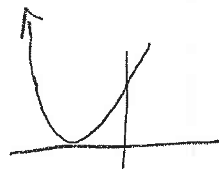
(4)

(1)(a) $\lim\limits_{x \to 3} \frac{3x+2}{-x+1} = \frac{3\cdot 3+2}{-3+1} = \frac{-11}{2}$

ze spojitosti

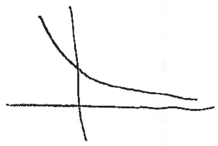
2a (b) $\lim\limits_{x \to \frac{\pi}{4}} \operatorname{tg} x = 1$

ze spojitosti



2b (c) $\lim\limits_{x \to -\infty} (x+3)^2 = \infty$

lim. polynomu



2c (d) $\lim\limits_{x \to \infty} e^{-x} = 0$

exponenciela

2d (e) $\lim\limits_{x \to \infty} \frac{3}{-8-x} = 0$

2e (f) $\lim\limits_{x \to 5} \frac{6}{x-5} \quad \nexists$

ale $\lim\limits_{x \to 5+} \frac{6}{x-5} = \infty$

$\lim\limits_{x \to 5-} \frac{6}{x-5} = -\infty$

2f (g) $\lim\limits_{x \to \infty} \frac{1}{\ln x + 1} \text{ "}=\text{ } \frac{1}{\infty}\text{"} = 0$

2g (h) $\lim\limits_{x \to 1} \frac{x^2+4x-5}{x-1} = \lim\limits_{x \to 1} \frac{(x-1)(x+5)}{x-1} = \lim\limits_{x \to 1} x+5 = \underline{6}$

2h (i) $\lim\limits_{x \to 1} \frac{x^2+4x-5}{(x-1)^2} = \lim\limits_{x \to 1} \frac{x+5}{x-1} \quad \nexists$

ale $\lim\limits_{x \to 1+} = \infty$

$\lim\limits_{x \to 1-} = -\infty$

2i (j) $\lim\limits_{x\to\infty} \dfrac{3x^2+3x+5+\frac{1}{x}}{-2x^2+4x-3} = \lim\limits_{x\to\infty} \dfrac{x^2}{x^2} \cdot \dfrac{3+\frac{3}{x}+\frac{5}{x^2}+\frac{1}{x^3}}{-2+\frac{4}{x}-\frac{3}{x^2}}$

$\overset{\text{VOAL}}{=} \dfrac{3+0+0+0}{-2+0-0} = \underline{\underline{-\frac{3}{2}}}$

(k) $\lim\limits_{x\to\infty} \dfrac{8x^3+7x^2+5+x^{10}+\frac{1}{x}}{2x^{10}+4x-9} =$

$= \lim\limits_{x\to\infty} \dfrac{x^{10}}{x^{10}} \cdot \dfrac{1+\frac{8}{x^7}+\frac{7}{x^8}+\frac{5}{x^{10}}+\frac{1}{x^{11}}}{2+\frac{4}{x^9}-\frac{9}{x^{10}}} \overset{\text{VOAL}}{=} \dfrac{1+0+0+0+0}{2+0-0} = \underline{\underline{\frac{1}{2}}}$

(l) $\lim\limits_{x\to-\infty} \dfrac{x^3+3x+5+\frac{1}{x}}{8x^3+4x^2-3} = \lim\limits_{x\to-\infty} \dfrac{x^3}{x^3} \cdot \dfrac{1+\frac{3}{x^2}+\frac{5}{x^3}+\frac{1}{x^4}}{8+\frac{4}{x}-\frac{3}{x^3}}$

$\overset{\text{VOAL}}{=} \dfrac{1+0+0+0}{8+0-0} = \underline{\underline{\frac{1}{8}}}$

(2)(a) $\lim\limits_{x\to-1} e^{\frac{x+2}{x+3}} = e^{\frac{-1+2}{-1+3}} = e^{\frac{1}{2}} = \underline{\underline{\sqrt{e}}}$

