject to} \qquad x(0) = 0, \quad x(T) = B.$$

Again $x'(t) \geq 0$ is needed for economic sense, and again this requirement is temporarily set aside. Compute $F_x = e^{-rt}c_2$ and $F_{x'} = 2e^{-rt}c_1 x'(t)$. The Euler equation is

$$e^{-rt}c_2 = d(2e^{-rt}c_1 x'(t))/dt.$$

It calls for balancing the present value of the marginal inventory cost at $t$ with the rate of change of the corresponding marginal production cost. Integrating this equation over a small increment of time, we get

$$\int_t^{t+\Delta} e^{-rt}c_2\, ds = \int_t^{t+\Delta} [d(2e^{-rt}c_1 x'(s))/ds]\, ds;$$

that is,

$$2e^{-rt}c_1 x'(t) + \int_t^{t+\Delta} e^{-rt}c_2\, ds = 2e^{-r(t+\Delta)}c_1 x'(t+\Delta).$$

The marginal cost of producing a unit at $t$ and holding it over the next little increment of time $\Delta$ equals the marginal cost of producing a unit at $t + \Delta$. Consequently, we are indifferent between producing a marginal unit at $t$ or a little later. No change in this production schedule can reduce total discounted cost.

## Section 5

# Solving the Euler Equation in Special Cases

The Euler equation can be difficult to solve. If any of the three arguments $(t, x, x')$ do not appear or if the integrand has a special structure, then hints for solutions are available. Some of these instances are mentioned below, both for specific guidance and for practice. Note that sometimes the "hints" are of little use and a direct solution to the Euler equation may be the easier route.

*Case 1.* $F$ depends on $t, x'$ only: $F = F(t, x')$.
Since $F$ does not depend on $x$, the Euler equation (3.11) reduces to

$$F_{x'} = \text{const}.$$

This is a first order differential equation in $(t, x')$ only and is referred to as a first integral of the Euler equation. Examples 4.1 and 4.3 fit this case.

1d

**Example.** The Euler equation for

$$\int_{t_0}^{t_1} (3x' - tx'^2)\,dt \qquad \text{subject to} \quad x(t_0) = x_0, \quad x(t_1) = x_1,$$

is

$$F_{x'} = 3 - 2tx' = c_0;$$

therefore

$$tx' = (c_0 - 3)/(-2) \equiv c_1.$$

Separate variables:

$$x' = c_1/t;$$

and integrate:

$$x = c_1 \ln t + c_2.$$



The constants of integration $c_1$ and $c_2$ satisfy the pair of equations

$$x_0 = c_1 \ln t_0 + c_2, \qquad x_1 = c_1 \ln t_1 + c_2.$$

*Case 2.* $F$ depends on $x, x'$ only; $F = F(x, x')$.
The Euler equation ~~(3.12) reduces to the *first integral*~~

$$F - x'F_{x'} = \text{const}, \qquad t_0 \le t \le t_1,$$

which is the first order differential equation to be solved.

1k

**Example.** Among the curves joining $(t_0, x_0)$ and $(t_1, x_1)$, find one generating a surface of minimum area when rotated about the $t$ axis. That is,

$$\min \int_{t_0}^{t_1} 2\pi x\left[1 + (x')^2\right]^{1/2} dt$$

$$\text{subject to} \qquad x(t_0) = x_0, \quad x(t_1) = x_1.$$

Since (ignoring the constant $2\pi$)

$$F_{x'} = xx'/\left[1 + (x')^2\right]^{1/2},$$

we solve

$$F - x'F_{x'} = x\left[1 + (x')^2\right]^{1/2} - x(x')^2/\left[1 + (x')^2\right]^{1/2} = c.$$

Manipulating algebraically,

$$x = c(1 + x'^2)^{1/2}, \qquad \text{or} \qquad x^2 = c^2 + c^2x'^2.$$

Rearranging, providing $c \neq 0$,

$$x'^2 = (x^2 - c^2)/c^2, \qquad \text{or} \qquad x' = \pm\left[(x^2 - c^2)/c^2\right]^{1/2}.$$

