(1) $\sum \underbrace{\sin \frac{1}{n} - \arcsin \frac{1}{n}}_{a_n}$ "pro $x = \frac{1}{n}$ zúrčíme funkci"

$\sin x = x - \frac{x^3}{6} + o(x^4)$

$x \to 0$

$\arcsin x = x + \frac{1}{6}x^3 + o(x^4)$

$f(x) = \sin x - \arcsin x$

$= x - \frac{x^3}{6} + o(x^4)$

$- x - \frac{1}{6}x^3 + o(x^4)$

$= -\frac{2}{6}x^3 + o(x^4)$

LSK s

$b_n = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{\left|\sin \frac{1}{n} - \arcsin \frac{1}{n}\right|}{\frac{1}{n^3}} = \frac{1}{3}$$

Heine $x_n = \frac{1}{n}$ $x_n \to 0$ $\frac{1}{n} \neq 0$ $\forall n \in \mathbb{N}$, $0 < \frac{1}{n} \leq 1$, tedy $\frac{1}{n} \in D_f$

$$\lim_{x\to 0} \frac{\left|-\frac{2}{6}x^3 + o(x^4)\right|}{x^3} = +\frac{1}{3} \in (0,\infty)$$

$\sum a_n$ AK $\iff$ $\sum b_n$ K $\sum \frac{1}{n^3}$ K

Závěr $\sum a_n$ AK

Pozn. $\sin x \leq \arcsin x$ na $x \in (0,1)$

tedy $a_n \leq 0$

$$\sum \quad 2\tan\left(\frac{1}{n^{1/5}}\right) - \sin\left(\frac{1}{n^{1/5}}\right) - \frac{1}{n^{3/5}}$$

$$f(x) = 2\tan x - \sin x - x^3 \qquad \left(x = \frac{1}{n^{1/5}}\right)$$

$$f(x) = 2\left[\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^6)\right) - \left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^6)\right)\right] - x^3$$

$$= 2x^5\left(\frac{2}{15} - \frac{1}{120}\right) + o(x^6) = \frac{1}{4}x^5 + o(x^6)$$

$$b_n = \frac{1}{n} \qquad \sum \frac{1}{n} \text{ D}$$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \frac{1}{4} \in (0,\infty) \qquad \text{Heine } x_n = \frac{1}{n^{1/5}}$$

$$= \lim_{x\to 0} \frac{\left|\frac{1}{4}x^5 + o(x^5)\right|}{x^5} = \frac{1}{4}$$

$$\sum |a_n| \text{ K} \iff \sum b_n \text{ K}$$

Zeiver $\sum |a_n|$ D

$$\sum \underbrace{\sin\left(\frac{1}{n} - \arcsin\frac{1}{n}\right)}_{a_n}$$

$(x = 1/n)$

$$f(x) = \sin(x - \arcsin x)$$

$$x - \arcsin x = x - x - \frac{x^3}{6} + o(x^4)$$

$$f(x) = -\frac{x^3}{6} + o(x^4) + o(x^3)$$

$b_n = \frac{1}{n^3}$ $\qquad \sum b_n$ K

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = 1/6 \in (0,\infty)$$

Heine $x_n = 1/n$

$$\lim_{x\to 0} \frac{\left|-\frac{x^3}{6} + o(x^3)\right|}{x^3} = 1/6$$

Závěr $\sum |a_n|$ K

$$(4) \sum \underbrace{\ln \frac{1}{n^\beta} - \ln \left(\sin \frac{1}{n^\beta}\right)}_{a_n} \qquad \beta > 0$$

$$a_n = \ln \frac{\frac{1}{n^\beta}}{\sin 1/n^\beta} = -\ln n^\beta \sin \frac{1}{n^\beta}$$

zavedme funkci $\qquad \left(x = \frac{1}{n^\beta}\right)$

$$f(x) = -\ln\left(\frac{1}{x} \sin x\right)$$

$$\frac{1}{x}\sin x = \left[\frac{1}{x}\left(x - \frac{x^3}{6} + o(x^4)\right)\right] \qquad x \to 0$$

$$= \left[1 \underbrace{- \frac{x^2}{6} + o(x^3)}\right]$$

$$\ln\left(\frac{1}{x}\sin x\right) = -\frac{x^2}{6} + \underbrace{o(x^3) + o\left(-\frac{x^2}{6} + o(x^3)\right)}_{o(x^2)}$$

$$b_n = \frac{1}{n^{2\beta}} \qquad \sum n^{-2\beta} \text{ K} \iff -2\beta < -1 \,;\; \beta > 1/2$$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \frac{1}{6} \in (0,\infty) \quad \text{Heine} \qquad x_n = \frac{1}{n^\beta},\; x_n \to 0,\; \frac{1}{n^\beta} \neq 0 \;\forall n$$

