(1)

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}$$

(1) $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ D_0 kontrolovat D_f

vyvozí D_0

$$f = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$$

(2) (d) $h_1 = \frac{1}{1}$ ok integrovatelná

$$h_n \overset{?}{\geq} h_{n+1} \qquad \frac{x^n}{n+1} \overset{?}{\geq} \frac{x^{n+1}}{n+2}$$

$$\frac{n+2}{n+1} \geq x \in [0,1] \quad \text{ok} \checkmark$$

→ lze prohodit $\sum$ a $\int$

(3) $\int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}\,dx = \sum_{n=0}^{\infty} \int_0^1 (-1)^n \frac{x^n}{n+1}\,dx =$

$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{x^{n+1}}{(n+1)^2}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = \underline{\underline{\frac{\pi^2}{12}}}$$

(2) $$\int_0^1 \frac{\ln(1-x)}{x} = -\frac{\pi^2}{6}$$

(1) $$\frac{\ln(1-x)}{x} = \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n \frac{(-x)^{n+1}}{n+1} = (-1)\sum_{n=0}^{\infty}\frac{x^n}{n+1}$$

(2) $\boxed{\text{Levi}}$ pro řady - nezáporné, měřitelné

(3) $$\int_0^1 \frac{\ln(1-x)}{x} = \int_0^1 \sum_{n=0}^{\infty}(-1)\frac{x^n}{n+1} \overset{\text{Levi}}{=} (-1)\sum_{n=0}^{\infty}\int_0^1 \frac{x^n}{n+1} = -\sum_{n=0}^{\infty}\left[\frac{x^{n+1}}{(n+1)^2}\right]_0^1 =$$

$$= -\sum_{n=0}^{\infty}\frac{1}{(n+1)^2} = -\frac{\pi^2}{6}$$

(3)

$$\int_0^\infty \frac{dx}{1+e^x}$$

(1) $$\frac{1}{1+e^x} = \frac{e^{-x}}{e^{-x}+1} = \underbrace{e^{-x}}_{a} \sum_{n=0}^{\infty} (\underbrace{-e^{-x}}_{q})^n$$

(2) (a) geom. Σ , $\int_0^\infty \frac{1}{1+e^x}$ konvergují z LSK

(3) $$\int_0^\infty \frac{1}{1+e^x} = \sum_{n=0}^{\infty} \int_0^\infty (-1)^n \underbrace{e^{-nx}\,e^{-x}}_{e^{-x(n+1)}} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{e^{-x(n+1)}}{-(n+1)}\right]_0^\infty$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = \ln 2$$

(4)

$$\int_0^\infty \frac{x}{e^x+1}\,dx$$

(1) $$\frac{x}{e^x(1+e^{-x})} = \frac{x}{e^x} \sum_{n=0}^{\infty} (-1)^n e^{-xn} = \sum_{n=0}^{\infty} \underbrace{(-1)^n x \; e^{-(n+1)x}}_{g_n(x)}$$

(2) $\boxed{\text{Lebesgue}}$ nebo (c)

$$\int_0^\infty \sum_{n=0}^\infty |g_n(x)|\,dx = \int_0^\infty \sum_{n=0}^\infty x e^{-(n+1)x} = \int_0^\infty \frac{-x}{e^x-1}\,dx < \infty$$

(3) $$\int f(x) = \sum_{n=0}^\infty \int_0^\infty (-1)^n x e^{-(n+1)x}\,dx = \sum_{n=0}^\infty (-1)^n \frac{1}{(n+1)^2} = \frac{\pi^2}{12}$$

Per partes

$$\int_0^\infty x \quad e^{-(n+1)x} = \underbrace{\left[\frac{x e^{-(n+1)x}}{-(n+1)}\right]_0^\infty}_{0} - \int_0^\infty \frac{e^{-(n+1)x}}{-(n+1)}\,dx$$

$u = x \qquad v' = e^{-(n+1)x}$

$u' = 1 \qquad v = \dfrac{e^{-(n+1)x}}{-(n+1)}$

$$= \left[\frac{e^{-(n+1)x}}{-(n+1)^2}\right]_0^\infty = \frac{1}{(n+1)^2}$$

(5)

$$\int_0^\infty \frac{x}{e^x - 1}\, dx$$

(1) $$f(x) = \frac{x}{e^x} \cdot \frac{1}{1 - e^{-x}} = x e^{-x} \sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty} x e^{-(n+1)x}$$

