is called **pivoting** the matrix about the element (number) 2. Similarly, we have pivoted about the element 2 in the second column of (7d), shown circled,

$$\begin{matrix} 2 \\ \textcircled{2} \\ 3 \end{matrix}$$

in order to obtain the augmented matrix (7g). Finally, pivoting about the element 11 in column 3 of (7g)

$$\begin{matrix} 7 \\ -2 \\ \textcircled{11} \end{matrix}$$

leads to the augmented matrix (7h), in which all columns to the left of the vertical line are in unit form. The element about which a matrix is pivoted is called the *pivot element*.

Before looking at the next example, let's introduce the following notation for the three types of row operations.

Notation for Row Operations

Letting R_i denote the ith row of a matrix, we write:

Operation 1 $R_i \leftrightarrow R_j$ to mean: Interchange row i with row j.

Operation 2 cR_i to mean: Replace row i with c times row i.

Operation 3 $R_i + aR_j$ to mean: Replace row i with the sum of row i and a times row j.

2a

EXAMPLE 4 Pivot the matrix about the circled element.

$$\left[\begin{array}{cc|c} \textcircled{3} & 5 & 9 \\ 2 & 3 & 5 \end{array}\right]$$

Solution Using the notation just introduced, we obtain

$$\left[\begin{array}{cc|c} 3 & 5 & 9 \\ 2 & 3 & 5 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & \frac{5}{3} & 3 \\ 2 & 3 & 5 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & \frac{5}{3} & 3 \\ 0 & -\frac{1}{3} & -1 \end{array}\right]$$

The first column, which originally contained the entry 3, is now in unit form, with a 1 where the pivot element used to be, and we are done.

Alternate Solution In the first solution, we used operation 2 to obtain a 1 where the pivot element was originally. Alternatively, we can use operation 3 as follows:

$$\left[\begin{array}{cc|c} 3 & 5 & 9 \\ 2 & 3 & 5 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 3 & 5 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -1 & -3 \end{array}\right]$$

■

Note In Example 4, the two matrices

$$\left[\begin{array}{cc|c} 1 & \frac{5}{3} & 3 \\ 0 & -\frac{1}{3} & -1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -1 & -3 \end{array}\right]$$

look quite different, but they are in fact equivalent. You can verify this by observing that they represent the systems of equations

$$\begin{aligned} x + \tfrac{5}{3}y &= 3 \\ -\tfrac{1}{3}y &= -1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + 2y &= 4 \\ -y &= -3 \end{aligned}$$

respectively, and both have the same solution: $x = -2$ and $y = 3$. Example 4 also shows that we can sometimes avoid working with fractions by using an appropriate row operation.

A summary of the Gauss–Jordan method follows.

The Gauss–Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows (operation 1), if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

Before writing the augmented matrix, be sure to write all equations with the variables on the left and constant terms on the right of the equal sign. Also, make sure that the variables are in the same order in all equations.

EXAMPLE 5 Solve the system of linear equations given by

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned} \qquad \textbf{(8)}$$

Solution Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{rrr|r} \textcircled{3} & -2 & 8 & 9 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1]{R_2 + 2R_1} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & 2 & 19 & 27 \\ 0 & 2 & -12 & -4 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & \textcircled{2} & -12 & -4 \\ 0 & 2 & 19 & 27 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 2 & 19 & 27 \end{array}\right]$$

To solve a system of equations using matrices, we transform the augmented matrix into a matrix in **row-echelon form** using row operations. For a consistent and independent system of equations, its **augmented matrix** is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

ROW-ECHELON FORM

For a consistent and independent system of equations, its augmented matrix is in **row-echelon form** when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right] \qquad a, b, c, d, e, f \text{ are real numbers}$$

Once we get the augmented matrix into row-echelon form, we can write the equivalent system of equations and read the value of at least one variable. We then substitute this value in another equation to continue to solve for the other variables. This process is illustrated in the next example.

How to Solve a System of Equations Using a Matrix

2b

Solve the system of equations using a matrix: $\left\{\begin{array}{l} 3x+4y=5 \\ x+2y=1 \end{array}\right.$

Answer

Step 1. Write the augmented matrix for the system of equations.		$\left\{\begin{array}{l} 3x+4y=5 \\ x+2y=1 \end{array}\right.$ $\left[\begin{array}{cc|c} 3 & 4 & 5 \\ 1 & 2 & 1 \end{array}\right]$
Step 2. Using row operations get the entry in row 1, column 1 to be 1.	Interchange the rows, so 1 will be in row 1, column 1.	$\begin{array}{l} R_2 \\ R_1 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 5 \end{array}\right]$
Step 3. Using row operations, get zeros in column 1 below the 1.	Multiply row 1 by −3 and add it to row 2.	$-3R_1+R_2 \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & 2 \end{array}\right]$
Step 4. Using row operations, get the entry in row 2, column 2 to be 1.	Multiply row 2 by $-\frac{1}{2}$.	$-\frac{1}{2}R_2 \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$
Step 5. Continue the process until the matrix is in row-echelon form.	The matrix is now in row-echelon form.	$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$
Step 6. Write the corresponding system of equations.	$\begin{array}{cc} x & y \end{array}$ $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$	$\left\{\begin{array}{r} x+2y=1 \\ y=-1 \end{array}\right.$
Step 7. Use substitution to find the remaining variables.	Substitute $y=-1$ into $x+2y=1$.	$y=-1$ $x+2y=1$ $x+2(-1)=1$ $x-2=1$ $x=3$

Step 8. Write the solution as an ordered pair or triple.		(3, –1)
Step 9. Check that the solution makes the original equations true.		We leave the check to you.

Example 21.6.14

Solve the system of equations using a matrix: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

The solution is $(4, -1)$.

Example 21.6.15

Solve the system of equations using a matrix: $\begin{cases} 2x+y=-4 \\ x-y=-2 \end{cases}$

Answer

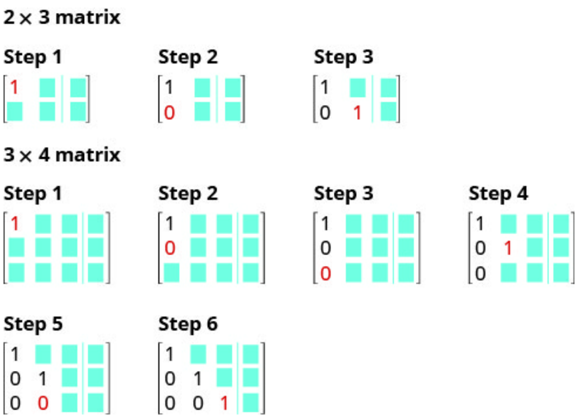
The solution is $(-2, 0)$.

