4. cvičení

http://www.karlin.mff.cuni.cz/~kuncova/
kytaristka@gmail.com

Příklady

7.

$$\lim_{x\to+\infty} x^{3/2}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)$$

Řešení: Protože na rozvíjení v nekonečnu nemáme vztahy, provedeme substituci $y=\frac{1}{x}$ a dostaneme

$$\lim_{x\to+\infty} x^{3/2}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)=\lim_{y\to0+} y^{-3/2}\left(\sqrt{\frac{1}{y}+1}+\sqrt{\frac{1}{y}-1}-2\sqrt{\frac{1}{y}}\right)=$$

$$=\lim_{y\to0+} y^{-2}\left(\sqrt{1+y}+\sqrt{1-y}-2\right)=$$

a nyní rozvineme obě odmocniny do druhého řádu

$$=\lim_{y\to0+} y^{-2}\left((1+\frac{1}{2}y+\frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2!}y^2+o(y^2))+(1-\frac{1}{2}y+\frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2!}y^2+o(y^2))-2\right)=$$

$$=\lim_{y\to0+}\left(-\frac{1}{8}-\frac{1}{8}+o(1)\right)=-\frac{1}{2}+0=-\frac{1}{4}.$$

8.

$$\lim_{x\to+\infty}\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5}$$

Řešení:

Provedeme substituci $y=\frac{1}{x}$.

$$\lim_{x\to+\infty}\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5}=\lim_{x\to+\infty} x\sqrt[6]{1+\frac{1}{x}}-x\sqrt[6]{1-\frac{1}{x}}=\lim_{y\to0+}\frac{\sqrt[6]{1+y}-\sqrt[6]{1-y}}{y}=$$

a nyní rozvineme odmocniny v čitateli, stačí do prvního řádu

$$=\lim_{y\to0+}\frac{(1+\frac{1}{6}y+o(y)-(1-\frac{1}{6}y+o(y))}{y}=\frac{1}{3}+\lim_{y\to0+}\frac{o(y)}{y}=\frac{1}{3}+0=\frac{1}{3}.$$

9.

$$\lim_{x\to+\infty}\left[\left(x^3-x^2+\frac{x}{2}\right)e^{1/x}-\sqrt{x^6+1}\right]$$

Řešení: Provedeme substituci $y=\frac{1}{x}$.

$$\lim_{x\to+\infty}\left[\left(x^3-x^2+\frac{x}{2}\right)e^{1/x}-\sqrt{x^6+1}\right]=\lim_{y\to0+}\frac{\left[\left(1-y+\frac{1}{2}y^2\right)e^y-\sqrt{1+y^6}\right]}{y^3}=$$

a nyní rozvineme exponencielu a odmocninu v čitateli do třetího řádu

$$= \lim_{y\to 0+} \frac{\left(1 - y + \frac{1}{2}y^2\right)\left(1 + y + \frac{y^2}{2} + \frac{y^3}{6} + o(y^3)\right) - 1 + o(y^6)}{y^3} =$$

$$= \lim_{y\to 0+} \frac{\left(1 - y + \frac{1}{2}y^2\right) + \left(y - y^2 + \frac{1}{2}y^3\right) + \frac{y^2}{2} - \frac{y^3}{2} + \frac{y^3}{6} + o(y^3) - 1}{y^3} =$$

$$= \lim_{y\to 0+} \frac{\frac{y^3}{6} + o(y^3)}{y^3} = \frac{1}{6} + \lim_{y\to 0+} \frac{o(y^3)}{y^3} = \frac{1}{6}.$$

10.

$$\lim_{x\to +\infty} \left[x - x^2 \ln\left(1 + \frac{1}{x}\right)\right]$$

Řešení: Provedeme substituci $y = \frac{1}{x}$.

$$\lim_{x\to +\infty} \left[x - x^2 \ln\left(1 + \frac{1}{x}\right)\right] = \lim_{y\to 0+} \left[\frac{1}{y} - \frac{1}{y^2} \ln(1+y)\right] =$$

a nyní rozvineme logaritmus do druhého řádu

$$= \lim_{y\to 0+} \left[\frac{1}{y} - \frac{1}{y^2}\left(y - \frac{y^2}{2} + o(y^2)\right)\right] =$$

$$= \lim_{y\to 0+} \left[\frac{1}{y} - \frac{1}{y} + \frac{1}{2} + o(1)\right] = \frac{1}{2}.$$