(2) Levi

(3) $$\int_0^\infty f(x) = \int \sum x e^{-(n+1)x} = \sum \int_0^\infty x e^{-(n+1)x}$$

$$\int x e^{-(n+1)x} = x \cdot \frac{e^{-(n+1)x}}{-(n+1)} - \int \frac{e^{-(n+1)x}}{-(n+1)}$$

$$\underset{u}{x}\, \underset{v'}{e^{-(n+1)x}}$$

$$u' = 1 \qquad v = \frac{e^{-(n+1)x}}{-(n+1)} \qquad\qquad \frac{e^{-(n+1)x}}{+(n+1)(n+1)}$$

$$= \sum \left[x \frac{e^{-(n+1)x}}{-(n+1)} - \frac{e^{-(n+1)x}}{(n+1)^2} \right]_0^\infty = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}$$

Per partes je Newton

(6)

$$\int_0^1 \frac{x^{p-1}}{1+x^q}\,dx$$

(1) $\frac{x^{p-1}}{1+x^q} = \sum_{n=0}^{\infty} (-1)^n x^{p-1} x^{qn} = \sum_{n=0}^{\infty} (-1)^n x^{p-1+nq}$

(2) (d) $h_1 = x^{p-1}$ $\quad p>0 \quad \int h_1$ konv.

$h_n \overset{?}{\geq} h_{n+1} \qquad x^{p-1+nq} \geq x^{p-1+(n+1)q}$

$1 \geq x^q \qquad q>0$ ✓

(3) $\int_0^1 f(x) = \sum (-1)^n \int_0^1 x^{p-1+nq}\,dx = \sum (-1)^n \left[\frac{x^{p-1+nq+1}}{p+nq}\right]_0^1$

$= \underline{\underline{\sum_{n=0}^{\infty} (-1)^n \frac{1}{p+nq}}}$

(7)

$$\int_0^1 \ln \frac{1}{1-x}$$

(1) $\ln \frac{1}{1-x} = -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

(2) Levi

(3) $\int f = \sum \int_0^1 \frac{x^n}{n} = \sum_{n=0}^{\infty} \left[\frac{x^{n+1}}{n(n+1)} \right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{n(n+1)} = \underline{1}$

(8) $\displaystyle\int_0^1 \ln \frac{1+x}{1-x}\, dx$

(1)

$$\left(\ln \frac{1+x}{1-x}\right)' = \frac{2}{1-x^2} \qquad \ln 1 = 0$$

$$\left(\ln \frac{1+x}{1-x}\right) = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

z mocninné Σ:

$$\frac{2}{1-x^2} = 2 \sum_{n=0}^{\infty} x^{2n} \qquad \| \text{ obě strany } \int$$

$$\ln \frac{1+x}{1-x} = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

(2) Levi

(3) $$\int f = 2 \sum_{n=0}^{\infty} \int_0^1 \frac{x^{2n+1}}{2n+1} = 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)}$$

$$= 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} - \frac{1}{2n+2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \underline{2 \ln 2}$$

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots$$

(9) $\displaystyle\int_0^1 \frac{1}{x} \ln \frac{1+x}{1-x}\, dx$

(1) $\displaystyle f(x) = 2 \cdot \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = 2 \sum \frac{x^{2n}}{2n+1}$

(2) Levi

(3) $\displaystyle \int f = \sum_{n=0}^{\infty} 2 \int_0^1 \frac{x^{2n}}{2n+1} = 2 \sum \left[\frac{x^{2n+1}}{(2n+1)^2} \right]_0^1 =$

$\displaystyle = 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \underline{\underline{\frac{\pi^2}{4}}}$

(10) $\int_0^\infty e^{-x} \cos\sqrt{x}\, dx =$

(1) $\cos y = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{y^{2n}}{(2n)!}$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n e^{-x} \frac{x^n}{(2n)!} \qquad x \in (0,\infty)$$

(2) (d) $h_1 = e^{-x} \frac{1}{2!} \quad \in L^1(0,\infty)$

$h_n \overset{?}{\geq} h_{n+1}$ NE ? $\quad \frac{e^{-x} x^n x}{(2n+2)!} \overset{?}{\leq} \frac{e^{-x} x^n}{2n!}$

NE $\qquad x \leq (2n+2)(2n+1)$

(e) $\int_0^\infty \sum_{n=0}^{\infty} e^{-x} \frac{x^n}{(2n)!}$?