The steps are summarized here.

SOLVE A SYSTEM OF EQUATIONS USING MATRICES.

1. Write the augmented matrix for the system of equations.
2. Using row operations get the entry in row 1, column 1 to be 1.
3. Using row operations, get zeros in column 1 below the 1.
4. Using row operations, get the entry in row 2, column 2 to be 1.
5. Continue the process until the matrix is in row-echelon form.
6. Write the corresponding system of equations.
7. Use substitution to find the remaining variables.
8. Write the solution as an ordered pair or triple.
9. Check that the solution makes the original equations true.

Here is a visual to show the order for getting the 1’s and 0’s in the proper position for row-echelon form.



2c

SOLUTION We start by writing the augmented matrix corresponding to system (4):

$$\left[\begin{array}{rr|r} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \qquad (5)$$

Our objective is to use row operations from Theorem 1 to try to transform matrix (5) into the form

$$\left[\begin{array}{rr|r} 1 & 0 & m \\ 0 & 1 & n \end{array}\right] \qquad (6)$$

where m and n are real numbers. Then the solution to system (4) will be obvious, since matrix (6) will be the augmented matrix of the following system (a row in an augmented matrix always corresponds to an equation in a linear system):

$$\begin{aligned} x_1 &= m \qquad & x_1 + 0x_2 &= m \\ x_2 &= n \qquad & 0x_1 + x_2 &= n \end{aligned}$$

Now we use row operations to transform matrix (5) into form (6).

Step 1 To get a 1 in the upper left corner, we interchange R_1 and R_2 (Theorem 1A):

$$\left[\begin{array}{rr|r} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{rr|r} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right]$$

Step 2 To get a 0 in the lower left corner, we multiply R_1 by (-3) and add to R_2 (Theorem 1C)—this changes R_2 but not R_1. Some people find it useful to write $(-3R_1)$ outside the matrix to help reduce errors in arithmetic, as shown:

$$\left[\begin{array}{rr|r} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right] \underset{\sim}{(-3)R_1 + R_2 \rightarrow R_2} \left[\begin{array}{rr|r} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right]$$

$$-3 \quad 6 \quad -21$$

Step 3 To get a 1 in the second row, second column, we multiply R_2 by $\frac{1}{10}$ (Theorem 1B):

$$\left[\begin{array}{rr|r} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right] \underset{\sim}{\tfrac{1}{10}R_2 \rightarrow R_2} \left[\begin{array}{rr|r} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

Step 4 To get a 0 in the first row, second column, we multiply R_2 by 2 and add the result to R_1 (Theorem 1C)—this changes R_1 but not R_2:

$$0 \quad 2 \quad -4$$

$$\left[\begin{array}{rr|r} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right] \underset{\sim}{2R_2 + R_1 \rightarrow R_1} \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

We have accomplished our objective! The last matrix is the augmented matrix for the system

$$\begin{aligned} x_1 &= 3 \qquad & x_1 + 0x_2 &= 3 \\ x_2 &= -2 \qquad & 0x_1 + x_2 &= -2 \end{aligned} \qquad (7)$$

Since system (7) is equivalent to system (4), our starting system, we have solved system (4); that is, $x_1 = 3$ and $x_2 = -2$.

CHECK

$$\begin{aligned} 3x_1 + 4x_2 &= 1 \qquad & x_1 - 2x_2 &= 7 \\ 3(3) + 4(-2) &\overset{?}{=} 1 \qquad & 3 - 2(-2) &\overset{?}{=} 7 \\ 1 &\overset{\checkmark}{=} 1 \qquad & 7 &\overset{\checkmark}{=} 7 \end{aligned}$$

The preceding process may be written more compactly as follows:

Step 1: Need a 1 here.

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

Step 2: Need a 0 here.

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right] \quad (-3)R_1 + R_2 \rightarrow R_2$$

$$\begin{array}{ccc} -3 & 6 & -21 \end{array}$$

Step 3: Need 1 here.

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right] \quad \tfrac{1}{10}R_2 \rightarrow R_2$$

$$\begin{array}{ccc} 0 & 2 & -4 \end{array}$$

Step 4: Need a 0 here.

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right] \quad 2R_2 + R_1 \rightarrow R_1$$

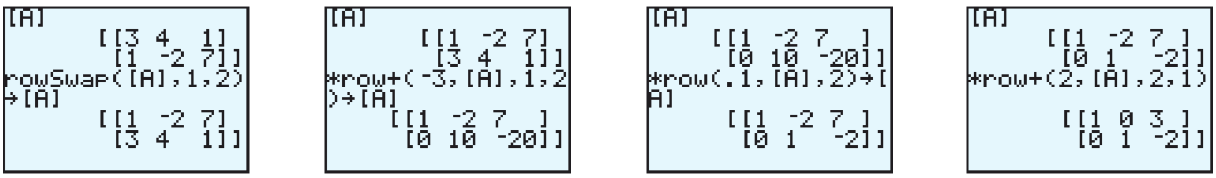
$$\sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

Therefore, $x_1 = 3$ and $x_2 = -2$.

Matched Problem 1 Solve using augmented matrix methods:

$$\begin{aligned} 2x_1 - x_2 &= -7 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

Many graphing calculators can perform row operations. Figure 3 shows the results of performing the row operations used in the solution of Example 1. Consult your manual for the details of performing row operations on your graphing calculator.



(A) $R_1 \leftrightarrow R_2$ (B) $(-3)R_1 + R_2 \rightarrow R_2$ (C) $\frac{1}{10}R_2 \rightarrow R_2$ (D) $2R_2 + R_1 \rightarrow R_1$

Figure 3 **Row operations on a graphing calculator**

Explore and Discuss 1 The summary following the solution of Example 1 shows five augmented matrices. Write the linear system that each matrix represents, solve each system graphically, and discuss the relationships among these solutions.

EXAMPLE 2 Solving a System Using Augmented Matrix Methods Solve using augmented matrix methods:

$$\begin{aligned} 2x_1 - 3x_2 &= 6 \\ 3x_1 + 4x_2 &= \frac{1}{2} \end{aligned}$$

SOLUTION

Step 1: Need a 1 here.

$$\left[\begin{array}{rr|r} 2 & -3 & 6 \\ 3 & 4 & \frac{1}{2} \end{array}\right] \quad \tfrac{1}{2}R_1 \to R_1$$

Step 2: Need a 0 here.

$$\sim \left[\begin{array}{rr|r} 1 & -\frac{3}{2} & 3 \\ 3 & 4 & \frac{1}{2} \end{array}\right] \quad (-3)R_1 + R_2 \to R_2$$

$$-3 \quad \tfrac{9}{2} \quad -9$$

Step 3: Need a 1 here.

$$\sim \left[\begin{array}{rr|r} 1 & -\frac{3}{2} & 3 \\ 0 & \frac{17}{2} & -\frac{17}{2} \end{array}\right] \quad \tfrac{2}{17}R_2 \to R_2$$

$$0 \quad \tfrac{3}{2} \quad -\tfrac{3}{2}$$

Step 4: Need a 0 here.

$$\sim \left[\begin{array}{rr|r} 1 & -\frac{3}{2} & 3 \\ 0 & 1 & -1 \end{array}\right] \quad \tfrac{3}{2}R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{rr|r} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -1 \end{array}\right]$$

So, $x_1 = \frac{3}{2}$ and $x_2 = -1$. The check is left for you.