2j (b) $\lim\limits_{x\to\frac{\pi}{2}} \operatorname{tg} x \ \nexists$

ale $\lim\limits_{x\to\frac{\pi}{2}+} \operatorname{tg} x = -\infty$ $\quad \lim\limits_{x\to\frac{\pi}{2}-} \operatorname{tg} x = \infty$

(c) $\lim\limits_{x\to\infty} \dfrac{1}{(x-4)^2} = 0$

2k (d) $\lim\limits_{x\to 2} \ln(x-3) \ \nexists$ není def na okolí, nemá smysl

2l (e) $\lim\limits_{x\to 1} \dfrac{x}{\sqrt{x^2-1}} \ \nexists$

ale $\lim\limits_{x\to 1+} \dfrac{x}{\sqrt{x^2-1}} = \infty$

(f) $\lim\limits_{x\to 1} \dfrac{x^2+3x-4}{x^2-2x+1} = \lim\limits_{x\to 1} \dfrac{(x-1)(x+4)}{(x-1)(x-1)} = \lim\limits_{x\to 1} \dfrac{x+4}{x-1}$

$\nexists$ ale $\lim\limits_{x\to 1+} = \infty$

$\lim\limits_{x\to 1-} = -\infty$

(g) $\lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x^2+3x+4} = 0$ spojitost

(2m) (h) $\lim\limits_{x\to 0} \dfrac{x^3-2x+x}{2x^3+x^2-2x} = \lim\limits_{x\to 0} \dfrac{x(x^2-2+1)}{x(2x^2+x-2)} \overset{\text{VOL}}{=} \dfrac{-1}{-2} = \underline{\underline{\dfrac{1}{2}}}$

④ Vachu

3a (1)(a) $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ 2 policajti

$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$ na $\mathbb{R}^+$

$-\frac{1}{x} \to 0$, $\frac{1}{x} \to 0$

(b) $\lim_{x\to\infty} \frac{\ln x}{x^2} = 0$

2 policajti

$\frac{0}{x^2} \le \frac{\ln x}{x^2} \le \frac{x-1}{x^2}$

pro $x \in (1,\infty)$

$\lim_{x\to\infty} 0 = 0$ $\lim_{x\to\infty} \frac{x-1}{x^2} = \lim_{x\to\infty} \frac{x^2}{x^2} \frac{\frac{1}{x} - \frac{1}{x^2}}{1} = 0$

3b (c) $\lim_{x\to\infty} e^{-x} \cos x = 0$

2 policajti

$-1 \cdot e^{-x} \le f \le e^{-x} \cdot 1$

$-1 \cdot e^{-x} \to 0$, $e^{-x} \cdot 1 \to 0$

3c (d) $\lim_{x\to\infty} \frac{x + \sin x}{x - \sin x} = \lim_{x\to\infty} \frac{x}{x} \cdot \frac{1 + \frac{\sin x}{x}}{1 - \frac{\sin x}{x}} = 1$

$\frac{\sin x}{x} \to 0$ viz c

$\frac{\sin x}{x} \to 0$ 2 policajti

3d (e) $\lim_{x\to 0+} x \cos\left(\frac{x+3}{\sqrt{x}-1}\right) = 0$ 2 policajti

$-x \le \quad \le x$

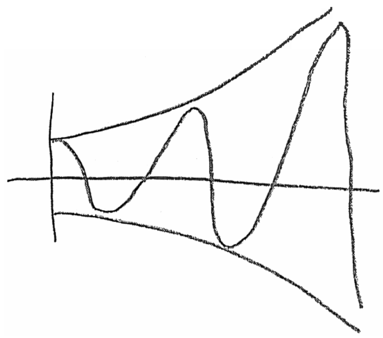
$-x \to 0$, $x \to 0$

3e (f) $\lim\limits_{x \to \infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x \to \infty} \dfrac{e^x}{e^x} \cdot \dfrac{1 + e^{-2x}}{1 - \underset{\downarrow \atop 0}{e^{-2x}}} \overset{\text{vote}}{=} \underline{1}$

3f (g) $\lim\limits_{x \to \infty} (2 + \cos x) \quad \nexists$ "graf:  "

3g (h) $\lim\limits_{x \to \infty} e^x \cos x \quad \nexists$

"graf:"