11.

$$\lim_{x\to 0} \frac{\sqrt{1+\operatorname{tg} x} - \sqrt{1+\sin x}}{x^3}$$

Řešení:

Protože jmenovatel je třetího řádu, budeme hledat rozvoj čitatele do třetího řádu. Je

$$\operatorname{tg} x = x + \frac{x^3}{3} + o(x^3)$$

odkud vyplývá

$$\sqrt{1+\operatorname{tg} x} = 1 + \frac{1}{2}\operatorname{tg} x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\operatorname{tg}^2 x + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}\operatorname{tg}^3 x + o(x^3) =$$

$$= 1 + \frac{1}{2}x + \frac{1}{6}x^3 - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

Na druhou stranu

$$\sqrt{1+\sin x} = 1 + \frac{1}{2}\sin x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\sin^2 x + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}\sin^3 x + o(x^3) =$$

$$= 1 + \frac{1}{2}x - \frac{1}{12}x^3 - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

odkud vyplývá, že

$$\sqrt{1 + \operatorname{tg} x} - \sqrt{1 + \sin x} = \left(\frac{1}{6} + \frac{1}{12}\right) x^3 + o(x^3) = \frac{1}{4}x^3 + o(x^3),$$

a tedy

$$\lim_{x \to 0} \frac{\sqrt{1 + \operatorname{tg} x} - \sqrt{1 + \sin x}}{x^3} = \frac{1}{4}.$$

12.

$$\lim_{x \to 0} \frac{\cos(xe^x) - \cos(xe^-x)}{x^3}$$

Řešení: Zřejmě stačí rozvést čitatel do třetího řádu.

$$\lim_{x \to 0} \frac{\cos(xe^x) - \cos(xe^-x)}{x^3} = \lim_{x \to 0} \frac{1 - \frac{1}{2}x^2e^{2x} - (1 - \frac{1}{2}x^2e^{-2x}) + o(x^3)}{x^3} =$$

$$= \lim_{x \to 0} \left(\frac{e^{-2x} - e^{2x}}{2x} + \frac{o(x^3)}{x^3} \right) = -2 + 0 = -2.$$

13.

$$\lim_{x \to 0} \frac{e^{x^2+5x^4} - e^{x^2-3x^4}}{(\cos x - 1)(\cosh x - 1)}$$

Řešení:

Protože platí

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}, \qquad \lim_{x \to 0} \frac{\cosh x - 1}{x^2} = \frac{1}{2},$$

pak, existuje-li limita napravo, platí

$$\lim_{x \to 0} \frac{e^{x^2+5x^4} - e^{x^2-3x^4}}{(\cos x - 1)(\cosh x - 1)} = \lim_{x \to 0} -4 \frac{e^{x^2+5x^4} - e^{x^2-3x^4}}{x^4} =$$

stačí tedy rozvést čitatel do čtvrtého stupně. Tak dostaneme

$$= \lim_{x \to 0} -4 \frac{(1 + x^2 + 5x^4 + \frac{x^4}{2}) - (1 + x^2 - 3x^4 + \frac{x^4}{2}) + o(x^4)}{x^4} = \lim_{x \to 0} -4 \frac{8x^4 + o(x^4)}{x^4} = -32.$$

14.

$$\lim_{x \to 0} \frac{\sinh(\operatorname{tg} x) - x}{x^3}$$

Řešení:

Čitatel musíme rozvést do třetího řádu. Platí, že

$$\sinh(\operatorname{tg} x) = \frac{e^{\operatorname{tg} x} - e^{-\operatorname{tg} x}}{2} = \frac{1 + \operatorname{tg} x + \frac{\operatorname{tg}^2 x}{2} + \frac{\operatorname{tg}^3 x}{6} + o(x^3) - (1 - \operatorname{tg} x + \frac{\operatorname{tg}^2 x}{2} - \frac{\operatorname{tg}^3 x}{6} + o(x^3))}{2} =$$

$$= \operatorname{tg} x + \frac{\operatorname{tg}^3 x}{6} + o(x^3)$$

dostáváme tak

$$\lim_{x\to 0} \frac{\sinh(\operatorname{tg} x) - x}{x^3} = \lim_{x\to 0} \frac{\operatorname{tg} x + \frac{\operatorname{tg}^3 x}{6} - x + o(x^3)}{x^3} = \lim_{x\to 0} \frac{\operatorname{tg} x - x}{x^3} + \lim_{x\to 0} \frac{1}{6}\frac{\operatorname{tg}^3 x}{x^3} + \lim_{x\to 0} \frac{o(x^3)}{x^3} =$$

A dále platí, že

$$\operatorname{tg} x = \frac{\sin x}{\cos x} = \frac{x - x^3/3! + o(x^4)}{1 - x^2/2 + o(x^3)} =$$

$$= (x - \frac{x^3}{3!} + o(x^4)) \cdot \sum_{k=0}^{\infty} (\frac{x^2}{2} + o(x^3))^k =$$

$$= x - \frac{x^3}{6} + \frac{x^3}{2} + o(x^3) = x + \frac{1}{3}x^3 + o(x^3)$$

a proto dostaneme, že

$$= \lim_{x\to 0} \frac{\operatorname{tg} x - x}{x^3} + \lim_{x\to 0} \frac{1}{6}\frac{\operatorname{tg}^3 x}{x^3} + \lim_{x\to 0} \frac{o(x^3)}{x^3} =$$

$$= \lim_{x\to 0} \frac{\frac{1}{3}x^3 + o(x^3)}{x^3} + \lim_{x\to 0} \frac{1}{6}\frac{\operatorname{tg}^3 x}{x^3} + \lim_{x\to 0} \frac{o(x^3)}{x^3} = \frac{1}{3} + \frac{1}{6} + 0 = \frac{1}{2}.$$

15. Najděte $a, b \in \mathbb{R}$ tak, aby $\lim_{x\to 0} \frac{x - (a + b\cos x)\sin x}{x^4} = 0$.

Řešení:

Platí

$$x - a\sin x - b\sin x\cos x = x - a\left(x - \frac{x^3}{6} + o(x^4)\right) - \frac{b}{2}\left(2x - \frac{(2x)^3}{6} + o(x^4)\right) =$$

$$= x - ax + \frac{ax^3}{6} - bx + \frac{4bx^3}{6} + o(x^4).$$

Aby limita byla nulová, musí být pro každé x z nějakého okolí nuly (a tedy všude)

$$x - ax - bx = 0 \implies 1 - a - b = 0$$

$$\frac{ax^3}{6} + \frac{4bx^3}{6} = 0 \implies a + 4b = 0$$

Odtud máme $a = -4b$ a $1 + 4b - b = 0$, tedy $b = -\frac{1}{3}$ a $a = \frac{4}{3}$.

16. $\lim_{x\to 0} \frac{(1+x)^x - 1}{x^n}$

Řešení:

Je $(1+x)^x = e^{x\ln(1+x)}$ a rozvoj je

$$e^{x\ln(1+x)} = e^{x(x-\frac{x^2}{2}+o(x^2))} = e^{x^2-x^3/2+o(x^3)} = 1+x^2-\frac{x^3}{2}+o(x^3) = 1+x^2+o(x^2),$$

tudíž

$$(1+x)^x - 1 = e^{x\ln(1+x)} - 1 = x^2 + o(x^2),$$

hledané n je rovno dvěma a platí

$$\lim_{x\to 0} \frac{(1+x)^x - 1}{x^2} = \lim_{x\to 0} \frac{x^2+o(x^2)}{x^2} = 1.$$

17. $\lim_{x\to 0} \frac{\ln^2(1+\sin x) - \ln^2(1+\arcsin x)}{x^n}$

Řešení:

Porovnáme rozvoje funkcí $\sin x$ a $\arcsin x$ a zjistíme první člen, kde se liší. Ten bude určující.

$$\sin x = x - \frac{x^3}{6} + o(x^3), \qquad \arcsin x = x + \frac{x^3}{6} + o(x^3).$$