Matched Problem 2 Solve using augmented matrix methods:

$$\begin{aligned} 5x_1 - 2x_2 &= 11 \\ 2x_1 + 3x_2 &= \frac{5}{2} \end{aligned}$$

EXAMPLE 3 **Solving a System Using Augmented Matrix Methods** Solve using augmented matrix methods:

$$\begin{aligned} 2x_1 - x_2 &= 4 \\ -6x_1 + 3x_2 &= -12 \end{aligned} \tag{8}$$

SOLUTION

$$\left[\begin{array}{rr|r} 2 & -1 & 4 \\ -6 & 3 & -12 \end{array}\right] \quad \begin{array}{l} \tfrac{1}{2}R_1 \to R_1 \text{ (to get a 1 in the upper left corner)} \\ \tfrac{1}{3}R_2 \to R_2 \text{ (this simplifies } R_2\text{)} \end{array}$$

$$\sim \left[\begin{array}{rr|r} 1 & -\frac{1}{2} & 2 \\ -2 & 1 & -4 \end{array}\right] \quad 2R_1 + R_2 \to R_2 \text{ (to get a 0 in the lower left corner)}$$

$$2 \quad -1 \quad 4$$

$$\sim \left[\begin{array}{rr|r} 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The last matrix corresponds to the system

$$\begin{aligned} x_1 - \frac{1}{2}x_2 &= 2 \qquad x_1 - \frac{1}{2}x_2 = 2 \\ 0 &= 0 \qquad 0x_1 + 0x_2 = 0 \end{aligned} \tag{9}$$

This system is equivalent to the original system. Geometrically, the graphs of the two original equations coincide, and there are infinitely many solutions. In general, if we end up with a row of zeros in an augmented matrix for a two-equation, two-variable system, the system is dependent, and there are infinitely many solutions.

We represent the infinitely many solutions using the same method that was used in Section 4.1; that is, by introducing a parameter. We start by solving

Exercises

1. `https://solveme.edc.org/mobiles/`

2. Solve the following systems of equations:

(a) $\begin{aligned} 3x + 5y &= 9 \\ 2x + 3y &= 5 \end{aligned}$

(b) $\begin{aligned} 3x + 4y &= 5 \\ x + 2y &= 1 \end{aligned}$

(c) $\begin{aligned} 3x + 4y &= 1 \\ x - 2y &= 7 \end{aligned}$

(d) $\begin{aligned} 2x - 3y &= 6 \\ 3x + 4y &= \tfrac{1}{2} \end{aligned}$

3. Find matrix for this system:

$$\begin{aligned} x + 2y &= 3 \\ 4y + 5x &= 6 \end{aligned}$$

(a) $\begin{pmatrix} 0 & 2 & | & 3 \\ 4 & 5 & | & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & | & 3 \\ 4 & 5 & | & 6 \end{pmatrix}$

(c) ✓ $\begin{pmatrix} 1 & 2 & | & 3 \\ 5 & 4 & | & 6 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 2 & | & 3 \\ 5 & 4 & | & 6 \end{pmatrix}$

4. Find matrix for this system:

$$\begin{aligned} x &= 6 \\ y &= 3 \end{aligned}$$

(a) $\begin{pmatrix} 1 & | & 6 \\ 1 & | & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & | & 9 \end{pmatrix}$

(c) ✓ $\begin{pmatrix} 1 & 0 & | & 6 \\ 0 & 1 & | & 3 \end{pmatrix}$

5. Solve the following systems of equations:

(a) $\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$

(b) $\begin{aligned} 2y + 3z &= 7 \\ 3x + 6y - 12z &= -3 \\ 5x - 2y + 2z &= -7 \end{aligned}$

(c) $\begin{aligned} 6x + 4y + 3z &= -6 \\ x + 2y + z &= \tfrac{1}{3} \\ -12x - 10y - 7z &= 11 \end{aligned}$

(d) $\begin{aligned} 3x + 8y + 2z &= -5 \\ 2x + 5y - 3z &= 0 \\ x + 2y - 2z &= -1 \end{aligned}$

look quite different, but they are in fact equivalent. You can verify this by observing that they represent the systems of equations

$$\begin{aligned} x + \frac{5}{3}y &= 3 \\ -\frac{1}{3}y &= -1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + 2y &= 4 \\ -y &= -3 \end{aligned}$$

respectively, and both have the same solution: $x = -2$ and $y = 3$. Example 4 also shows that we can sometimes avoid working with fractions by using an appropriate row operation.

A summary of the Gauss–Jordan method follows.

The Gauss–Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows (operation 1), if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

Before writing the augmented matrix, be sure to write all equations with the variables on the left and constant terms on the right of the equal sign. Also, make sure that the variables are in the same order in all equations.

5a

EXAMPLE 5 Solve the system of linear equations given by

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned} \tag{8}$$

Solution Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{rrr|r} \textcircled{3} & -2 & 8 & 9 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1]{R_2 + 2R_1} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & 2 & 19 & 27 \\ 0 & 2 & -12 & -4 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & \textcircled{2} & -12 & -4 \\ 0 & 2 & 19 & 27 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 2 & 19 & 27 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 31 & 31 \end{array}\right]$$

$$\xrightarrow{\frac{1}{31}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 + 6R_3]{R_1 - 9R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The solution to System (8) is given by $x = 3$, $y = 4$, and $z = 1$. This may be verified by substitution into System (8) as follows:

$$\begin{aligned} 3(3) - 2(4) + 8(1) &= 9 \quad ✓ \\ -2(3) + 2(4) + 1 \quad &= 3 \quad ✓ \\ 3 \quad + 2(4) - 3(1) &= 8 \quad ✓ \end{aligned}$$

When searching for an element to serve as a pivot, it is important to keep in mind that you may work only with the row containing the potential pivot or any row *below* it. To see what can go wrong if this caution is not heeded, consider the following augmented matrix for some linear system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right]$$

Observe that column 1 is in unit form. The next step in the Gauss–Jordan elimination procedure calls for obtaining a nonzero element in the second position of row 2. If you use row 1 (which is *above* the row under consideration) to help you obtain the pivot, you might proceed as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 0 & 0 & 3 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 2 & 1 & -2 \end{array}\right]$$

As you can see, not only have we obtained a nonzero element to serve as the next pivot, but it is already a 1, thus obviating the next step. This seems like a good move. But beware, we have undone some of our earlier work: Column 1 is no longer a unit column where a 1 appears first. The correct move in this case is to interchange row 2 with row 3 in the first augmented matrix.