3h (i) $\lim\limits_{x \to 0} \dfrac{x^2}{e^x} \overset{\text{vote}}{=} \dfrac{0}{1} = \underline{\underline{0}}$

3i (j) $\lim\limits_{x \to \infty} \dfrac{x}{\sin x} \quad \nexists$ "-čas od času dotiko 0"

(4) (a) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x} = \lim\limits_{x\to\infty} \dfrac{x}{x}\cdot\dfrac{\sqrt{1+\frac{1}{x^2}}}{1} \overset{\text{VOAL}}{=} \underline{\underline{1}}$

$\sqrt{\lim} = \lim\sqrt{\ }$

(b) $\lim\limits_{x\to\infty} \sqrt{x+2}-\sqrt{x} = \lim\limits_{x\to\infty} \dfrac{x+2-x}{\sqrt{x+2}+\sqrt{x}} =$

$= \lim\limits_{x\to\infty} \dfrac{2}{\underbrace{\sqrt{x+2}}_{\infty}+\underbrace{\sqrt{x}}_{\infty}} = 0$

(c) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+1}}{x} = \lim\limits_{x\to-\infty} \dfrac{|x|}{x}\cdot\dfrac{\sqrt{1+\frac{1}{x^2}}}{1} = \underline{\underline{-1}}$

(d) $\lim\limits_{x\to\infty} \sqrt{x+2}+\sqrt{x} = {}``\infty+\infty" = \underline{\underline{\infty}}$

(e) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-1}{x} = \lim\limits_{x\to 0} \dfrac{x+1-1}{x\,(\sqrt{x+1}+1)} = \lim\limits_{x\to 0} \dfrac{1}{\sqrt{x+1}+1}$

$\overset{\text{VOAL}}{=} \dfrac{1}{\sqrt{0+1}+1} = \underline{\underline{\dfrac{1}{2}}}$

(f) $\lim\limits_{x\to-2} \dfrac{\sqrt[3]{x-6}+2}{x^3+8} = \lim\limits_{x\to-2} \dfrac{x-6+2^3}{(x^3+8)\left(\sqrt[3]{x-6}^{\,2}-2\sqrt[3]{x-6}+2^2\right)}$

$= \lim\limits_{x\to-2} \dfrac{\cancel{(x+2)}}{\cancel{(x+2)}(x^2-x(+2)+4)\left(\sqrt[3]{(x-6)^2}-2\sqrt[3]{x-6}+4\right)} =$

$= \lim\limits_{x\to-2} \dfrac{1}{\underbrace{x^2}_{4}\underbrace{-2x}_{+4}\underbrace{+4}_{+4}} \cdot \dfrac{1}{\left(\underbrace{\sqrt[3]{(x-6)}^{\,2}}_{4}-\underbrace{2\sqrt[3]{x-6}}_{4}+\underbrace{4}_{4}\right)} \overset{\text{VOAL}}{=} \dfrac{1}{12^2} = \underline{\underline{\dfrac{1}{144}}}$

(4) (g) $\lim\limits_{x\to\infty} x\left(\sqrt{x^2+1}-x\right) = \lim\limits_{x\to\infty} \dfrac{x\left(x^2+1-x^2\right)}{\sqrt{x^2+1}+x} =$

$= \lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}+x} = \lim\limits_{x\to\infty} \dfrac{x}{x}\cdot\dfrac{1}{\sqrt{1+\underset{\downarrow \atop 0}{\frac{1}{x^2}}}+1} \overset{\text{věta}}{=} \underline{\underline{\dfrac{1}{2}}}$

(4) (h) $\lim\limits_{x\to\infty} x^3-x^2+3x-8 = \lim x^3\left(1-\frac{1}{x}+\frac{3}{x^2}-\frac{8}{x^3}\right) = \text{"}\infty\cdot 1\text{"}$

$= \underline{\infty}$

~~(i)~~ $\lim\limits_{x\to\infty} x+\sin x = \infty$

policajti

$x-1 \le \qquad \le x+1$

5a

5.

$$\lim_{x\to 2}\frac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}}$$

Řešení:

Rozložte polynomy na součin a pak teprve umocněte.

$$\lim_{x\to 2}\frac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}}=\lim_{x\to 2}\frac{(x-2)^{20}(x+1)^{20}}{[(x-2)^2]^{10}(x+4)^{10}}=\lim_{x\to 2}\frac{(x+1)^{20}}{(x+4)^{10}}=\frac{3^{20}}{6^{10}}=\frac{3^{10}}{2^{10}}.$$

5b

6.

$$\lim_{x\to 1}\frac{x^m-1}{x^n-1},$$

kde m, n jsou přirozená čísla.