$$0 < \frac{1}{n^\beta} \leq 1 \quad \to \quad \frac{1}{n^\beta} \in D_f$$

$$\lim_{x\to 0} \frac{\left|-\frac{x^2}{6} + o(x^2)\right|}{x^2} = \frac{1}{6}$$

$$\sum |a_n| \text{ K} \iff \sum b_n \text{ K} \iff \beta > 1/2$$

Závěr $\quad \sum a_n$ AK $\iff \beta > 1/2$

$$\sum \underbrace{\left(\sin\frac{1}{n} - \frac{1}{n}\right)\frac{1}{n^\alpha}}_{a_n}$$

pomocná funkce $\left(x = \frac{1}{n}\right)$

$f(x) = \sin x - x$

rozvoj: $f(x) = -x + x - \frac{x^3}{3!} + o(x^3)$

$= -\frac{x^3}{3!} + o(x^3)$

$b_n = \frac{1}{n^\alpha} \cdot \frac{1}{n^3} = n^{-\alpha-3}$ $\quad \sum n^{-\alpha-3}$ k $\iff -\alpha-3 < -1$

$\boxed{-2 < \alpha}$

$\lim \frac{|a_n|}{b_n} = \lim \frac{\left|\left(\sin\frac{1}{n} - \frac{1}{n}\right)\frac{1}{n^\alpha}\right|}{\frac{1}{n^3}\cdot\frac{1}{n^\alpha}} = 1/6 \in (0,\infty)$

Heine $\quad x_n = \frac{1}{n} \quad x_n \to 0 \quad \frac{1}{n} \neq 0 \quad \frac{1}{n} \in D_f \ (=\mathbb{R})$

$\lim_{x\to 0} \frac{\left|-\frac{x^3}{6} + o(x^3)\right|}{x^3} = +\frac{1}{6}$

$\sum |a_n|$ k $\iff \sum b_n$ k

Závěr: $\quad \sum a_n$ Ak $\iff \alpha > -2$

$$\sum \underbrace{\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}}_{a_n}$$

$$\sqrt{n}\left(\sqrt{1+\frac{2}{n}} - 2\sqrt{1+\frac{1}{n}} + 1\right)$$

$(x = \frac{1}{n})$

$$f(x) = \sqrt{1+2x} - 2\sqrt{1+x} + 1 \qquad x \to 0$$

$$= \cancel{1} + \frac{1}{2}(\cancel{2x}) - \frac{1}{8}(2x)^2 + o(x^2) - 2\left(\cancel{1} + \frac{1}{2}\cancel{x} - \frac{1}{8}x^2 + o(x^2)\right) + \cancel{1}$$

$$\binom{1/2}{2} = \frac{\frac{1}{2}(-\frac{1}{2})}{2!} = -\frac{1}{8}$$

$$= x^2\left(-\frac{4}{8} + \frac{2}{8}\right) + o(x^2)$$

$$= -\frac{x^2}{4} + o(x^2)$$

$$b_n = \sqrt{n} \cdot \frac{1}{n^2} = \frac{1}{n^{3/2}} \qquad \sum \frac{1}{n^{3/2}} \quad K$$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim \frac{\left|\sqrt{n}\left(\sqrt{1+2/n} - 2\sqrt{1+1/n} + 1\right)\right|}{\sqrt{n} \cdot \frac{1}{n^2}} = \frac{1}{4} \in (0,\infty)$$

Heine $\quad x_n = \frac{1}{n} \quad x_n \to 0$

$$\lim_{x\to 0} \frac{\left|-\frac{x^2}{4} + o(x^2)\right|}{x^2} = \frac{1}{4}$$

$\sum |a_n|$ K $\Leftrightarrow \sum b_n$ K

Závěr: $\quad \sum |a_n|$ K

$$\sum \underbrace{\left[e - \left(1+\frac{1}{n}\right)^n\right]^p}_{a_n}$$

$$e - e^{n \ln(1+\frac{1}{n})} = -e\left(1 - e^{n\ln(1+\frac{1}{n}) - 1}\right)$$

$(x = 1/n)$

$$f(x) = -e\left[1 - e^{\frac{1}{x}\ln(1+x) - 1}\right]$$

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\frac{1}{x}\ln(1+x) - 1 = 1 - \frac{x}{2} + \frac{1}{x}o(x^2) - 1 = -\frac{x}{2} + o(x)$$

$$e^{1/x \ln(1+x) - 1} = 1 + \left(-\frac{x}{2}\right) + o(x) + o\left(-\frac{x}{2} + o(x)\right) - 1 \qquad x \to 0$$

$$= -\frac{x}{2} + o(x)$$

$$f(x) = -e\left(-\frac{x}{2} + o(x)\right)$$

$$b_n = \left(\frac{1}{n}\right)^p$$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim \frac{\left|\left[e - \left(1+\frac{1}{n}\right)^n\right]^p\right|}{\frac{1}{n^p}}$$