(b) $\sum \int_0^\infty e^{-x} \frac{x^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \cdot n! < \infty$

d'Al $\quad \dfrac{\frac{(n+1)!}{(2n+2)!}}{\frac{n!}{2n!}} = \frac{2n(n+1)}{(2n+2)(2n+1)} \to \frac{1}{2}$ ✓

(3) $\int_0^\infty f = \sum_{n=0}^{\infty} (-1)^n \int_0^\infty e^{-x} \frac{x^n}{(2n)!} = \underline{\underline{\sum_{n=0}^{\infty} (-1)^n \frac{n!}{(2n)!}}}$

Vime: $\int_0^\infty e^{-x} x^n\, dx = n!$

they ask for $f \in L^1(0,\infty)$, instead of $f \in L^1(\mathbb{R})$.

JPE, Sept 1997. Evaluate

$$\sum_{n=0}^{\infty} \int_{1/2}^{\infty} (1-e^{-t})^n e^{-t^2}\, dt.$$

Since the integrands are non-negative functions, the summation and integration "commute", hence

$$\begin{aligned}\sum_{n=0}^{\infty} \int_{1/2}^{\infty} (1-e^{-t})^n e^{-t^2}\, dt &= \int_{1/2}^{\infty} \sum_{n=0}^{\infty} (1-e^{-t})^n e^{-t^2}\, dt \\ &= \int_{1/2}^{\infty} e^{-t^2} \sum_{n=0}^{\infty} (1-e^{-t})^n\, dt \\ &= \int_{1/2}^{\infty} e^{-t^2} \frac{1}{1-(1-e^{-t})}\, dt \\ &= \int_{1/2}^{\infty} e^{t-t^2}\, dt\end{aligned}$$

The last integral is computed by "completing the square" and changing variable:

$$\int_{1/2}^{\infty} e^{t-t^2}\, dt = e^{1/4} \int_{1/2}^{\infty} e^{-(t-1/2)^2}\, dt = e^{1/4} \int_0^{\infty} e^{-s^2}\, ds = e^{1/4}\sqrt{\pi}/2.$$

Note: apparently it is assumed here that the students know the value of the integral $\int_0^\infty e^{-s^2}\, ds$, or can quickly compute it.

JPE, Sept 1995 and Oct 1991. Evaluate

$$\sum_{n=0}^{\infty} \int_0^{\pi/2} (1-\sqrt{\sin x})^n \cos x\, dx.$$

Since the integrands are non-negative functions, the summation and integration

"commute", hence

$$\begin{aligned}\sum_{n=0}^{\infty} \int_0^{\pi/2} (1-\sqrt{\sin x})^n \cos x\, dx &= \int_0^{\pi/2} \sum_{n=0}^{\infty} (1-\sqrt{\sin x})^n \cos x\, dx \\ &= \int_0^{\pi/2} \cos x \sum_{n=0}^{\infty} (1-\sqrt{\sin x})^n\, dx \\ &= \int_0^{\pi/2} \cos x \frac{1}{1-(1-\sqrt{\sin x})}\, dx \\ &= \int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\, dx\end{aligned}$$

The last integral is rather elementary:

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\, dx = \int_0^{\pi/2} \frac{d \sin x}{\sqrt{\sin x}} = 2\sqrt{\sin x}\,\Big|_0^{\pi/2} = 2.$$

8 Riemann integral vs Lebesgue integral

JPE, Sept 2011. Is it true that the characteristic function of the Cantor set is Lebesgue integrable in $[0,1]$ but not Riemann integrable?

False. The characteristic function of the Cantor set is continuous on the complement to the Cantor set (that complement consists of open intervals on which the function is identically zero). Thus the set of discontinuity points is exactly the Cantor set, which measure zero. This implies Riemann integrability.

JPE, Sept 2004. Let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \text{ is irrational} \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that f is measurable.
(ii) Is f Lebesgue integrable? If yes, find its Lebesgue integral.
(iii) Is f Riemann integrable? If yes, find its Riemann integral.

(i) just like in some homework exercises.

(ii) yes, because f is measurable and bounded. Changing f on a set of measure zero will not affect its Lebesgue integral, so we can replace f with $g(x) = \sqrt{x}$ for all $x \in [0,1]$. Now

$$\int_{[0,1]} f\, dm = \int_{[0,1]} g\, dm = \int_0^1 \sqrt{x}\, dx = \frac{2}{3}.$$