(1) $f_n(x) = \sqrt[2n]{x^n + |\ln x|}$ $\qquad x \in (0, \infty)$

(a) fix $x \in (0,\infty)$

$$\lim_{n\to\infty} \sqrt[2n]{x^n + |\ln x|} = \begin{cases} 1 & x \in (0,1] \\ \sqrt{x} & x \in (1,\infty) \end{cases}$$

$x \in (0,1)$ $\qquad \underset{\downarrow \atop 1}{\sqrt[2n]{|\ln x|}} \le \quad \le \underset{\downarrow \atop 1}{\sqrt[2n]{\underbrace{1+|\ln x|}_{c\in(1,\infty)}}}$

$x \in (1,\infty)$ $\qquad \underset{\downarrow \atop \sqrt{x}}{\sqrt[2n]{x^n}} \le \quad \le \underset{\downarrow \atop \sqrt{x}}{\sqrt[2n]{x^n + x^n}}$

(b) $x \in (0,1]$

fix n

$$\sigma_n = \sup_{x\in(0,1]} |f_n(x) - f(x)| = \sup \Big| \underbrace{\sqrt[2n]{x^n + |\ln x|}}_{g_n(x)} - 1 \Big| \underset{=\infty}{\not\to} 0$$

$x^n - \ln x$

pro $x \to 0$ $\qquad$; $\lim_{x\to 0+} x^n + |\ln x| = \underline{\underline{0 + \infty}}$ $\qquad f_n \not\rightrightarrows f$ na $(0,1]$

(c) $x \in [1,\infty)$, fix n

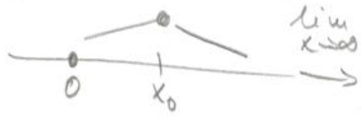
$$\sigma_n = \sup_{x\in[1,\infty)} \Big| \underbrace{\sqrt[2n]{x^n + |\ln x|} - \sqrt{x}}_{g_n(x) \ge 0} \Big|$$

$$g_n(x) = \sqrt{x}\left(\sqrt[2n]{1 + \frac{\ln x}{x^n}} - 1\right) \le \sqrt{x}\left(1 + \frac{\ln x}{x^n} - 1\right) = \underbrace{x^{\frac{1}{2}-n} \ln x}_{h(x)}$$

$$(x^{\frac12 - n} \ln x)' = (\tfrac12 - n)\, x^{\frac12 - n - 1} \ln x + x^{\frac12 - n} \cdot \frac{1}{x} = \ln x \cdot (\tfrac12 - n)\, x^{-n-\frac12} + x^{-n-\frac12}$$

$$= x^{-n-\frac12}\left(1 + (\tfrac12 - n)\ln x\right)$$

$h(x) \ge 0$



$\ln x = \dfrac{-1}{\frac12 - n}$

$x_0 = e^{\frac{1}{n-\frac12}}$

$$h(x_0) = \left(e^{\frac{1}{n-\frac12}}\right)^{\frac12 - n} \cdot \frac{+1}{n-\frac12} = \frac{1}{e} \cdot \frac{1}{n-\frac12}$$

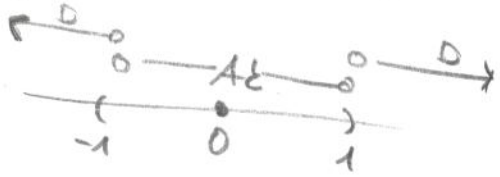
$$r_n \leq \frac{1}{e(n-1/2)} \qquad \lim_{n\to\infty} r_n = 0$$

tedy $\quad f_n \rightrightarrows f \quad$ na $[1,\infty)$

(2) $\displaystyle\sum_{n=1}^{\infty} \underbrace{(-1)^n \frac{1}{n(n+1)}}_{a_n} x^{n+1}$

$\displaystyle\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n(n+1)}} = 1$

$\rho = \frac{1}{1} = 1 \qquad x_0 = 0$



$\displaystyle f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n(n+1)} x^{n+1} \qquad f(0) = 0$

$\displaystyle f'(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} \qquad f'(0) = 0$

$\displaystyle f'' = \sum_{n=1}^{\infty} (-1)^n x^{n-1} = \sum_{n=1}^{\infty} -(-x)^{n-1} = \sum_{n=0}^{\infty} -(-x)^n = \frac{-1}{1+x}$

pak $f' = -\ln|1+x| + c = -\ln(1+x)$

$f: \int -\ln(1+x) \underset{P.P.}{=} -x\ln(1+x) - \ln(1+x) + x + c$

$c = 0$

$f = -x\ln(1+x) - \ln(1+x) + x = -\ln(1+x)\cdot(1+x) + x$

krajní body: $\displaystyle\sum \frac{(-1)^n \cdot (\pm 1)^{n+1}}{n(n+1)}$ AK LSK s $\frac{1}{n^2}$

z Abelovy věty

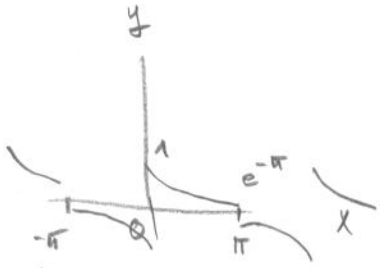
$\displaystyle f(1) = \lim_{x\to 1-} f(x) = 1 - 2\ln 2$

$\displaystyle f(-1) = \lim_{x\to -1+} x - \ln(1+x)(1+x) = -1 + 0 = -1$

Závěr: $\sum$ AK k $-\ln(1+x)(1+x) + x$ na $[-1,1]$

(3) $e^{-x} = \sum_{n=1}^{\infty} b_n \sin(nx)$ $x \in (0,\pi)$

Neboli rozvoj do sin. Σ fce e^{-x}



$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-x}\sin(nx)\,dx = \frac{2}{\pi}\left[e^{-x}\cdot\frac{-n\cos(nx) - \sin(nx)}{n^2+1}\right]_0^{\pi}$$

2x P.P

$$= \frac{2}{\pi}\left(\frac{-e^{-\pi}\, n\cos(\pi n)}{n^2+1} + \frac{n}{n^2+1}\right) = \frac{2}{\pi}\cdot\frac{1}{n^2+1}\left(n - ne^{-\pi}(-1)^n\right)$$

f je monotónní na $(-\pi,0)$, $(0,\pi)$

$f \in BV([-\pi,\pi])$ $V_{-\pi}^{\pi}(f) = 2(1-e^{-\pi}) + 2$

f spoj na $(0,\pi)$.

Pak z J-D kritéria $s_n \rightrightarrows f$ na $(0,\pi)$