The next example illustrates how to handle a situation in which the first entry in row 1 of the augmented matrix is zero.

Explore & Discuss

1. Can the phrase "a nonzero constant multiple of itself" in a type-2 row operation be replaced by "a constant multiple of itself"? Explain.
2. Can a row of an augmented matrix be replaced by a row obtained by adding a constant to every element in that row without changing the solution of the system of linear equations? Explain.

EXAMPLE 6 Solve the system of linear equations given by

$$\begin{aligned} 2y + 3z &= 7 \\ 3x + 6y - 12z &= -3 \\ 5x - 2y + 2z &= -7 \end{aligned}$$

Solution Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 7 \\ 3 & 6 & -12 & -3 \\ 5 & -2 & 2 & -7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} \boxed{3} & 6 & -12 & -3 \\ 0 & 2 & 3 & 7 \\ 5 & -2 & 2 & -7 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 5 & -2 & 2 & -7 \end{array}\right]$$

$$\xrightarrow{R_3 - 5R_1} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & \boxed{2} & 3 & 7 \\ 0 & -12 & 22 & -2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & -12 & 22 & -2 \end{array}\right]$$

$$\xrightarrow[R_3 + 12R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & \boxed{40} & 40 \end{array}\right]$$

$$\xrightarrow{\frac{1}{40}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 - \frac{3}{2}R_3]{R_1 + 7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The solution to the system is given by $x = -1$, $y = 2$, and $z = 1$; this may be verified by substitution into the system.

APPLIED EXAMPLE 7 Manufacturing: Production Scheduling Complete the solution to Example 1 in Section 2.1, page 70.

Solution To complete the solution of the problem posed in Example 1, recall that the mathematical formulation of the problem led to the following system of linear equations:

$$\begin{aligned} 2x + y + z &= 180 \\ x + 3y + 2z &= 300 \\ 2x + y + 2z &= 240 \end{aligned}$$

where x, y, and z denote the respective numbers of type-A, type-B, and type-C souvenirs to be made.

5c

$$\begin{aligned} 6x+4y+3z &= -6 \\ x+2y+z &= \frac{1}{3} \\ -12x-10y-7z &= 11 \end{aligned}$$

Solution

Write the augmented matrix for the system of equations.

$$\left[\begin{array}{ccc|c} 6 & 4 & 3 & -6 \\ 1 & 2 & 1 & \frac{1}{3} \\ -12 & -10 & -7 & 11 \end{array}\right]$$

On the matrix page of the calculator, enter the augmented matrix above as the matrix variable $[A]$.

$$[A]=\left[\begin{array}{ccc|c} 6 & 4 & 3 & -6 \\ 1 & 2 & 1 & \frac{1}{3} \\ -12 & -10 & -7 & 11 \end{array}\right]$$

Use the **rref(** function in the calculator, calling up the matrix variable $[A]$.

rref([A])

Use the MATH --> FRAC option in the calculator to express the matrix elements as fractions.

Evaluate

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & -4 \end{array}\right] \rightarrow \begin{aligned} x+0y+0z &= -\frac{2}{3} \\ y+0z &= \frac{5}{2} \\ z &= -4 \end{aligned}$$

Thus the solution, which can easily be read from the right column of the reduced row-echelon form of the matrix, is $\left(-\frac{2}{3}, \frac{5}{2}, -4\right)$.

Exercise 3.3.3

Solve the system of equations.

$$\begin{aligned} 4x-7y+2z &= -5 \\ -x+3y-8z &= -10 \\ -5x-4y+6z &= 19 \end{aligned}$$

Answer

Write the augmented matrix for the system of equations.

$$\left[\begin{array}{ccc|c} 4 & -7 & 2 & -5 \\ -1 & 3 & -8 & -10 \\ -5 & -4 & 6 & 19 \end{array}\right]$$

On the matrix page of the calculator, enter the augmented matrix above as the matrix variable $[A]$.

$$[A]=\left[\begin{array}{ccc|c} 4 & -7 & 2 & -5 \\ -1 & 3 & -8 & -10 \\ -5 & -4 & 6 & 19 \end{array}\right]$$

We use the same procedure when the system of equations has three equations.

Example 21.6.16

Solve the system of equations using a matrix: $\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$

Answer

	$\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{rrr\|r} 3 & 8 & 2 & -5 \\ 2 & 5 & -3 & 0 \\ 1 & 2 & -2 & -1 \end{array}\right]$
Interchange row 1 and 3 to get the entry in row 1, column 1 to be 1.	$\begin{array}{c} R_3 \\ \\ R_1 \end{array} \left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 2 & 5 & -3 & 0 \\ 3 & 8 & 2 & -5 \end{array}\right]$
Using row operations, get zeros in column 1 below the 1.	$\begin{array}{c} \\ -2R_1+R_2 \\ \\ \end{array} \left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 3 & 8 & 2 & -5 \end{array}\right]$
	$\begin{array}{c} \\ \\ -3R_1+R_3 \end{array} \left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 8 & -2 \end{array}\right]$
The entry in row 2, column 2 is now 1.	
Continue the process until the matrix is in row-echelon form.	$\begin{array}{c} \\ \\ -2R_2+R_3 \end{array} \left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & -6 \end{array}\right]$
	$\begin{array}{c} \\ \\ \frac{1}{6}R_3 \end{array} \left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$
The matrix is now in row-echelon form.	$\left[\begin{array}{rrr\|r} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$
Write the corresponding system of equations.	$\begin{cases} x+2y-2z=-1 \\ y+z=2 \\ z=-1 \end{cases}$
Use substitution to find the remaining variables.	$y+z=2$ $y+(-1)=2$ $y=3$
	$x+2y-2z=-1$ $x+2(3)-2(-1)=-1$ $x+6+2=-1$ $x=-9$
Write the solution as an ordered pair or triple.	$(-9, 3, -1)$
Check that the solution makes the original equations true.	We leave the check for you.

