1) $\displaystyle\int_0^{\pi/2} \underbrace{\frac{\sin t}{t^{1-1/\alpha}}}_{\geq 0}\, dt \qquad \alpha \in \mathbb{R}^+$

LSK $\quad g(x) = \dfrac{t}{t^{1-1/\alpha}} = \dfrac{1}{t^{-1/\alpha}}$ $\qquad \displaystyle\int_0^{\pi/2} \frac{1}{t^{-1/\alpha}}\, dt$ AK $\Leftrightarrow -1/\alpha < 1$

$\alpha > -1$

$$\lim_{t \to 0} \frac{\frac{\sin t}{t^{1-1/\alpha}}}{\frac{t}{t^{1-1/\alpha}}} = 1$$

z LSK $\displaystyle\int_0^{\pi/2} \frac{\sin t}{t^{1-1/\alpha}}\, dt$ AK pro $\alpha > -1$

jinak D

(1b) $\displaystyle\int_{\pi/2}^{\infty} \frac{\sin t}{t^{1-1/\alpha}}\, dt$ $\qquad$ Fakt: AK $\qquad 1 - 1/\alpha > 1 \to$ nikdy

NAK $\qquad 0 < 1 - 1/\alpha \leq 1$

$\frac{1}{\alpha} < 1 \qquad 0 \leq \frac{1}{\alpha}$ ✓

$\boxed{1 < \alpha}$

(1c) $\displaystyle\int_0^{\infty} \frac{\sin t}{t^{1-1/\alpha}}\, dt$ $\qquad$ NAK pro $\alpha > 1$

(1d) $\displaystyle\int_0^{\infty} \sin x^\alpha\, dx = \int_0^{\infty} \frac{1}{\alpha} \frac{\sin y}{y^{1-1/\alpha}}\, dy$

$x^\alpha = y$ $\qquad$ $\begin{array}{ccc} x & 0 & \infty \\ y & 0 & \infty \end{array}$ $\Big\}$ NAK pro $\alpha > 1$

$x = y^{1/\alpha}$

$dx = \frac{1}{\alpha} y^{1/\alpha - 1}\, dy$

(2a) $\int_0^{42} \arcsin\frac{x}{x^2+1} \ln x \cos x \, dx$

v 0 $\exists$ spojité rozšíření, neboť

$$\lim_{x\to 0+} \arcsin\frac{x}{x^2+1} \ln x \cos x =$$

$$= \lim_{x\to 0+} \frac{\arcsin\frac{x}{x^2+1}}{\frac{x}{x^2+1}} \cdot \frac{x}{x^2+1} \ln x \cos x =$$

$$= \lim_{x\to 0+} \underbrace{\frac{\arcsin\frac{x}{x^2+1}}{\frac{x}{x^2+1}}}_{\downarrow \atop 1} \cdot \lim \underbrace{\frac{\cos x}{x^2+1}}_{\downarrow \atop \frac{1}{0+1}} \cdot \lim \underbrace{x \ln x}_{\downarrow \atop 0} = 0$$

$$\lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} \overset{\text{l'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$$

spoj. na kpt $\to$ AK

(2b) $\int_{42}^{\infty} \frac{\ln x}{x} \cos x \, dx$

AK: $\left|\frac{\ln x}{x}\cos x\right| \overset{SK}{\geq} \left|\frac{\cos x}{x}\right| \qquad \int_{42}^{\infty} \frac{\cos x}{x}$ D

NAK: $\left.\begin{array}{l} \frac{\ln x}{x} \to 0 \text{ , monotónně} \\ \cos x \quad \text{má om. PF} \end{array}\right\}$ Dirichlet $\Longrightarrow \int_{42}^{\infty} \frac{\ln x}{x} \cos x$ NAK

(2c) $$\int_{42}^{\infty} \underbrace{\frac{\arcsin\frac{x}{x^2+1}}{\frac{x}{x^2+1}}}_{\text{Ábel}} \cdot \underbrace{\frac{x}{x^2+1} \cdot x}_{\text{Ábel}} \cdot \underbrace{\frac{\ln x}{x} \cos x}_{\text{NAK}} \, dx$$

$\frac{x^2}{x^2+1} \to 1$, zjevně monotónní, má vl. lim, tedy je na okolí ∞ omezená

$$\frac{\arcsin \frac{x}{x^2+1}}{\frac{x}{x^2+1}} \to 1$$

monotonie : $\frac{\arcsin y}{y}$ je monot

$\frac{x}{x^2+1}$ je monot.

$$\frac{x^2+1-x\,2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{pro dost velké } x$$

složení 2 monot. je monot.