Řešení:

Sledujte výpočet. Uvědomte si, že proměnnou je x, nikoliv m, n, která jsou danými parametry. Opět by šlo použít l'Hospitalovo pravidlo.

$$\lim_{x\to 1}\frac{x^m-1}{x^n-1}=\lim_{x\to 1}\frac{(x-1)(x^{m-1}+x^{m-2}+\ldots+x+1)}{(x-1)(x^{n-1}+x^{n-2}+\ldots+x+1)}=\lim_{x\to 1}\frac{x^{m-1}+x^{m-2}+\ldots+x+1}{x^{n-1}+x^{n-2}+\ldots+x+1}=\frac{m}{n}.$$

Jiné řešení skýtá fakt, že

$$\lim_{x\to 1}\frac{x^m-1}{x-1}=\lim_{x\to 1}mx^{m-1}=m,$$

což víme ze vzorců pro derivaci funkce x^m, vlastně počítáme derivaci této funkce v bodě 1. Známe-li tento vzorec, lze tento postup použít i pro m, n reálná, různá od nuly.

5c

7.

$$\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}$$

Řešení:

Vytknutím dostaneme

$$\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}=\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x}}\frac{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}}{\sqrt{1+\frac{1}{x}}}=1.$$

5d

8.

$$\lim_{x\to+\infty}\frac{\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}}{\sqrt{2x-1}}$$

5d

Řešení:

Vytknutím dostaneme

$$\lim_{x\to\infty}\frac{\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}}{\sqrt{2x+1}}=\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x}}\frac{1+\sqrt[6]{\frac{1}{x}}+\sqrt[4]{\frac{1}{x}}}{\sqrt{2+\frac{1}{x}}}=\frac{1}{\sqrt{2}}.$$

5e

9.

$$\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$$

Řešení:

Rozšířením dostaneme

$$\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}=\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}\frac{\sqrt{x}+2}{\sqrt{x}+2}\frac{\sqrt{1+2x}+3}{\sqrt{1+2x}+3}=$$

$$=\lim_{x\to 4}\frac{1+2x-9}{x-4}\frac{\sqrt{x}+2}{\sqrt{1+2x}+3}=\lim_{x\to 4}\frac{2(x-4)}{x-4}\frac{\sqrt{x}+2}{\sqrt{1+2x}+3}=\lim_{x\to 4}2\frac{\sqrt{x}+2}{\sqrt{1+2x}+3}=2\cdot\frac{2+2}{3+3}=\frac{4}{3}.$$

5f

10.

$$\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}},$$

kde $a>0$

Řešení:

Úpravou dostaneme

$$\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}=\lim_{x\to a}\frac{1}{\sqrt{x+a}}\cdot\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x-a}}=$$

$$=\frac{1}{\sqrt{2a}}\cdot\left[\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x-a}}+1\right]=\frac{1}{\sqrt{2a}}+\frac{1}{\sqrt{2a}}\cdot\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x-a}}=\frac{1}{\sqrt{2a},}$$

neboť

$$\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x-a}}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x-a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\lim_{x\to a}\frac{x-a}{\sqrt{x-a}}\cdot\frac{1}{\sqrt{x}+\sqrt{a}}=$$

$$=\lim_{x\to a}\frac{\sqrt{x-a}}{1}\cdot\frac{1}{\sqrt{x}+\sqrt{a}}=\frac{0}{1}\cdot\frac{1}{\sqrt{2a}}=0.$$

5g

11.