Historical Note

Maxime Bôcher (1867–1918)

The first known use of augmented matrices appeared between 200 B.C. and 100 B.C. in a Chinese manuscript entitled *Nine Chapters of Mathematical Art*. The coefficients were arranged in columns rather than in rows, as today, but remarkably the system was solved by performing a succession of operations on the columns. The actual use of the term *augmented matrix* appears to have been introduced by the American mathematician Maxime Bôcher in his book *Introduction to Higher Algebra*, published in 1907. In addition to being an outstanding research mathematician and an expert in Latin, chemistry, philosophy, zoology, geography, meteorology, art, and music, Bôcher was an outstanding expositor of mathematics whose elementary textbooks were greatly appreciated by students and are still in demand today.

[Image: HUP Bocher, Maxime (1), olvwork650836]

The basic method for solving a linear system is to perform algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These are called ***elementary row operations*** on a matrix.

In the following example we will illustrate how to use elementary row operations and an augmented matrix to solve a linear system in three unknowns. Since a systematic procedure for solving linear systems will be developed in the next section, do not worry about how the steps in the example were chosen. Your objective here should be simply to understand the computations.

EXAMPLE 6 | Using Elementary Row Operations

In the left column we solve a system of linear equations by operating on the equations in the system, and in the right column we solve the same system by operating on the rows of the augmented matrix.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad\qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add −2 times the first equation to the second to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add −2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add −3 times the first equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

Add −3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add −3 times the second equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Add −3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third equation by −2 to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Multiply the third row by −2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add −1 times the second equation to the first to obtain

$$\begin{aligned} x \quad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add −1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

$$\begin{aligned} x \quad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution $x = 1,\ y = 2,\ z = 3$ is now evident.

The solution in this example can also be expressed as the ordered triple (1, 2, 3) with the understanding that the numbers in the triple are in the same order as the variables in the system, namely, x, y, z.

Exercise Set 1.1

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3.

a. $x_1 + 5x_2 - \sqrt{2}x_3 = 1$ **b.** $x_1 + 3x_2 + x_1x_3 = 2$

c. $x_1 = -7x_2 + 3x_3$ **d.** $x_1^{-2} + x_2 + 8x_3 = 5$

e. $x_1^{3/5} - 2x_2 + x_3 = 4$ **f.** $\pi x_1 - \sqrt{2}x_2 = 7^{1/3}$

2. In each part, determine whether the equation is linear in x and y.

a. $2^{1/3}x + \sqrt{3}y = 1$ **b.** $2x^{1/3} + 3\sqrt{y} = 1$

c. $\cos\left(\frac{\pi}{7}\right)x - 4y = \log 3$ **d.** $\frac{\pi}{7}\cos x - 4y = 0$

e. $xy = 1$ **f.** $y + 7 = x$

Systems of linear equations can be represented by matrices. Operations on equations (for eliminating variables) can be represented by appropriate row operations on the corresponding matrices. For example,

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & -2x_3 & = & 1 \\ 2x_1 & -3x_2 & +x_3 & = & -8 \\ 3x_1 & +x_2 & +4x_3 & = & 7 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 2 & -3 & 1 & -8 \\ 3 & 1 & 4 & 7 \end{array}\right]$$

$$\begin{array}{c} [\text{Eq } 2] - 2[\text{Eq } 1] \\ [\text{Eq } 3] - 3[\text{Eq } 1] \end{array} \qquad \begin{array}{c} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & -2x_3 & = & 1 \\ & -5x_2 & +5x_3 & = & -10 \\ & -2x_2 & +10x_3 & = & 4 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & -5 & 5 & -10 \\ 0 & -2 & 10 & 4 \end{array}\right]$$

$$\begin{array}{c} (-1/5)[\text{Eq } 2] \\ (-1/2)[\text{Eq } 3] \end{array} \qquad \begin{array}{c} (-1/5)R_2 \\ (-1/2)R_3 \end{array}$$

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & -2x_3 & = & 1 \\ & x_2 & -x_3 & = & 2 \\ & x_2 & -5x_3 & = & -2 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -5 & -2 \end{array}\right]$$

$$[\text{Eq } 3] - [\text{Eq } 2] \qquad R_3 - R_2$$

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & -2x_3 & = & 1 \\ & x_2 & -x_3 & = & 2 \\ & & -4x_3 & = & -4 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -4 & -4 \end{array}\right]$$

$$(-1/4)[\text{Eq } 3] \qquad (-1/4)R_3$$

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & -2x_3 & = & 1 \\ & x_2 & -x_3 & = & 2 \\ & & x_3 & = & 1 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} [\text{Eq } 1] + 2[\text{Eq } 3] \\ [\text{Eq } 2] + [\text{Eq } 3] \end{array} \qquad \begin{array}{c} R_1 + 2R_3 \\ R_2 + R_3 \end{array}$$

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & & = & 3 \\ & x_2 & & = & 3 \\ & & x_3 & = & 1 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$[\text{Eq } 1] - [\text{Eq } 2] \qquad R_1 - R_2$$

$$\left\{\begin{array}{rrrcr} x_1 & & & = & 0 \\ & x_2 & & = & 3 \\ & & x_3 & = & 1 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Elementary row operations

Definition 1.3. There are three kinds of elementary row operations on matrices:

(a) Adding a multiple of one row to another row;

(b) Multiplying all entries of one row by a nonzero constant;

(c) Interchanging two rows.

Definition 1.4. Two linear systems in same variables are said to be **equivalent** if their solution sets are the same. A matrix A is said to be **row equivalent** to a matrix B , written

$$A \sim B,$$

if there is a sequence of elementary row operations that changes A to B.

Theorem 2.4. *Every matrix is row equivalent to one and only one reduced row echelon matrix. In other words, every matrix has a unique reduced row echelon form.*

Proof. The Row Reduction Algorithm show the existence of reduced row echelon matrix for any matrix M. We only need to show the uniqueness. Suppose A and B are two reduced row echelon forms for a matrix M. Then the systems $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$ have the same solution set. Write $A = [a_{ij}]$ and $B = [b_{ij}]$.

We first show that A and B have the same pivot columns. Let $i_1, \ldots, i_k$ be the pivot columns of A, and let $j_1, \ldots, j_l$ be the pivot columns of B. Suppose $i_1 = j_1, \ldots, i_{r-1} = j_{r-1}$, but $i_r \neq j_r$. Assume $i_r < j_r$. Then the i_rth row of A is

$$\left[0,\ \ldots,\ 0,\ 1,\ a_{r,i_r+1},\ \ldots, a_{r,j_r},\ a_{r,j_r+1},\ \ldots,\ a_{r,n}\right].$$

While the j_rth row of B is

$$\left[0,\ \ldots,\ 0,\ 1,\ b_{r,j_r+1},\ \ldots,\ b_{r,n}\right].$$

Since $i_{r-1} = j_{r-1}$ and $i_r < j_r$, we have $j_{r-1} < i_r < j_r$. So x_{i_r} is a free variable for $B\boldsymbol{x} = \mathbf{0}$. Let

$$u_{i_1} = -b_{1,i_r}, \quad \ldots, \quad u_{i_{r-1}} = -b_{r-1,i_r}, \quad u_{i_r} = 1, \quad \text{and} \quad u_i = 0 \quad \text{for} \quad i > i_r.$$

Then $\boldsymbol{u}$ is a solution of $B\boldsymbol{x} = \mathbf{0}$, but is not a solution of $A\boldsymbol{x} = \mathbf{0}$. This is a contradiction. Of course, $k = l$.