We can deal with the positive root only because of the symmetry of the problem. Separate variables:

$$dx/(x^2 - c^2)^{1/2} = dt/c,$$

provided $x \neq c$. Integrate (using an integral table):

$$\ln\left[\left(x + (x^2 - c^2)^{1/2}\right)/c\right] = (t + k)/c,$$

where $k$ is the constant of integration. Note that the derivative of $\ln c = 0$ so that when we differentiate both sides we get back the original differential equation. Taking antilogs gives

$$x + (x^2 - c^2)^{1/2} = ce^{(t+k)/c}.$$

Now

$$x - (x^2 - c^2)^{1/2} = \left[x - (x^2 - c^2)^{1/2}\right]\left[x + (x^2 - c^2)^{1/2}\right] / \left[x + (x^2 - c^2)^{1/2}\right]$$
$$= \left[x^2 - (x^2 - c^2)\right] / \left[x + (x^2 - c^2)^{1/2}\right]$$
$$= c^2 / ce^{(t-k)/c} = ce^{-(t+k)/c}.$$

That last step follows by substitution for $x + (x^2 - c^2)^{1/2}$. Finally, by addition we get

$$x = c\left[e^{(t+k)/c} + e^{-(t+k)/c}\right]/2.$$

This is the equation of a figure called a catenary; $c$ and $k$ can be found using the conditions $x(t_0) = x_0$ and $x(t_1) = x_1$.

1i

<u>Example. The brachistochrone problem.</u>

$$\min \int_{x_0}^{x_1} \left[(1 + y'^2)/y\right]^{1/2} dx,$$

where the constant $(2g)^{-1/2}$ is ignored. As the integrand does not involve $x$ explicitly (i.e., here the general form of the integrand is $F(x, y(x), y'(x))$ rather than $F(t, x(t), x'(t))$ as in our previous examples), we have

$$F - y' F_{y'} = \left[(1 + y'^2)/y\right]^{1/2} - y'^2\left[y(1 + y'^2)\right]^{-1/2}$$
$$= \left[y(1 + y'^2)\right]^{-1/2}\left[1 + y'^2 - y'^2\right] = \text{a constant}.$$

This implies that

$$\left[y(1 + y'^2)\right]^{-1/2} = \text{a constant}$$

which, in turn, means that

$$y(1 + y'^2) = 2k,$$

where $k$ is a constant, so

$$y' = \left[(2k - y)/y\right]^{1/2}.$$

Separating variables gives

$$\left[y/(2k - y)\right]^{1/2} dy = dx.$$

Multiply the numerator and denominator in the bracketed expression by $y$ to get

$$y\,dy/(2ky - y^2)^{1/2} = dx.$$



Both sides can now be integrated to get

$$x = -(2ky - y^2)^{1/2} + k \arccos(1 - y/k) + c,$$

where $c$ is a constant. This is the equation of a cycloid.

**Example.** Newton's Second Law of Motion is $F = ma = md^2x/dt^2$, a second order differential equation. The problem is to find the integral for which this differential equation is the Euler equation. It turns out to be the integral

$$\int_{t_0}^{t_1} \left(mx'^2/2 - V(x)\right] dt.$$

Since the integrand does not involve $t$,

$$mx'^2/2 - V(x) - mx'^2 = c,$$

a constant. Thus,

$$-V(x) - mx'^2/2 = c.$$

Differentiation with respect to $t$ yields

$$-V'x' - mx'x'' = 0$$

or

$$-V'(x) = mx''.$$

Identifying $-V'(x) = F$ yields the desired result.

The term $mx'^2/2$ in the integrand is a particle's kinetic energy, while $V(x)$ is defined as its potential energy. Physicists let $T = mx'^2/2$ and call

$$\int_{t_0}^{t_1} (T - V)\,dt = \int_{t_0}^{t_1} L\,dt$$

the *action*, or Hamilton's integral, where $L = T - V$ is called the Lagrangian. The description of the motion of a particle through space in terms of the Euler equation of this integral is referred to as the Principle of Least Action or Hamilton's Principle of Stationary Action. This principle plays a unifying role in theoretical physics in that laws of physics that are described by differential equations have associated with them an appropriate action integral. Indeed, discovering the action integral whose Euler equation is the desired physical law is a major achievement.