Odtud vidíme, že pravděpodobně budeme muset rozvíjet do třetího řádu. Je

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + o(y^3),$$

a proto

$$\ln(1+x\pm\frac{x^3}{6}+o(x^3)) = \left(x \pm \frac{x^3}{6} + o(x^3)\right) - \frac{1}{2}\left(x \pm \frac{x^3}{6} + o(x^3)\right)^2 + \frac{1}{3}\left(x \pm \frac{x^3}{6} + o(x^3)\right)^3 + o(x^3) =$$

$$= x - \frac{1}{2}x^2 \pm \frac{x^3}{6} + \frac{1}{3}x^3 + o(x^3),$$

z čehož vyplývá, že

$$\ln^2(1+x\pm\frac{x^3}{6}+o(x^3)) = x^2 - x^3 + \frac{1}{4}x^4 \pm \frac{x^4}{3} + o(x^4).$$

Není třeba pokračovat do vyšších mocnin, hledali jsme první člen, kde se rozvoje budou lišit. Odtud vyplývá

$$\ln^2(1+\sin x) - \ln^2(1+\arcsin x) = \ln^2(1+x-\frac{x^3}{6}+o(x^3)) - \ln^2(1+x+\frac{x^3}{6}+o(x^3)) =$$

$$= \left(x^2 - x^3 + \frac{1}{4}x^4 - \frac{x^4}{3} + o(x^4)\right) - \left(x^2 - x^3 + \frac{1}{4}x^4 + \frac{x^4}{3} + o(x^4)\right) = -\frac{2}{3}x^4 + o(x^4).$$

Odtud vyplývá, že hledané $n = 4$ a platí

$$\lim_{x \to 0} \frac{\ln^2(1 + \sin x) - \ln^2(1 + \arcsin x)}{x^4} = \lim_{x \to 0} \frac{-\frac{2}{3}x^4 + o(x^4)}{x^4} = -\frac{2}{3}.$$

EXAMPLE 2 Find the sums of the following series:

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$. (b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

SOLUTION

(a) Recall that

$$x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} - \cdots = \ln(1+x).$$

Substituting in $x = 1$ yields

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln(2).$$

(b) Recall that

$$x - \frac{x}{3} + \frac{x}{5} - \frac{x}{7} + \frac{x}{9} - \cdots = \tan^{-1}(x).$$

Substituting in $x = 1$ yields

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \tan^{-1}(1) = \frac{\pi}{4}.$$

This is known as the **Gregory-Leibniz formula** for π. ∎

Limits Using Power Series

When taking a limit as $x \to 0$, you can often simplify things by substituting in a power series that you know.

EXAMPLE 3 Evaluate $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

SOLUTION We simply plug in the Taylor series for $\sin x$:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x - x}{x^3} &= \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x}{x^3} \\
&= \lim_{x\to 0} \frac{-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^3} \\
&= \lim_{x\to 0} -\frac{1}{3!} + \frac{1}{5!}x^2 - \frac{1}{7!}x^4 + \cdots = -\frac{1}{3!} = -\frac{1}{6}
\end{aligned}$$

∎

EXAMPLE 4 Evaluate $\lim\limits_{x\to 0}\dfrac{x^2e^x}{\cos x-1}$.

SOLUTION We simply plug in the Taylor series for e^x and $\cos x$:

$$\begin{aligned}\lim_{x\to 0}\frac{x^2e^x}{\cos x-1} &= \lim_{x\to 0}\frac{x^2\left(1+x+\frac{1}{2}x^2+\cdots\right)}{\left(1-\frac{1}{2}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6\cdots\right)-1}\\ &= \lim_{x\to 0}\frac{x^2+x^3+\frac{1}{2}x^4+\cdots}{-\frac{1}{2}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots}\\ &= \lim_{x\to 0}\frac{1+x+\frac{1}{2}x^2+\cdots}{-\frac{1}{2}+\frac{1}{24}x^4-\frac{1}{6!}x^2+\cdots} = \frac{1}{-1/2} = -2\end{aligned}$$

■

Sometimes a limit will involve a more complicated function, and you must determine the Taylor series:

EXAMPLE 5 Evaluate $\lim\limits_{x\to 0}\dfrac{\ln(\cos x)}{x^2}$.

