1st lesson

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Theory

Definition 1. A system of m *equations in* n *unknowns* $x_1, \dots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m, \end{aligned} \tag{S}$$

where $a_{ij} \in \mathbb{R}$, $b_i \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$. The matrix form is

$$\mathbf{A}\vec{x} = \vec{b},$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \in M(m \times n)$, is called the *coefficient matrix*, $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in M(m \times 1)$ is called the *vector of the right-hand side* and $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M(n \times 1)$ is the *vector of unknowns.*

Definition 2. The matrix

$$(\mathbf{A}|\vec{b}) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

is called the *augmented matrix of the system* (S).

Remarks 3. Equivalent row operations:

1. Interchange two rows.
2. Multiple a row by a nonzero constant.
3. Replace any row by the sum of that row and a constant multiple of any other row.

Remarks 4. Algorithm - the Gauss elimination method:

1. Find the augmented matrix for the system.
2. Use row operations on the matrix. We need the first column full of zeros - except the first row.
3. Continue with the row operations to obtain the matrix in row echelon form (low left triangle full of zeros).
4. Write the final matrix as the system of equations. Solve it.
5. Check the solution - check it in the original equations.

Exercises

1. `https://solveme.edc.org/mobiles/`

2. Solve the following systems of equations:

(a) $$\begin{aligned} 3x + 5y &= 9 \\ 2x + 3y &= 5 \end{aligned}$$

(b) $$\begin{aligned} 3x + 4y &= 5 \\ x + 2y &= 1 \end{aligned}$$

(c) $$\begin{aligned} 3x + 4y &= 1 \\ x - 2y &= 7 \end{aligned}$$

(d) $$\begin{aligned} 2x - 3y &= 6 \\ 3x + 4y &= \tfrac{1}{2} \end{aligned}$$

3. Find matrix for this system:

$$\begin{aligned} x + 2y &= 3 \\ 4y + 5x &= 6 \end{aligned}$$

(a) $\begin{pmatrix} 0 & 2 & | & 3 \\ 4 & 5 & | & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & | & 3 \\ 4 & 5 & | & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & | & 3 \\ 5 & 4 & | & 6 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 2 & | & 3 \\ 5 & 4 & | & 6 \end{pmatrix}$

4. Find matrix for this system:

$$\begin{aligned} x &= 6 \\ y &= 3 \end{aligned}$$

(a) $\begin{pmatrix} 1 & | & 6 \\ 1 & | & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & | & 9 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & | & 6 \\ 0 & 1 & | & 3 \end{pmatrix}$

5. Solve the following systems of equations:

(a) $$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$

(b) $$\begin{aligned} 2y + 3z &= 7 \\ 3x + 6y - 12z &= -3 \\ 5x - 2y + 2z &= -7 \end{aligned}$$

(c) $$\begin{aligned} 6x + 4y + 3z &= -6 \\ x + 2y + z &= \tfrac{1}{3} \\ -12x - 10y - 7z &= 11 \end{aligned}$$

(d) $$\begin{aligned} 3x + 8y + 2z &= -5 \\ 2x + 5y - 3z &= 0 \\ x + 2y - 2z &= -1 \end{aligned}$$

(e) $\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$

(f) $\begin{aligned} x + y - 2z &= 1 \\ 2x - 3y + z &= -8 \\ 3x + y + 4z &= 7 \end{aligned}$

(g) $\begin{aligned} x + 2y - z &= 1 \\ 2x + y + 4z &= 2 \\ 3x + 3y + 4z &= 1 \end{aligned}$

6. Fill the blank space according to the hints:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 3 & 8 & 3 & 7 \\ 2 & -3 & 1 & -10 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right] \xrightarrow{-R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & -9 & -1 & -16 \end{array}\right] \xrightarrow[R_3 + 9R_2]{R_1 - 3R_2}$$

$$\left[\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot \end{array}\right] \xrightarrow[-R_3]{R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Source 1: `http://www.bumatematikozelders.com/altsayfa/matrix_theory/system_of_linear_equations_and_matrices.pdf`

7. Solve graphically, then compare with the matrix solution:

(a) $\begin{aligned} y &= -\tfrac{3}{2}x + \tfrac{1}{2} \\ 2x + 3y &= -6 \end{aligned}$

(b) $\begin{aligned} 4x &= 8 \\ 6y &= -3x + 6 \end{aligned}$

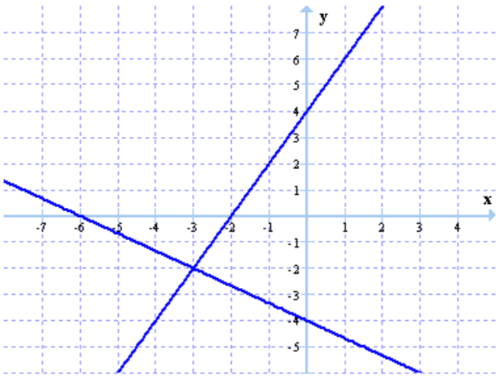
(c) $\begin{aligned} 4 &= -4y \\ -3x - y &= -4 \end{aligned}$

(d) $\begin{aligned} -x + 3y &= -6 \\ 6y &= 2x + 6 \end{aligned}$

(e) $\begin{aligned} 2(y - x) &= 0 \\ -x + y &= -3 \end{aligned}$

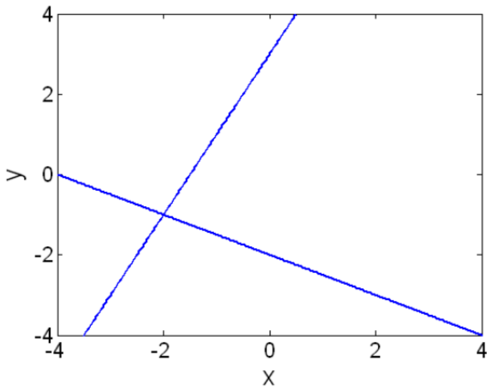
(f) $\begin{aligned} x + 4y &= 8 \\ y &= -\tfrac{1}{4}x + 2 \end{aligned}$

8. Find solution of this system:



Source 2: http://mathquest.carroll.edu/libraries/ALG.student.edition.pdf

9. Find system for this image:



(a) $3x + 3y = -6, 4x + 2y = 3$

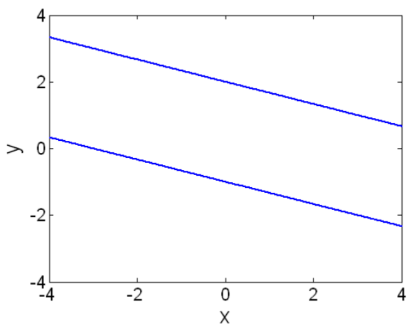
(b) $x - y = -5, 2x + 4y = 4$

(c) $-8x + 4y = 12, 2x + 4y = -8$

(d) $-x + 3y = 9, 2x - y = 4$

Source 3: http://mathquest.carroll.edu/libraries/ALG.student.edition.pdf

10. Find system for this image:



(a) $-x + 3y = 6, 2x + 6y = -6$

(b) $-x + 3y = 6, 2x + 6y = 12$

(c) $x + 3y = 6, 2x + 6y = 12$

(d) $x + 3y = 6, x + 3y = -3$

Source 4: http://mathquest.carroll.edu/libraries/ALG.student.edition.pdf

Source 5: `https://www.mashupmath.com/blog/2017/12/12/can-your-students-solve-these-star-wars-math-problems-192be`