$y \in (-1,1)$ $z \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\frac{\arcsin y}{y}$ pro $y = \sin z$ $\frac{z}{\sin z}$

$$\left(\frac{z}{\sin z}\right)' = \frac{\sin z - z\cos x}{\sin^2 z}$$

$\sin z - z\cos z \leq 0$

$\frac{\sin z}{\cos z} \leq z$ ✓

tedy z Abela

$$\int_{42}^{\infty} \arcsin \frac{x}{x^2+1} \ln x \cos x \, dx \quad NAK$$

pro AK srovnáme s $\left|\frac{1}{x} \ln x \cos x\right|$, což D

∃ tedy jen AK

$$(201) \int_{0}^{\infty} \arcsin \frac{x}{x^2+1} \ln x \cos x \, dx$$

→ jen NAK

(3a) $\int_0^{\pi/4} \underbrace{\frac{\sin x \ \sin 2x}{x^\alpha}}_{\geq 0} dx \qquad \alpha \in \mathbb{R}^+$

LSK s $\frac{1}{x^{\alpha-2}}$

$$\lim_{x \to 0+} \frac{\sin x \ \sin 2x}{x^\alpha} \cdot x^{\alpha-2} = 2$$

$\int_0^{\pi/4} x^{2-\alpha}$ AK $\Leftrightarrow$ $2-\alpha > -1$

$\boxed{3 > \alpha}$

(3b) $\int_{\pi/4}^{\infty} \frac{\sin x \ \sin 2x}{x^\alpha} dx$

$$\sin x \ \sin 2x = \frac{\cos x - \cos 3x}{2}$$

AK $\int_{\pi/4}^{\infty} \frac{|\cos x|}{2x^\alpha}$

$\int_{\pi/4}^{\infty} \frac{|-\cos 3x|}{2x^\alpha}$

Fakt $\Big\}$ AK pro $\alpha > 1$
součet konv. $\Rightarrow$ $\int_{\pi/4}^{\infty} \frac{\cos x - \cos 3x}{2x^\alpha}$ AK pro $\alpha > 1$

NAK $\int_{\pi/4}^{\infty} \frac{\cos x}{2x^\alpha}$

$\int_{\pi/4}^{\infty} \frac{-\cos 3x}{2x^\alpha}$

Fakt $\Big\}$ NAK pro $0 < \alpha \leq 1$
součet $\Rightarrow$ konv. $\int_{\pi/4}^{\infty} \frac{\cos x - \cos 3x}{2x^\alpha}$ NAK pro $0 < \alpha \leq 1$

- vyloučení AK pro $\alpha \leq 1$
  a vyloučení NAK pro $\alpha \leq 0$ $\Big\}$ BC podmínka
  ↳ nutná, $\alpha > 0$

(3) $\displaystyle\int_0^\infty \frac{\sin x \sin 2x}{x^\alpha}$ $\qquad 0 < \alpha \leq 1$

BC (problém u nekonečne)

$$\left|\int_{0+2k\pi}^{2\pi+2k\pi} \left|\frac{1}{x^\alpha} \sin x \sin 2x\right| dx\right| \geq \frac{1}{2\pi(k+1)} \int_{\frac{3}{2}\pi+2k\pi}^{2\pi+2k\pi} |\sin x \sin 2x| \, dx$$

$$= \frac{1}{2\pi(k+1)} \frac{1}{2} \int_{\frac{3}{2}\pi+2k\pi}^{2\pi+2k\pi} \cos x - \cos 3x \, dx = \frac{2}{3} \frac{1}{2\pi(k+1)} = \frac{1}{3\pi(k+1)}$$

$$\left[\sin x - \frac{1}{3}\sin 3x\right]_{\frac{3}{2}\pi}^{2\pi} = -\left(-1 - \frac{1}{3}(1)\right) = \frac{4}{3}$$