The "special methods" are *not* always the easiest way to solve a problem. Sometimes an applicable, special form of the Euler equation is easier to work with than the ordinary form and sometimes it is more difficult. The easiest route is determined by trial and error. For example, consider finding extremals for

$$\int_{t_0}^{t_1} 2x^2 + 3xx' - 4(x')^2\big]\,dt \qquad \text{subject to} \quad x(t_0) = x_0, \quad x(t_1) = x_1.$$

Since the integrand does not depend on $t$, we could write the Euler equation in the form $F - x'F_{x'} = c$; that is

$$2x^2 + 4(x')^2 = c. \tag{1}$$

This nonlinear differential equation is not readily solved.

On the other hand, the standard form of the Euler equation ($F_x = dF_{x'}/dt$) for this problem is the second order linear differential equation

$$2x'' + x = 0, \tag{2}$$

whose solution is easily found (see Section B3). The characteristic equation associated with this differential equation is $2r^2 + 1 = 0$, with roots $r = \pm i/2^{1/2}$. Hence extremals are of the form

$$x(t) = c_1 \sin t/2^{1/2} + c_2 \cos t/2^{1/2}.$$

The constants $c_1$ and $c_2$ are found using the given boundary conditions. (Differentiating the Euler equation found first (1) totally with respect to $t$ leads to the second form (2).) Note that several of the exercises of this section are of the form of this illustration and are more readily solved by the standard Euler equation.

*Case 3. F depends on $x'$ only: $F = F(x')$.*

The Euler equation is $F_{x'x'}x'' = 0$. Thus along the extremal at each $t$, either $F_{x'x'}(x') = 0$ or $x''(t) = 0$. In the latter case, integration twice indicates that the extremal is of the linear form $x(t) = c_1 t + c_2$. In the former case, either $F_{x'x'}(x') \equiv 0$ or else $x'$ is constant, i.e., $F_{x'} = 0$. The case of $x'$ constant was just considered. If $F$ is linear in $x'$, $F(x') = a + bx'$, the Euler equation is an identity and any $x$ satisfies it trivially (see also Case 5 to come).

We conclude that if the integrand $F$ depends solely on $x'$ but is not linear, then graphs of extremals are straight lines. Even if the functional form of $F$ or $F_{x'x'}$ appears complicated, we know that extremals must be linear. Boundary conditions determine the constants.

This result may be applied immediately to Example 1.4 to conclude that the *shortest distance between two points* in a plane is the straight line connecting them.

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**Example.** Extremals of

$$\int_{t_0}^{t_1} (x')^2 \exp(-x')\, dt \qquad \text{subject to} \quad x(t_0) = x_0, \quad x(t_1) = x_1$$

must be of the form $x(t) = c_1 t + c_2$.

*Case 4. F depends on $t$, $x$ only: $F = F(t, x)$.*

The Euler equation is $F_x(t, x) = 0$, which is not a differential equation. It calls for optimizing the integrand at each $t$. The dynamic problem is degenerate. (Review (1.4), for example).



*Case 5. F is linear in $x'$: $F = A(t, x) + B(t, x)x'$.*

The Euler equation is $A_x + B_x x' = B_t + B_x x'$; that is, $A_x(t, x) = B_t(t, x)$, which is not a differential equation. This may be viewed as an implicit equation for $x$ in terms of $t$. If the solution $x(t)$ of this equation satisfies the boundary conditions, it may be the optimal solution.

Alternatively, the Euler equation $A_x = B_t$ may be an identity, $A_x(t, x) \equiv B_t(t, x)$, satisfied by any function $x$. Then, according to the integrability theorem for exact differential equations (see Appendix B), there is a function $P(t, x)$ such that $P_t \equiv A$, $P_x \equiv B$ (so $P_{tx} \equiv A_x \equiv B_t$) and

$$F(t, x, x') = A + Bx' = P_t + P_x x' = dP/dt.$$

Thus, the integrand is the total derivative of a function $P$ and

$$\int_{t_0}^{t_1} (A + Bx')\, dt = \int_{t_0}^{t_1} (dP/dt)\, dt = P(t_1, x(t_1)) - P(t_0, x(t_0)).$$

The value of the integral depends only on the endpoints; the path joining them is irrelevant in this case. Any feasible path is optimal. This is analogous to the problem of maximizing a constant function; any feasible point would yield the same value.