Next we show that for $1 \leq r \leq k = l$, we have

$$a_{rj} = b_{rj}, \quad j_r + 1 \leq j \leq j_{r+1} - 1.$$

Otherwise, we have $a_{r_0 j_0} \neq b_{r_0 j_0}$ such that r_0 is smallest and then j_0 is smallest. Set

$$u_{j_0} = 1, \quad u_{i_1} = -a_{1,j_0}, \quad \ldots, \quad u_{r_0} = -a_{r_0 j_0}, \quad \text{and} \quad u_j = 0 \quad \text{otherwise.}$$

Then $\boldsymbol{u}$ is a solution of $A\boldsymbol{x} = \mathbf{0}$, but is not a solution of $B\boldsymbol{x} = \mathbf{0}$. This is a contradiction. □

Solving linear system

Example 2.1. Find all solutions for the linear system

$$\begin{cases} x_1 & +2x_2 & -x_3 & = & 1 \\ 2x_1 & +x_2 & +4x_3 & = & 2 \\ 3x_1 & +3x_2 & +4x_3 & = & 1 \end{cases}$$

Solution. Perform the row operations:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 2 \\ 3 & 3 & 4 & 1 \end{array}\right] \begin{array}{c} R_2 - 2R_1 \\ \sim \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 6 & 0 \\ 0 & -3 & 7 & -2 \end{array}\right] \begin{array}{c} (-1/3)R_2 \\ \sim \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ \sim \\ R_2 + 2R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \sim \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The system is equivalent to

$$\begin{cases} x_1 & = & 7 \\ x_2 & = & -4 \\ x_3 & = & -2 \end{cases}$$

which means the system has a unique solution.

(e) $$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

(f) $$\begin{aligned} x + y - 2z &= 1 \\ 2x - 3y + z &= -8 \\ 3x + y + 4z &= 7 \end{aligned}$$

(g) $$\begin{aligned} x + 2y - z &= 1 \\ 2x + y + 4z &= 2 \\ 3x + 3y + 4z &= 1 \end{aligned}$$

6. Fill the blank space according to the hints:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 3 & 8 & 3 & 7 \\ 2 & -3 & 1 & -10 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & -9 & -1 & -16 \end{array}\right] \xrightarrow{-R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & -9 & -1 & -16 \end{array}\right] \xrightarrow[R_3 + 9R_2]{R_1 - 3R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 2 \end{array}\right] \xrightarrow[-R_3]{R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Source 1: `http://www.bumatematikozelders.com/altsayfa/matrix_theory/system_of_linear_equations_and_matrices.pdf`

7. Solve graphically, then compare with the matrix solution:

(a) $$\begin{aligned} y &= -\tfrac{3}{2}x + \tfrac{1}{2} \\ 2x + 3y &= -6 \end{aligned}$$

(b) $$\begin{aligned} 4x &= 8 \\ 6y &= -3x + 6 \end{aligned}$$

(c) $$\begin{aligned} 4 &= -4y \\ -3x - y &= -4 \end{aligned}$$

(d) $$\begin{aligned} -x + 3y &= -6 \\ 6y &= 2x + 6 \end{aligned}$$

(e) $$\begin{aligned} 2(y - x) &= 0 \\ -x + y &= -3 \end{aligned}$$

(f) $$\begin{aligned} x + 4y &= 8 \\ y &= -\tfrac{1}{4}x + 2 \end{aligned}$$

3. Solving Systems of Linear Equations by Graphing

Example 2 **Solving a System of Linear Equations by Graphing**

Solve the system by graphing both linear equations and finding the point(s) of intersection.

$$y = -\frac{3}{2}x + \frac{1}{2}$$
$$2x + 3y = -6$$

Solution:

To graph each equation, write the equation in slope-intercept form $y = mx + b$.

First equation:

$$y = -\frac{3}{2}x + \frac{1}{2} \qquad \text{Slope: } -\tfrac{3}{2}$$

Second equation:

$$2x + 3y = -6$$
$$3y = -2x - 6$$
$$\frac{3y}{3} = \frac{-2x}{3} - \frac{6}{3}$$
$$y = -\frac{2}{3}x - 2 \qquad \text{Slope: } -\tfrac{2}{3}$$

From their slope-intercept forms, we see that the lines have different slopes, indicating that the lines must intersect at exactly one point. Using the slope and y-intercept we can graph the lines to find the point of intersection (Figure 3-2).

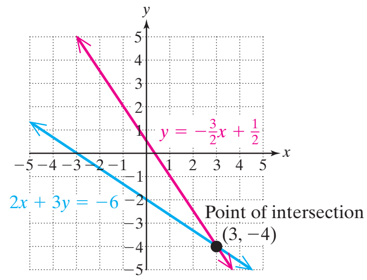


Figure 3-2

The point $(3, -4)$ appears to be the point of intersection. This can be confirmed by substituting $x = 3$ and $y = -4$ into both equations.

$$y = -\frac{3}{2}x + \frac{1}{2} \longrightarrow -4 \stackrel{?}{=} -\frac{3}{2}(3) + \frac{1}{2} \longrightarrow -4 = -\frac{9}{2} + \frac{1}{2} \checkmark \text{ True}$$

$$2x + 3y = -6 \longrightarrow 2(3) + 3(-4) \stackrel{?}{=} -6 \longrightarrow 6 - 12 = -6 \checkmark \text{ True}$$

The solution is $(3, -4)$.

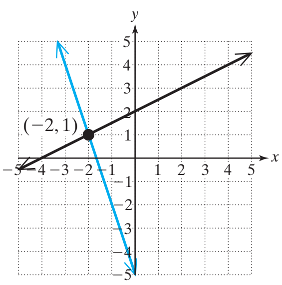
Skill Practice

2. Solve by using the graphing method.

$$3x + y = -5$$
$$x - 2y = -4$$

Skill Practice Answers

2.



TIP: In Example 2, the lines could also have been graphed by using the x- and y-intercepts or by using a table of points. However, the advantage of writing the equations in slope-intercept form is that we can compare the slopes *and* y-intercepts of each line.