Heine $x_n = 1/n$, $1/n \to 0$, $1/n \neq 0$

$0 < \frac{1}{n} \leq 1$ $\quad \frac{1}{n} \in D_f$

$$\lim_{x\to 0} \frac{\left|\left[+e\left(\frac{x}{2} + o(x)\right)\right]^p\right|}{x^p} = \lim \frac{e^p\left(\frac{x^p}{2^p} + (o(x))^p\right)}{x^p} = \left(\frac{e}{2}\right)^p \in (0, \infty)$$

$\to \sum |a_n|$ K $\Leftrightarrow \sum \frac{1}{n^p}$ K $\Leftrightarrow p > 1$

Záver: $\sum a_n$ AK $\Leftrightarrow p > 1$

$$\sum \left(e^{1/n} - 1 - 1/n\right)\left(\arcsin 1/n - \frac{1}{\sqrt{n}}\right)$$

$$\left(x = \frac{1}{\sqrt{n}}\right)$$

$$f(x) = \left(e^{x^2} - 1 - x^2\right)\left(\arcsin x^2 - x\right)$$

$$= \left(\cancel{1} + \cancel{x^2} + \frac{x^4}{2!} + o(x^4) - \cancel{1} - \cancel{x^2}\right)\left(x^2 + \frac{1}{6}x^6 + o(x^7) - x\right)$$

$$= -\frac{x^5}{2} + o(x^5)$$

$$b_n = \left(\frac{1}{\sqrt{n}}\right)^5 = n^{-5/2} \qquad \sum b_n \ \text{K}$$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = 1/2 \in (0,\infty)$$

Heine $x_n = \frac{1}{\sqrt{n}}$

$$\lim_{x\to 0} \frac{\left|-\frac{x^5}{2} + o(x^5)\right|}{x^5} = 1/2$$

Závěr: $\sum |a_n|$ K

$$\sum \sin\left(\frac{1}{\sqrt{n}}\right) - \ln\left(1 + \frac{1}{\sqrt[3]{n}}\right)$$

$$\left(x = \frac{1}{\sqrt[6]{n}}\right)$$

$$f(x) = \sin x^3 - \ln(1 + x^2)$$

$$= x^3 + o(x^3) - (x^2 + o(x^2))$$

$$= -x^2 + o(x^2)$$

LSK $\quad b_n = \left(\frac{1}{\sqrt[6]{n}}\right)^2 = \frac{1}{n^{1/3}} \qquad \sum b_n$ D

$$\lim_{n\to\infty} \frac{|a_n|}{b_n}$$

Heine $\quad x_n = \frac{1}{\sqrt[6]{n}}$

$$\lim_{x\to 0} \frac{|-x^2 + o(x^2)|}{x^2} = 1 \in (0,\infty)$$

Závěr $\quad \sum |a_n|$ D

$$\sum \frac{1}{\sqrt{n}} + \ln\left(\sqrt{1+1/n} - \frac{1}{\sqrt{n}}\right)$$

$$\left(\frac{1}{\sqrt{n}} = x\right)$$

$$f(x) = x + \ln\left(\sqrt{1+x^2} - x\right)$$

$$-x + \sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4) - x$$

$$x + \ln\left(\sqrt{1+x^2} - x\right) = x + \left(-x + \frac{x^2}{2} - \frac{1}{8}x^4 + o(x^4)\right) - \frac{1}{2}\left(-x + \frac{x^2}{2} - \frac{1}{8}x^4 + o(x^4)\right)^2$$
$$+ \frac{1}{3}\left(-x + \frac{x^2}{2} - \frac{x^4}{8} + o(x^4)\right)^3 + o\left(\left(-x + \frac{x^2}{2} - \frac{x^4}{8} + o(x^4)\right)^3\right)$$

$$= \cancel{\frac{x^2}{2}} - \frac{x^4}{8} + o(x^4) - \frac{1}{2}\left(\cancel{x^2} - x^3 + \frac{x^4}{4} + o(x^4)\right)$$
$$+ \frac{1}{3}\left(-x^3 + \frac{3}{2}x^4 + o(x^4)\right) + o(x^3)$$

$$= x^3\left(\frac{1}{2} - \frac{1}{3}\right) + o(x^3) = \frac{1}{6}x^3 + o(x^3)$$

$$b_n = \frac{1}{\sqrt{n}^3} = n^{-3/2} \qquad \sum b_n \ k$$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = 1/6 \in (0,\infty) \qquad \text{Heine } x_n = \frac{1}{\sqrt{n}}$$

$$\lim_{x\to 0} \frac{\left|\frac{1}{6}x^3 + o(x^3)\right|}{x^3} = \frac{1}{6}$$

Závěr: $\underline{\underline{\sum |a_n| \ k}}$