(1) Rozviňte do Laurentovy řady na oblasti

$1 < |z| < 2$

$$f(z) = \frac{1}{(z+1)(z+2)}$$

$$= \frac{1}{z+1} - \frac{1}{z+2}$$

$$\frac{1}{z+1} = \frac{1}{z}\cdot\frac{1}{\left(1+\frac{1}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n$$ ne $\left|\frac{1}{z}\right| < 1$

$1 < |z|$

$$\frac{-1}{2+z} = \frac{-1}{2\left(1+\frac{z}{2}\right)} = \sum_{n=0}^{\infty}\frac{(-1)}{2}\cdot\left(-\frac{1}{2}\right)^n z^n$$ pro $\left|\frac{z}{2}\right| < 1$

$|z| < 2$

celkem $$\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}}z^n$$

(2) Rozvinte do Laurentovy řady v bodě 0 pro $|z| > 0$

$$\frac{e^{z^2}-1}{z^3} = \frac{1}{z^3}\left(\sum_{n=0}^{\infty}\frac{(z^2)^n}{n!} - 1\right) = \frac{1}{z^3}\sum_{n=1}^{\infty}\frac{z^{2n}}{n!} =$$

$$= \underline{\underline{\sum_{n=1}^{\infty}\frac{z^{2n-3}}{n!}}}$$

(3) klasifikujte typ singularity v bodě $z_0 = 0$

$$z^2 \sin\left(\frac{1}{z}\right)$$

$$z^2 \sin \frac{1}{z} = z^2 \sum_{n=0}^{\infty} (-1)^n \frac{\frac{1}{z^{2n+1}}}{(2n+1)!} \quad \text{pro } z \in \mathbb{C}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2-2n-1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\frac{1}{z^{2n-1}}}{(2n+1)!}$$

$\to$ podstatná singularita

(4) Spočítajte rezíduum

$$\operatorname{res}_{1+i} \frac{z}{(z^2-2z+2)^2}$$

$$\frac{z}{(z^2-2z+2)^2} = \frac{z}{(z-1-i)^2(z-1+i)^2}$$

pól rádu 2. (netreba overovať)

$$\lim_{z\to 1+i} \frac{1}{(2-1)!}\left((z-1-i)^2 \frac{z}{(z-1-i)^2(z-1+i)^2}\right)' =$$

$$= \lim_{z\to 1+i}\left(\frac{z}{(z-1+i)^2}\right)' = \lim \frac{(z-1+i)^2 - z\cdot 2(z-1+i)}{(z-1+i)^4}$$

$$= \frac{(1+i-1+i)^2 - (1+i)\,2\,(1+i-1+i)}{(1+i-1+i)^4} =$$

$$= \frac{(2i)^2 - 2(1+i)\,2i}{(2i)^4} = \frac{1}{16}(-4-4i+4) = -\frac{i}{4}$$

$$\operatorname{res}_{1+i} \frac{z}{(z^2-2z+2)^2} = -\frac{i}{4}$$