$$\lim_{x\to -2}\frac{\sqrt[3]{x-6}+2}{x^3+8}$$

Řešení:

5g

Úpravou dostaneme

$$\lim_{x\to-2}\frac{\sqrt[3]{x-6}+2}{x^3+8}=\lim_{x\to-2}\frac{\sqrt[3]{x-6}+2}{(x+2)(x^2-2x+4)}\cdot\frac{\sqrt[3]{(x-6)^2}-2\sqrt[3]{x-6}+4}{\sqrt[3]{(x-6)^2}-2\sqrt[3]{x-6}+4}=$$

$$=\lim_{x\to-2}\frac{x-6+8}{(x+2)(x^2-2x+4)}\cdot\frac{1}{\sqrt[3]{(x-6)^2}-2\sqrt[3]{x-6}+4}=$$

$$=\lim_{x\to-2}\frac{1}{x^2-2x+4}\cdot\frac{1}{\sqrt[3]{(x-6)^2}-2\sqrt[3]{x-6}+4}=\frac{1}{12}\cdot\frac{1}{4+4+4}=\frac{1}{144}.$$

5h

12.

$$\lim_{x\to+\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-x\right)$$

Řešení:

Počítejme

$$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)=\lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}=$$

$$=\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}=$$

vytknutím $\sqrt{x}$ v čitateli i jmenovateli a krácením dostaneme

$$=\lim_{x\to\infty}\frac{\sqrt{1+\sqrt{\frac{1}{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}+1}=\frac{1}{2}.$$

5i

13.

$$\lim_{x\to0+}\left(\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}\right)$$

Řešení:

V příkladu je dobré provést substituci $\frac{1}{x}=y$. Přitom je důležité, že původní limita je jednostranná, totiž že $x\to0+$, a proto $y=\frac{1}{x}\to+\infty$. Platí tedy:

$$\lim_{x\to0+}\left(\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}\right)=\lim_{y\to+\infty}\left(\sqrt{y+\sqrt{y+\sqrt{y}}}-\sqrt{y-\sqrt{y+\sqrt{y}}}\right)=$$

Klasicky rozšiřujeme.

$$= \lim_{x\to 0+} \frac{\frac{1}{x} + \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}} - (\frac{1}{x} - \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}})}{\sqrt{\frac{1}{x} + \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}}} + \sqrt{\frac{1}{x} - \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}}}} = \lim_{x\to 0+} \frac{\sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}} + \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}}}{\sqrt{\frac{1}{x} + \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}}} + \sqrt{\frac{1}{x} - \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}}}} =$$

Vytknutím $\sqrt{1/x}$ v čitateli i jmenovateli a zkrácením dostaneme

$$= \lim_{x\to 0+} \frac{\sqrt{1+\sqrt{x}} + \sqrt{1+\sqrt{x}}}{\sqrt{1+\sqrt{x}+\sqrt{x^3}} + \sqrt{1-\sqrt{x+\sqrt{x^3}}}} = \frac{2}{2} = 1.$$

14.

$$\lim_{x\to+\infty} x^{1/3}\left[(x+1)^{2/3} - (x-1)^{2/3}\right]$$

Řešení:

Počítejme. Pokud vám vadí zápis odmocnin pomocí racionálních exponentů, přepište si jej.

$$\lim_{x\to\infty} x^{1/3}[(x+1)^{2/3}-(x-1)^{2/3}] = \lim_{x\to\infty} x^{1/3}\cdot\frac{(x+1)^2-(x-1)^2}{(x+1)^{4/3}+(x+1)^{2/3}(x-1)^{2/3}+(x-1)^{4/3}} =$$

$$= \lim_{x\to\infty} \frac{4x\cdot x^{1/3}}{(x+1)^{4/3}+(x+1)^{2/3}(x-1)^{2/3}+(x-1)^{4/3}} =$$

$$= \lim_{x\to\infty} \frac{4}{(1+\frac{1}{x})^{4/3}+(1+\frac{1}{x})^{2/3}(1-\frac{1}{x})^{2/3}+(1-\frac{1}{x})^{4/3}} = \frac{4}{1+1+1} = \frac{4}{3}.$$