1. If the slopes differ, the lines are different and nonparallel and must cross in exactly one point.
2. If the slopes are the same and the y-intercepts are different, the lines are parallel and do not intersect.
3. If the slopes are the same and the y-intercepts are the same, the two equations represent the same line.

Example 3 Solving a System of Linear Equations by Graphing

Solve the system by graphing.

5b

$$4x = 8$$
$$6y = -3x + 6$$

Solution:

The first equation $4x = 8$ can be written as $x = 2$. This is an equation of a vertical line. To graph the second equation, write the equation in slope-intercept form.

First equation:

$$4x = 8$$
$$x = 2$$

Second equation:

$$6y = -3x + 6$$
$$\frac{6y}{6} = \frac{-3x}{6} + \frac{6}{6}$$
$$y = -\frac{1}{2}x + 1$$

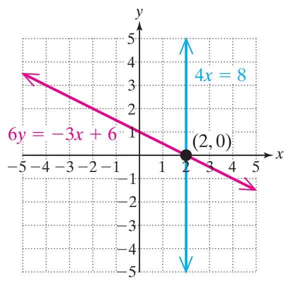


Figure 3-3

The graphs of the lines are shown in Figure 3-3. The point of intersection is (2, 0). This can be confirmed by substituting (2, 0) into both equations.

$$4x = 8 \longrightarrow 4(2) = 8 \ ✓ \text{ True}$$
$$6y = -3x + 6 \longrightarrow 6(0) = -3(2) + 6 \ ✓ \text{ True}$$

The solution is (2, 0).

Skill Practice

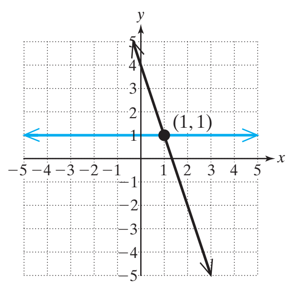
3. Solve the system by graphing.

5e

$$-4 = -4y$$
$$-3x - y = -4$$

Skill Practice Answers

3.



Example 4 Solving a System of Equations by Graphing

Solve the system by graphing.

$$-x + 3y = -6$$
$$6y = 2x + 6$$

Solution:

To graph the line, write each equation in slope-intercept form.

First equation:	Second equation:
$-x + 3y = -6$	$6y = 2x + 6$
$3y = x - 6$	
$\frac{3y}{3} = \frac{x}{3} - \frac{6}{3}$	$\frac{6y}{6} = \frac{2x}{6} + \frac{6}{6}$
$y = \frac{1}{3}x - 2$	$y = \frac{1}{3}x + 1$

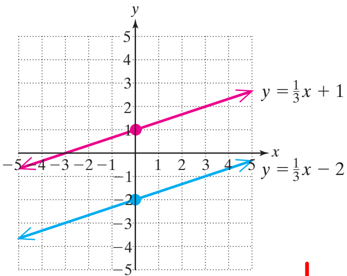


Figure 3-4

Because the lines have the same slope but different y-intercepts, they are parallel (Figure 3-4). Two parallel lines do not intersect, which implies that the system has no solution. The system is inconsistent.

Skill Practice

4. Solve the system by graphing.

$$2(y - x) = 0$$
$$-x + y = -3$$

Example 5 Solving a System of Linear Equations by Graphing

Solve the system by graphing.

$$x + 4y = 8$$
$$y = -\frac{1}{4}x + 2$$

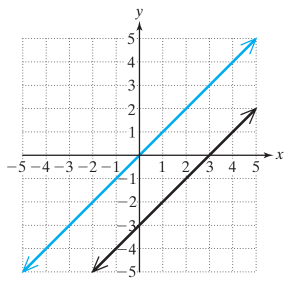
Solution:

Write the first equation in slope-intercept form. The second equation is already in slope-intercept form.

First equation:	Second equation:
$x + 4y = 8$	$y = -\frac{1}{4}x + 2$
$4y = -x + 8$	
$\frac{4y}{4} = \frac{-x}{4} + \frac{8}{4}$	
$y = -\frac{1}{4}x + 2$	

Skill Practice Answers

4. No solution; inconsistent system



Notice that the slope-intercept forms of the two lines are identical. Therefore, the equations represent the same line (Figure 3-5). The system is dependent, and the solution to the system of equations is the set of all points on the line.

Because not all the ordered pairs in the solution set can be listed, we can write the solution in set-builder notation. Furthermore, the equations $x + 4y = 8$ and $y = -\frac{1}{4}x + 2$ represent the same line. Therefore, the solution set may be written as $\{(x, y) \mid y = -\frac{1}{4}x + 2\}$ or $\{(x, y) \mid x + 4y = 8\}$.

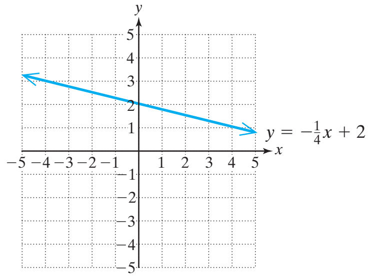


Figure 3-5

Skill Practice

5. Solve the system by graphing.

$$y = \frac{1}{2}x + 1$$
$$x - 2y = -2$$

Calculator Connections

The solution to a system of equations can be found by using either a *Trace* feature or an *Intersect* feature on a graphing calculator to find the point of intersection between two curves.

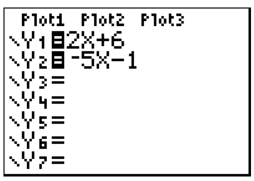
For example, consider the system

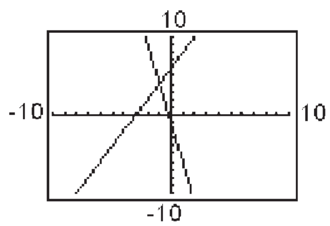
$$-2x + y = 6$$
$$5x + y = -1$$

First graph the equations together on the same viewing window. Recall that to enter the equations into the calculator, the equations must be written with the y-variable isolated. That is, be sure to solve for y first.