Case 5 is the only instance in which an Euler equation is an identity. To understand this, suppose that (3.10) is an identity, satisfied for any set of four values $t, x, x', x''$. The coefficient of $x''$ must be zero if (3.10) is to hold for every possible value of $x''$. Thus, $F_{x'x'} \equiv 0$. Then $F_x - F_{x't} - x'F_{xx'} \equiv 0$ for any $t, x, x'$. The first identity implies that $F$ must be linear in $x'$, so $F$ has the form $A(t, x) + B(t, x)x'$. Then the second identity becomes $A_x \equiv B_t$, as was to be shown.

Two integrands that appear quite different can lead to the same Euler equation and thus have the same extremals. This happens if the integrands differ by an exact differential. For example, let $P(t, x)$ be a twice differentiable function and define

$$Q(t, x, x') \equiv dP/dt = P_t(t, x) + P_x(t, x)x'(t).$$

Then for any twice differentiable function $F(t, x, x')$, the two integrals

$$\int_{t_0}^{t_1} F(t, x, x')\, dt \qquad \text{and} \qquad \int_{t_0}^{t_1} [F(t, x, x') + Q(t, x, x')]\, dt$$

subject to $\quad x(t_0) = x_0, \quad x(t_1) = x_1$

differ by a constant (namely, $P(t_1, x_1) - P(t_0, x_0)$) and the Euler equations associated with the respective integrals are identical.

1ée

**Example 1.** $\int_{t_0}^{t_1} x'(t)\,dt$ subject to $x(t_0) = x_0$, $x(t_1) = x_1$. Since $F = x'$, $F_x = 0$ and $F_{x'} = 1$, so that the Euler equation is $0 = 0$, satisfied always. The integrand is an exact differential, so

$$\int_{t_0}^{t_1} x'(t)\,dt = x(t_1) - x(t_0) = x_1 - x_0$$

for any differentiable function. The value of the integral depends only on the endpoint conditions and not on the path connecting the endpoints.

**Example 2.** Suppose the production cost in Example 1.1 is linear:

$$\min \int_0^T [c_1 x'(t) + c_2 x(t)]\,dt$$

$$\text{subject to} \quad x(0) = 0, \quad x(T) = B.$$

Then $F_x = c_2$ and $F_{x'} = c_1$, so that the Euler equation is $c_2 = 0$. This means that there is no optimal production plan if there is a positive holding cost ($c_2 > 0$) and that any feasible plan is optimal if the holding cost is zero. The underlying economics is as follows. If the unit production cost is constant, then the total cost of manufacturing $B$ units is $c_1 B$ irrespective of the time schedule:

$$\int_0^T c_1 x'(t)\,dt = c_1[x(T) - x(0)] = c_1 B.$$

If the cost of holding inventory is zero, then all feasible production plans are equally good. If the inventory holding cost is positive, then postponing production to the last moment reduces the total storage cost. The limiting answer is $x(t) = 0$, $0 < t < T$, $x(T) = B$, which is a plan that can be approached but is not itself a continuous function.

**Example 3.** To find extremals for

$$\int_0^T tx x'\,dt \qquad \text{subject to} \quad x(0) = 0, \quad x(T) = B,$$

compute $F_x = tx'$, $F_{x'} = tx$, $dF_{x'}/dt = x + tx'$. The Euler equation is $tx' = x + tx'$ or $0 = x$, which can be satisfied only if $B = 0$.

To verify that there is no solution to the problem, integrate by parts by letting $xx'\,dt = dv$ and $t = u$, sc $x^2/2 = v$ and $dt = du$. Then

$$\int_0^T txx'\,dt = \left(B^2 T - \int_0^T x^2\,dt\right)\Big/ 2 \le B^2 T/2.$$



The upper bound of $B^2T/2$ can be realized only by setting $x(t) = 0$, $0 \le t \le T$. This function satisfies the Euler equation but not the boundary conditions (except if $B = 0$). Evidently, there is no minimum; the integral can be made arbitrarily small.