Pak $\displaystyle\int_{2k\pi}^{\infty} \left|\frac{1}{x^\alpha} \sin x \sin 2x\right| dx \geq \sum_{n=2k+1}^{\infty} \frac{1}{3\pi(n+1)}$ což Diverguje

Tedy $\exists \varepsilon : \forall b' \in (0, \infty) \quad \exists x_1 = 2k\pi ; \; 2k\pi \geq \lceil b' \rceil$

a $x_2 = m$ aby $\displaystyle\sum_{n=2k\pi}^{m} \frac{1}{3\pi(n+1)} \geq \varepsilon$

$$\int_{2k\pi}^{m} |f(x)| \geq \varepsilon$$

$\Downarrow$

tedy diverguje

(5) $\int_{-\infty}^{\infty} \sin e^x \, dx = \int_0^{\infty} \frac{\sin y}{y} \, dy$ NAK (víme)

$y = e^x$

$dy = e^x dx$

(4)(a) $\int_1^{\infty} \frac{x^a}{e^{x^2}-1} \sin \frac{1}{x^2} \, dx$

LSK s e^{-x}

$$\lim_{x\to\infty} \frac{\left| \frac{x^a}{e^{x^2}-1} \sin \frac{1}{x^2} \right|}{\frac{1}{e^x}} = \lim \frac{e^x x^a}{e^{x^2}-1} \underbrace{\sin \frac{1}{x^2}}_{\to 0} =$$

$$= \lim_{x\to\infty} \underbrace{\frac{x^a}{e^x}}_{\downarrow\, 0} \; \underbrace{\frac{1}{e^{x^2-2x}-1}}_{\downarrow\, 0} \; \underbrace{\sin \frac{1}{x^2}}_{\downarrow\, 0} = 0$$

$\to$ AK $\forall a \in \mathbb{R}$

$\int_0^1 \frac{\sin t}{t^{\frac{a+1}{2}}} \, dt$ NAK pro $a > -1$

AK $a > 1$

$\int_0^1 x^{a-2} \sin \frac{1}{x^2} \, dx \to$ substituce

$\int_0^1 \underbrace{\frac{x^2}{e^{x^2}-1}}_{\to 1} \frac{x^a}{x^2} \sin \frac{1}{x^2}$ Závěr AK $a > 1$

NAK $a > -1$

monotónně $\left(\frac{y}{e^y-1}\right)' = \frac{e^y-1-ye^y}{(e^y-1)^2}$ $e^y(1-y) \le 1$

$(1-y)e^y \le (1-y)(1+y) =$

$= 1-y^2 \le 1$

$$\int_0^\infty \frac{\sin\left(\frac{1}{x}\right)\arctan x}{x}\, dx$$

(a) $$\int_0^{\pi/2} \sin\frac{1}{x}\, \frac{\arctan x}{x}\, dx = \int_{2/\pi}^{\infty} \sin y\, \frac{\arctan \frac{1}{y}}{1/y} \cdot \frac{1}{y^2}\, dy$$

$$y = \frac{1}{x} \qquad x = \frac{1}{y}$$

$$dy = -\frac{1}{x^2}\, dx \qquad dx = -\frac{1}{y^2}\, dy$$

$$= \int_{2/\pi}^{\infty} \underbrace{\frac{\sin y}{y}}_{\text{konv. z Dir.}}\ \underbrace{\arctan 1/y}_{\geq 0 \text{ monot.}}\, dy$$

$\int$ K z Abela

(b) $$\int_{\pi/2}^{\infty} \frac{\sin\frac{1}{x}}{x} \arctan x\, dx$$

srovnáme s $\frac{1}{x} \cdot \frac{\arctan x}{x}$

$$\lim_{x\to\infty} \frac{\frac{\sin\frac{1}{x}}{\cancel{x}}\,\cancel{\arctan x}}{\frac{1}{x}\,\frac{\cancel{\arctan x}}{\cancel{x}}} = 1$$

neb $$\int_{\pi/2}^{\infty} \frac{\arctan x}{x^2}\, dx \leq \int_{\pi/2}^{\infty} \frac{1}{x^2}\, dx \quad K$$

tedy i pův. $\int$ konv.