SOLUTION Using the Taylor series formula, the first few terms of the Taylor series for $\ln(\cos x)$ are:

$$\ln(\cos x) = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \cdots.$$

(Really, we only need that first term.) Therefore,

$$\lim_{x\to 0}\frac{\ln(\cos x)}{x^2} = \lim_{x\to 0}\frac{-\frac{1}{2}x^2-\frac{1}{12}x^4+\cdots}{x^2} = \lim_{x\to 0}-\frac{1}{2}-\frac{1}{12}x^2+\cdots = -\frac{1}{2}$$

■

Limits as $x \to a$ can be obtained using a Taylor series centered at $x = a$:

EXAMPLE 6 Evaluate $\lim\limits_{x\to 1}\dfrac{\ln x}{x-1}$.

SOLUTION Recall that

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$$

Plugging this in gives

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots}{x-1}$$
$$= \lim_{x\to 1}\left(1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 + \cdots\right) = 1 \qquad \blacksquare$$

Taylor Polynomials

A partial sum of a Taylor series is called a **Taylor polynomial**. For example, the Taylor polynomials for e^x are:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= 1 + x \\ T_2(x) &= 1 + x + \frac{1}{2}x^2 \\ T_3(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \\ &\vdots \end{aligned}$$

You can approximate any function $f(x)$ by its Taylor polynomial:

$$f(x) \approx T_n(x)$$

If you use the Taylor polynomial centered at a, then the approximation will be particularly good near $x = a$.

TAYLOR POLYNOMIALS

Let $f(x)$ be a function. The **Taylor polynomials** for $f(x)$ centered at $x = a$ are:

$$\begin{aligned} T_0(x) &= f(a) \\ T_1(x) &= f(a) + f'(a)(x-a) \\ T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\vdots \end{aligned}$$

You can approximate $f(x)$ using a Taylor polynomial.

6 Applications

The first application of Taylor polynomials is to the calculation of limits of functions and sequences. Through *limit comparison* for improper integrals and series and the root/ratio test, Taylor polynomials become a very powerful tool to study the convergence of improper integrals and series.

Example 6.1. We start from the motivating example of the first section:

$$\lim_{x\to 0}\frac{\sin x - x + 2x^3}{x^2} = \lim_{x\to 0}\frac{x + O(x^2) - x + 2x^3}{x^2} = \lim_{x\to 0}\frac{O(x^2)+2x^3}{x^2} = \lim_{x\to} O(1){+}2x = \lim_{x\to 0} O(1) = ???$$

We have ended up with an $O(1)$, this means that the precision doesn't suffice. We try to take a better approximation of $\sin x$:

$$\lim_{x\to 0}\frac{\sin x - x + 2x^3}{x^2} = \lim_{x\to 0}\frac{x - \frac{x^3}{6} + O(x^4) - x + 2x^3}{x^2} = \lim_{x\to 0}\frac{O(x^4)+\frac{11}{6}x^3}{x^2} = \lim_{x\to 0} O(x^2) + \frac{11}{6}x = 0$$

Note that in the previous example we could have used de l'Hôpital twice and get the same result:

$$\lim_{x\to 0}\frac{\sin x - x + 2x^3}{x^2} = \lim_{x\to 0}\frac{\cos x - 1 + 6x}{2x} = \lim_{x\to 0}\frac{-\sin x + 6}{2} = 0$$

Sometimes, however, the method of de l'Hôpital will lead to very complicated formulas. The following example illustrates this:

Example 6.2. Consider the limit

$$\lim_{x\to 0}\frac{\cos(x)\sin(x\ln(1+x))}{x^2}.$$

The derivative of the numerator alone is:

$$-\sin(x)\sin(x\ln(1+x)) + \cos(x)\cos(x\ln(1+x))\left(\ln(1+x) + \frac{x}{1+x}\right)$$

and this still tends to 0 as x approaches 0. So to compute the original limit we would have to differentiate the above formula once more, making it very easy to do mistakes. Instead, we can use the big O notation as follows:

$$\cos(x)\sin(x\ln(1+x)) = (1+O(x^2))\left(x\ln(1+x) + O(x^3\ln^3(1+x))\right) =$$

$$(1+O(x^2))\left(x(x+O(x^2)) + O(x^3)\right) = (1+O(x^2))(x^2+O(x^3)) = x^2 + O(x^3),$$

so we obtain:

$$\lim_{x\to 0}\frac{x^2+O(x^3)}{x^2} = \lim_{x\to 0} 1 + O(x) = 1.$$

Caveat: expanding a composition of functions can be tricky. In the previous example we had to calculate $\sin(x\ln(1+x))$. In the example we have decided to expand sine first, and then log. In general it is faster to expand the inner function first, but it's also more delicate. Suppose we wanted to expand $\sin(\ln(1+x))$ up to $O(x^4)$:

$$\sin(\ln(1+x)) = \sin(x+O(x^2)) = x + O(x^2) + \ldots$$

no metter how much you expand sine you cannot improve $O(x^2)$. To reach the precision of $O(x^4)$ we first need to expand log up to $O(x^4)$:

$$\sin(\ln(1+x)) = \sin\Big(x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)\Big) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4) - \frac{1}{6}\Big(x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)\Big)^3 =$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4) - \frac{1}{6}x^3$$

In the expansion of sine we cannot stop at the first order, we have to calculate any order that contributes monomials up to degree four. In this specific case we need to calculate the third order but we don't need to take any higher order.

Example 6.3.

$$\lim_{x\to 0} \frac{\log(1 + x\arctan x) - e^{x^2} + 1}{\sqrt{1+x^4} - 1}$$

We can try to solve the limit just by taking Taylor polynomials of length 1:

$$\begin{aligned}
&= \lim_{x\to 0} \frac{\log(1 + x(x + O(x^3))) - 1 - x^2 + O(x^4) + 1}{\frac{1}{2}x^4 + O(x^8)} = \\
&= \lim_{x\to 0} \frac{\log(1 + x^2 + O(x^4))) - x^2 + O(x^4)}{\frac{1}{2}x^4 + O(x^8)} = \\
&= \lim_{x\to 0} \frac{x^2 + O(x^4) + O((x^2 + O(x^4))^2) - x^2 + O(x^4)}{\frac{1}{2}x^4 + O(x^8)} = \\
&= \lim_{x\to 0} \frac{O(x^4)}{\frac{1}{2}x^4 + O(x^8)} = \lim_{x\to 0} \frac{O(1)}{\frac{1}{2} + O(x^4)} = ???
\end{aligned}$$

The calculation is inconclusive because we have $O(1)$ at the numerator. This means that we have to increase the precision of the numerator:

$$\begin{aligned}
&= \lim_{x\to 0} \frac{\log(1 + x(x - \frac{1}{3}x^3 + O(x^5))) - x^2 - \frac{1}{2}x^4 + O(x^6)}{\frac{1}{2}x^4 + O(x^8)} = \\
&= \lim_{x\to 0} \frac{x^2 - \frac{1}{3}x^4 + O(x^6) - \frac{1}{2}(x^2 - \frac{1}{3}x^4 + O(x^6))^2 + O(x^6) - x^2 - \frac{1}{2}x^4 + O(x^6)}{\frac{1}{2}x^4 + O(x^8)} = \\
&= \lim_{x\to 0} \frac{-\frac{1}{3}x^4 - \frac{1}{2}x^4 - \frac{1}{2}x^4 + O(x^6)}{\frac{1}{2}x^4 + O(x^8)} = \\
&= \lim_{x\to 0} \frac{-\frac{4}{3} + O(x^2)}{\frac{1}{2} + O(x^4)} = -\frac{8}{3}
\end{aligned}$$

In the calculation we have used the following expansions:

$$\begin{aligned}
\arctan x &= x - \frac{1}{3}x^3 + O(x^5) \\
e^{x^2} &= 1 + x^2 + \frac{1}{2}x^4 + O(x^6) \\
\log(1+x) &= x - \frac{1}{2}x^2 + O(x^3) \\
\sqrt{1+x^4} &= 1 + \frac{1}{2}x^4 + O(x^8)
\end{aligned}$$