**Example 4.** For

$$\int_{t_0}^{t_1} e^{-rt}(x' - ax)\,dt \qquad \text{subject to} \quad x(t_0) = x_0, \quad x(t_1) = x_1,$$

we compute

$$F_x = -ae^{-rt}, \qquad F_{x'} = e^{-rt}, \qquad dF_{x'}/dt = -re^{-rt}.$$

The Euler equation is $a = r$. If, in fact $a = r$, then the Euler equation is an identity and the integrand is an exact differential, namely $d(e^{-rt}x(t))/dt$. The value of the integral is $e^{-rt_1}x_1 - e^{-rt_0}x_0$, independent of the path between the given endpoints. On the other hand, if $a \ne r$, then the Euler equation cannot be satisfied; therefore there is no optimum. To verify this, add $rx - rx$ to the integrand and then use the observation just made:

$$\int_{t_0}^{t_1} e^{-rt}(x' - rx + rx - ax)\,dt$$

$$= \int_{t_0}^{t_1} e^{-rt}(x' - rx)\,dt + \int_{t_0}^{t_1} e^{-rt}(rx - ax)\,dt$$

$$= x_1 e^{-rt_1} - x_0 e^{-rt_0} + (r - a)\int_{t_0}^{t_1} e^{-rt} x\,dt.$$

If $r = a$, all feasible paths give the same value. If $r \ne a$, the value of the integral may be made arbitrarily large or small by choosing a path with a very high peak or low trough.

**Example 5.** Consider the discounted profit maximization problem (compare Example 1.3),

$$\max \int_0^T e^{-rt}[p(t)f(K(t)) - c(t)(K' + bK)]\,dt \tag{3}$$

$$\text{subject to} \quad K(0) = K_0, \quad K(T) = K_T,$$

where $c(t)$ is the cost per unit of gross investment, $p(t)$ the price per unit of output (given functions of time), $K(t)$ the stock of productive capital, $f(K)$ output, and $I = K' + bK$ gross investment (net investment plus depreciation). Compute

$$F_K = e^{-rt}[pf'(K) - cb] \qquad \text{and} \qquad F_{K'} = -e^{-rt}c.$$

**Motivace:** *Jakou trasu má zvolit loď plující z jednoho přístavu do druhého, aby její dráha byla co nejkratší?*

*Matematická formulace této úlohy není složitá. Pluje-li loď po dráze parametrizované souřadnicí $x$ (její okamžitá poloha je $(x, y(x), z(x))$), pak pro délku dráhy platí*

$$l(y,z) = \int_{x_0}^{x_1} \sqrt{1 + \left(y'(x)\right)^2 + \left(z'(x)\right)^2}\,dx. \tag{14}$$

*Podmínkou v naší úloze je použití lodě. Jinými slovy, cesta se musí odehrávat po hladině moře, tj. po kulové ploše s poloměrem asi 6400km. Funkce $y$ a $z$ musí v každém bodě dráhy splňovat podmínku*

$$x^2 + \left(y(x)\right)^2 + \left(z(x)\right)^2 = 6400^2 \tag{15}$$

*a dále musí platit okrajové podmínky - tj. odkud kam vede dráha lodě.*

# 2 Příklady

## 2.1 Úlohy s pevnými koncovými body

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**Příklad 2.1** Nalezněte extremálu funkcionálu

$$J\big(x(t)\big) = \int_0^1 \dot{x}(t)\cdot t + [\dot{x}(t)]^2\,dt$$

s koncovými podmínkami extremály $x(0) = 0$ a $x(1) = 2$.

*Řešení:* Nejprve určíme podle (2) Eulerovu - Lagrangeovu rovnici

$$\frac{\partial g}{\partial x} = 0, \qquad \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} = \frac{d}{dt}\big(t + 2\dot{x}(t)\big) = 1 + 2\ddot{x}(t) \qquad \Rightarrow \qquad 0 = -1 - 2\ddot{x}(t)\,.$$

Nyní provedeme integraci

$$\frac{d^2x(t)}{dt^2} = -\frac{1}{2} \quad \to \quad \frac{dx(t)}{dt} = -\frac{1}{2}t + a \quad \to \quad x(t) = -\frac{1}{4}t^2 + a\,t + b.$$

Koeficienty $a$ a $b$ určíme z koncových podmínek extremály.

$$\begin{aligned} x(0) = 0 \quad &\Rightarrow \quad 0 = b \\ x(1) = 2 \quad &\Rightarrow \quad 2 = -\tfrac{1}{4}\cdot 1 + a \quad \Rightarrow \quad a = \tfrac{9}{4} \end{aligned}$$

Extremála, pro kterou zadaný funkcionál $J\big(x(t)\big)$ nabývá extrému, tedy je

$$x(t) = -\frac{1}{4}t^2 + \frac{9}{4}t.$$

□



1h

**Příklad 2.2** Nalezněte funkci $x \in C^2(0,1)$, pro kterou nabývá funkcionál

$$J\big(x(t)\big) = \int_0^{\pi/2} [\dot{x}(t)]^2 - [x(t)]^2\,dt$$

extrému za podmínky $x(0) = 0$ a $x(\pi/2) = 1$.