Isolate y.

$$-2x + y = 6 \longrightarrow y = 2x + 6$$
$$5x + y = -1 \longrightarrow y = -5x - 1$$

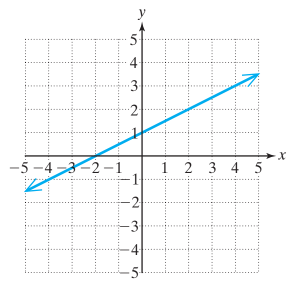




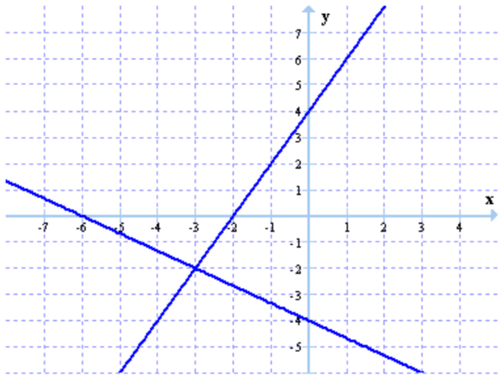
By inspection of the graph, it appears that the solution is $(-1, 4)$. The *Trace* option on the calculator may come close to $(-1, 4)$ but may not show the exact solution (Figure 3-6). However, an *Intersect* feature on a graphing calculator may provide the exact solution (Figure 3-7). See your user's manual for further details.

Skill Practice Answers

5. $\{(x, y) \mid y = \frac{1}{2}x + 1\}$; infinitely many solutions; dependent system



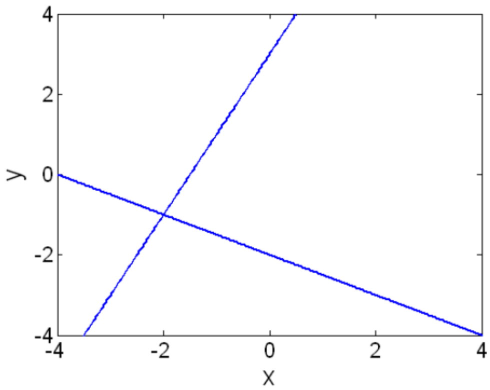
8. Find solution of this system:



[-3, -2]

Source 2: http://mathquest.carroll.edu/libraries/ALG.student.edition.pdf

9. Find system for this image:



(a) $3x + 3y = -6, 4x + 2y = 3$

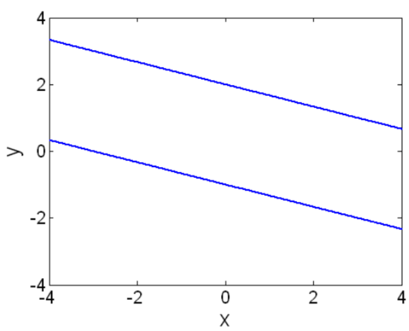
(b) $x - y = -5, 2x + 4y = 4$

(c) $-8x + 4y = 12, 2x + 4y = -8$

(d) $-x + 3y = 9, 2x - y = 4$

Source 3: http://mathquest.carroll.edu/libraries/ALG.student.edition.pdf

10. Find system for this image:



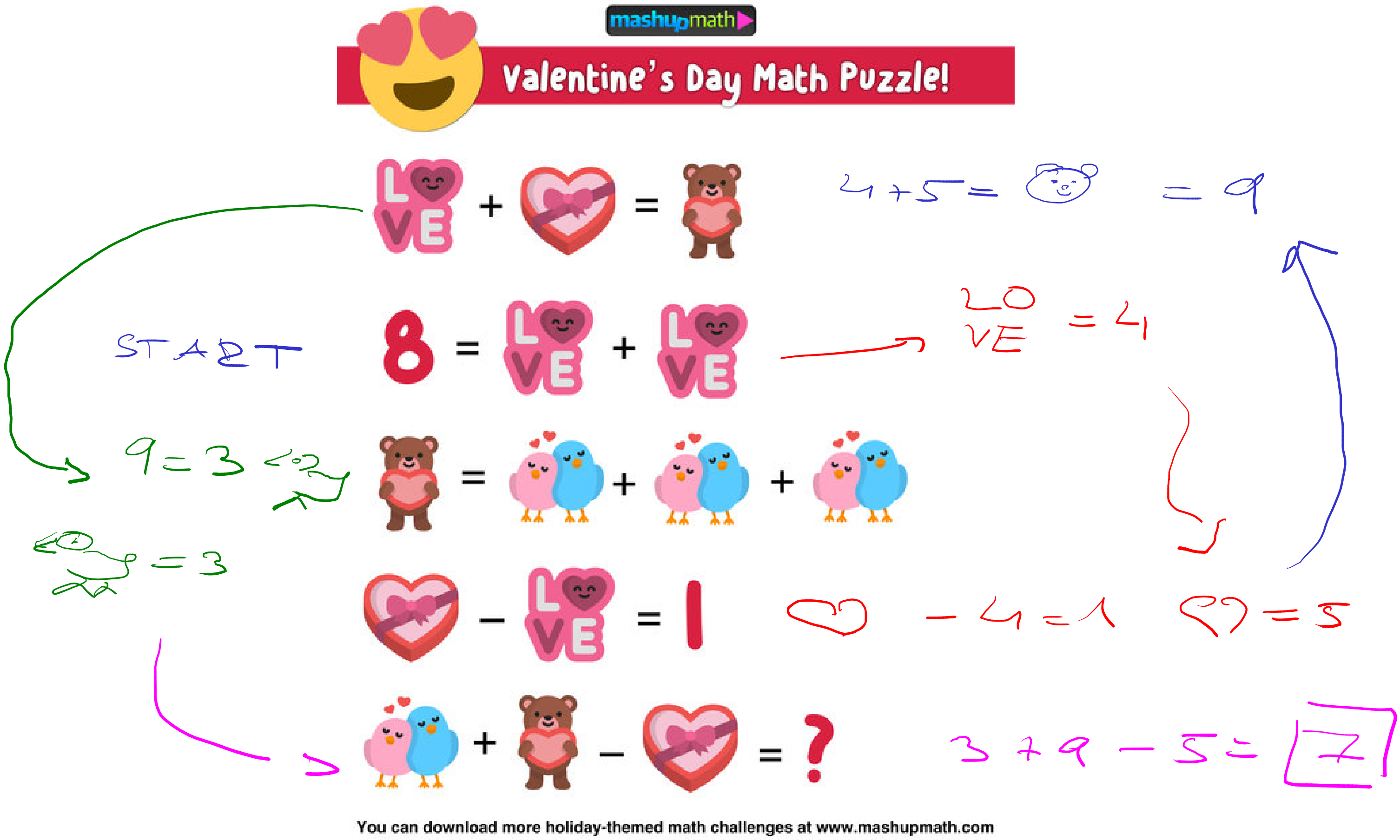
(a) $-x + 3y = 6, 2x + 6y = -6$

(b) $-x + 3y = 6, 2x + 6y = 12$

(c) $x + 3y = 6, 2x + 6y = 12$

(d) $x + 3y = 6, x + 3y = -3$

Source 4: http://mathquest.carroll.edu/libraries/ALG.student.edition.pdf



Source 5: `https://www.mashupmath.com/blog/2017/12/12/can-your-students-solve-these-star-wars-math-problems-192be`