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$$\lim_{n\to\infty} \int_0^\infty e^{-nx^2}\,dx$$

fix $x \in (0,\infty)$ $\quad f_n = e^{-nx^2}$. $\quad$ spoj. $\Rightarrow$ měř.

$\downarrow$

$\lim_{n\to\infty} e^{-nx^2} = 0$ $\quad$ tedy spoj.

$$\int_0^\infty 0\,dx = 0$$

Pro $x \in (0,\infty)$

$e^{-nx^2} \leq e^{-x^2}$ $\qquad -nx^2 \leq -x^2$

$\qquad x^2 \leq nx^2$

$g = e^{-x^2}$ majoranta $\Rightarrow$ není v tabulce, ale ukázali jsme ve 2. cvičení

Tedy z Leb. lze prohodit

$$\lim_{n\to\infty} \int_0^\infty \underbrace{\frac{e^{-nx}}{1+x^2}}_{f_n} \, dx = 0$$

f_n spoj. $\to$ měř.

fix x: $\quad \lim\limits_{n\to\infty} \frac{e^{-nx}}{1+x^2} = 0$

$$\frac{e^{-nx}}{1+x^2} \leq \frac{1}{1+x^2} =: g(x)$$ integr. majoranta (víme z minulých cvičení)

Tedy z Lebesguea

$$\lim_{n\to\infty} \int_0^\infty \frac{e^{-nx}}{1+x^2} \, dx = \int_0^\infty 0 = 0$$

$$\lim_{n\to\infty} \int_0^1 \underbrace{\frac{n \sin x}{1+n^2\sqrt{x}}}_{f_n \text{ spoj.} \to \text{ měř.}} \, dx = 0$$

0 spoj. $\to$ měř.

fix $x \in (0,1)$: $$\lim_{n\to\infty} \frac{n \sin x}{1+n^2\sqrt{x}} = \lim_{n\to\infty} \frac{1}{n} \frac{\sin x}{\frac{1}{n^2} + \sqrt{x}} = 0$$

Lebesgue

$$\left| \frac{n \sin x}{1+n^2\sqrt{x}} \right| \le \frac{n}{1+n^2\sqrt{x}} \le \frac{n}{n^2\sqrt{x}} = \frac{1}{n\sqrt{x}} \le \frac{1}{\sqrt{x}} \text{ majoranta}$$

$$\lim_{n\to\infty} \int_0^1 \underbrace{n\sqrt{x}\, e^{-n^2x^2}}_{f_n} dx = 0$$

f_n spoj., 0 spoj. $\Rightarrow$ měř.

pro $x \in (0,1)$: $\lim_{n\to\infty} f_n = 0$

(ze školy)

fix $x \in (0,1)$; zavedeme $h_x(n) = n\sqrt{x}\, e^{-n^2x^2}$ $n \in [1,\infty)$

$$h_x'(n) = \sqrt{x}\, e^{-n^2x^2} + n\sqrt{x}\, e^{-n^2x^2}(-2nx^2) = 0$$

$$\sqrt{x}\, e^{-n^2x^2}(1 - 2n^2x^2) = 0$$

$$\frac{1}{2x^2} = n_0^2$$

$$\frac{1}{\sqrt{2}\, x} = n_0$$

pak $h_x(n_0) = e^{-1/2} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{x}}$

majoranta: $g(x) = \max \left\{ \sqrt{x}\, e^{-x^2} ; \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{x}} ; 0 \right\}$

$\sqrt{x}\, e^{-x^2}$ $\downarrow$ integrov.

$\frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{x}}$ $\downarrow$ integrov.

max 2 integrovatelných = integrovatelná

Z Lebesguea lze prohodit

Problem 2. If $f \in L^1(\mathbb{R})$, prove that

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} f\, dx = 0.$$

Give an example to show that this result need not be true if f is not integrable on $\mathbb{R}$.

Solution.

- Let

$$f_n = \frac{1}{2n}\chi_{[-n,n]} f,$$

where $\chi_{[-n,n]}$ is the characteristic function of the interval $[-n, n]$. Then

$$\int f_n\, dx = \frac{1}{2n} \int_{-n}^{n} f\, dx.$$

- We have $f_n(x) \to 0$ as $n \to \infty$ whenever $f(x) \neq \pm\infty$, so $f_n \to 0$ pointwise a.e. on $\mathbb{R}$. Also, for $n \geq 1$,

$$|f_n| \leq \frac{1}{2}|f| \in L^1(\mathbb{R}).$$

- The Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} \int f_n\, dx = \int \lim_{n\to\infty} f_n\, dx = \int 0\, dx = 0,$$

which proves the result

- If $f = 1$, then

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} f\, dx = 1.$$

In this case the sequence

$$f_n = \frac{1}{2n}\chi_{[-n,n]}$$

converges pointwise (and even uniformly) to 0 on $\mathbb{R}$ as $n \to \infty$, but the integrals do not. Note that the convergence is not monotone and the sequence (f_n) is not dominated by any integrable function.

Příklad od H. Malčíka

Jako příklad na rozmyšlenou: Máme posloupnost nezáporných integrovatelných funkcí a lze zaměnit limitu a integrál, existuje integrovatelná majoranta?

Protipříklad je třeba toto:

Nechť $f_n(x) = \frac{1}{n \log n} e^{-\frac{x}{n}}\, dx$. Potom $f_n \to 0$ a $\int_0^\infty f_n = \frac{1}{\log n} \to 0$. Nechť g je majoranta. Potom

$$\int_n^{n+1} g(x)\, dx \geq \int_n^{n+1} f_n(x)\, dx = \frac{1}{\log n} \int_1^{1+\frac{1}{n}} e^{-t}\, dt \geq \frac{c}{n \log n},$$

tedy

$$\int_0^\infty g(x)\, dx \geq c \sum_{n=1}^{\infty} \frac{1}{n \log n} = \infty.$$

$t = \frac{x}{n}$

$dt = \frac{1}{n}\, dx$

x	n	$n+1$
t	1	$1+\frac{1}{n}$

$$\left[-e^{-t}\right]_1^{1+\frac{1}{n}} = -e^{-(1+\frac{1}{n})} + e^{-1} = e^{-1}\left(1 - e^{-1/n}\right)$$

$$= -e^{-1}\left(e^{-1/n} - 1\right)$$

$$\geq -e^{-1} \cdot \left(-1/n\right)$$

$$= \frac{e^{-1}}{n} = \frac{c}{n}$$

$e^x - 1 \geq x \quad (\forall x)$