*Řešení:* Eulerova - Lagrangeova rovnice je

$$0 = 2x(t) + 2\ddot{x}(t).$$

Obecným řešením této rovnice je funkce

$$x(t) = c_1 \sin t + c_2 \cos t.$$

Integrační konstanty $c_1$, $c_2$ určíme z okrajových podmínek.

$$\begin{aligned} x(0) = 0 \quad &\Rightarrow \quad 0 = c_1 \sin 0 + c_2 \cos 0 = c_2 \\ x(\pi/2) = 1 \quad &\Rightarrow \quad 1 = c_1 \sin(\pi/2) = c_1 \end{aligned}$$

Extremála, pro kterou zadaný funkcionál $J\big(x(t)\big)$ nabývá extrému, tedy je

$$x(t) = \sin t.$$

□

## 2.2 Úlohy s volnými koncovými body

### 2.2.1 Pevný koncový čas, volná koncová hodnota

**Příklad 2.3** Nalezněte extremálu funkcionálu

$$J\big(x(t)\big) = \int_0^1 \dot{x}(t)\cdot t + [\dot{x}(t)]^2\,dt$$

s koncovými podmínkami extremály $x(0) = 0$ a $x(1)$ je volné.

*Řešení:* Eulerova - Lagrangeova rovnice je shodná jako v příkladě 2.1, tj.

$$0 = -1 - 2\ddot{x}(t)\,.$$

Po integraci opět dostaneme

$$x(t) = -\frac{1}{4}t^2 + a\,t + b.$$

The Euler equation is

$$d\left[-e^{-rt}c(t)\right]/dt = e^{-rt}\left[p(t)f'(K(t)) - c(t)b\right].$$

To interpret this, integrate over a small interval of time:

$$e^{-rt}c(t) - e^{-r(t+\Delta)}c(t+\Delta) = \int_t^{t+\Delta} e^{-rs}\left[p(s)f'(K(s)) - c(s)b\right]\,ds.$$

The cost difference between purchasing a marginal unit of capital at $t$ rather than at $t+\Delta$ is just offset by the marginal profit earned by that capital over the period $[t, t+\Delta]$.

Performing the indicated differentiation in the Euler equation yields the equivalent requirement

$$e^{-rt}\left[pf'(K) - cb\right] = e^{-rt}\left[rc - c'\right],$$

so the rule for choosing the optimal level of capital $K^*(t)$ is

$$p(t)f'(K^*(t)) = (r+b)c(t) - c'(t). \tag{4}$$

This is a static equation for $K^*(t)$, not a differential equation. It is feasible only if $K^*(0) = K_0$ and $K^*(T) = K_T$. It says that, if possible, capital stock should be chosen so that the value of the marginal product of capital at each moment equals the cost of employing it. The "user cost" of capital $(r+b)c - c'$ includes not only foregone interest on the money invested in capital and decline in money value because of its physical deterioration but also capital gains (or losses). Capital gains may occur, for example, if the unit price of capital rises, thereby increasing the value of the capital stock held by the firm. On the other hand, capital loss is possible, for instance, through the invention of a new productive method that makes the firm's capital stock outmoded, diminishing its value.

1m

**Example 6.** The Euler equation for

$$\int_0^1 (x'^2 - 2xx' + 10tx)\,dt \qquad \text{subject to} \quad x(0) = 1, \quad x(1) = 2$$

is $x'' = 5t$ with solution $x(t) = 5t^3/6 + t/6 + 1$. This problem, its Euler equation, and its solution should be compared with Example 4.2. The Euler equations and solutions are identical. The integrands differ by the term $-2xx' = d(-x^2)/dt$, an exact differential. The value of the two integrals evaluated along the extremal will differ by

$$-x^2(1) + x^2(0) = -4 + 1 = -3.$$



## EXERCISES

1. Find candidates to maximize or minimize

$$\int_{t_0}^{t_1} [t(1 + (x')^2)]^{1/2}\,dt \qquad \text{subject to} \quad x(t_0) = x_0, \quad x(t_1) = x_1.$$

You need not find the constants of integration.

2. Find candidates to maximize or minimize

$$\int_{t_0}^{t_1} F(t, x, x')\,dt \qquad \text{subject to} \quad x(t_0) = x_0, \quad x(t_1) = x_1$$

(but do not find the constants of integration) where
a. $F(t, x, x') = x^2 + 4xx' + 2(x')^2$,
b. $F(t, x, x') = x^2 - 3xx' - 2(x')^2$,
c. $F(t, x, x') = x'(\ln x')^2$,
d. $F(t, x, x') = -x^2 + 3xx' + 2(x')^2$,
e. $F(t, x, x') = te^{x'}$

3. Find candidates to maximize or minimize

$$\int_{t_0}^{t_1} [x^2 + axx' + b(x')^2]\,dt \qquad \text{subject to} \quad x(t_0) = x_0, \quad x(t_1) = x_1.$$

Consider the cases $b = 0$, $b > 0$, and $b < 0$. How does the parameter $a$ affect the solution? Why?

4. A monopolist believes that the number of units $x(t)$ he can sell depends not only on the price $p(t)$ he sets, but also on the rate of change of price, $p'(t)$:

$$x = a_0 p + b_0 + c_0 p'. \tag{5}$$

His cost of producing at rate $x$ is

$$C(x) = a_1 x^2 + b_1 x + c_1. \tag{6}$$

Given the initial price $p(0) = p_0$ and required final price $p(T) = p_1$, find the price policy over $0 \le t \le T$ to maximize profits

$$\int_0^T [px - C(x)]\,dt$$

given (5), (6), and the boundary conditions above. (Caution: this problem involves much messy algebra; it has been included for its historic interest. See the suggestions for further reading that follow.)

5. Suppose a mine contains an amount $B$ of a mineral resource (such as coal, copper, or oil). The profit rate that can be earned from selling the resource at rate $x$ is $\ln x$. Find the rate at which the resource should be sold over the fixed period $[0, T]$ to maximize the present value of profits from the mine. Assume the discount

(2k) $\int_a^b 2\pi y\sqrt{1+y'^2}\,dx$

$$f = 2\pi\, y\sqrt{1+y'^2} \qquad \frac{\partial f}{\partial y} = 2\pi\sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y'} = 2\pi y\,\cancel{\tfrac{1}{2}}\,\frac{1}{\sqrt{1+y'^2}}\cdot\cancel{2}y'$$

$$\cancel{2\pi}\, y\left(\frac{\sqrt{1+y'^2} - y'\,\cancel{\tfrac{1}{2}}\frac{1}{\sqrt{1+y'^2}}\cdot\cancel{2}y'}{1+y'^2}\right)\cdot y''$$

$$+\ \frac{\cancel{2\pi}\, y'\cdot y'}{\sqrt{1+y^2}} - \cancel{2\pi}\sqrt{1+y'^2} = 0$$

zkratka $\quad y'\frac{\partial f}{\partial y'} - f = c$

$$y'\,\cancel{2\pi}y\frac{1}{\sqrt{1+y'^2}}\cdot y' - \cancel{2\pi}\, y\sqrt{1+y'^2} = c$$

pak $\quad y\left(\dfrac{y'^2 - (1+y'^2)}{\sqrt{1+y'^2}}\right) = c$

$$y = c_1\sqrt{1+y'^2} \qquad |^2$$

$$y^2 = c_2 + c_2 y'^2$$

$$y'^2 = \frac{y^2-c_2}{c_2} \quad\to\quad y' = \pm\frac{\sqrt{y^2-c_2}}{c_2}$$

stačí pro kladné

$$\frac{y'}{\sqrt{y^2-c_2}} = \frac{1}{c_2}.$$

pak $$\ln\left(\frac{y+(y^2-c_2)^{1/2}}{\sqrt{c_2}}\right) = \frac{x+k}{c_2}$$

tedy $$y + \sqrt{y^2-c_2} = c\,e^{\frac{x+k}{c}}$$

$$\rightsquigarrow \quad y = \frac{c}{2}\left(e^{\frac{x+k}{c}} + e^{-\frac{x+k}{c}}\right)$$